Opuscula Math. 38, no. 4 (2018), 463–482
https://doi.org/10.7494/OpMath.2018.38.4.463 Opuscula Mathematica

POSITIVE DEFINITE FUNCTIONS AND DUAL PAIRS OF LOCALLY CONVEX SPACES

Daniel Alpay and Saak Gabriyelyan

Communicated by Palle E.T. Jorgensen

Abstract. Using pairs of locally convex topological vector spaces in duality and topologies defined by directed families of sets bounded with respect to the duality, we prove general factorization theorems and general dilation theorems for operator-valued positive definite functions.

Keywords: positive definite function, locally convex space, dual pair, the (strong) factorization property, dilation theory.

Mathematics Subject Classification: 42A82, 47A20, 47A68.

1. INTRODUCTION

Complex-valued positive definite functions and their associated reproducing kernel Hilbert spaces play a key role in stochastic processes and related topics. Also of key importance in this circle of ideas are the notions of positive operator from a Hilbert space into itself, and of a dilation of a positive definite function. The framework of complex-valued functions, or even functions taking values in a Hilbert space (see for instance [5]), is too restrictive for various applications, and the case of functions taking values in a locally convex topological vector space has been considered in various publications, originating with Petrick's unpublished report [9]. We also mention Masani's paper [6] where anti-linear operators from a Banach space into its dual (or conversely, from the dual of a Banach space into the Banach space itself). The case of Banach spaces is also considered in e.g. [4]. In the papers [1, 2, 8] and books [3, 10], the Banach space is replaced by a general locally convex topological vector space (with appropriate hypothesis). In the present paper we consider factorization and dilation theorems in a general setting, and prove results which encompass, to the best of our knowledge, all previous factorization and dilation results of this kind.

To present our results we first need some notation and definitions. We will use the general framework of spaces in duality. Let (X,Y) be a dual pair of complex locally convex spaces (lcs, for short). We denote by (y,x) the value of $y\in Y$ at $x\in X$ and by $\overline{L}(X,Y)$ the family of all antilinear operators from X to Y.

Definition 1.1. An antilinear operator T from X to Y is called *positive* if $(Tx,x)\geq 0$ for every $x\in X$.

Definition 1.2. Let Z be a set and let (X,Y) be a dual pair of complex lcs. A function $K:Z\times Z\to \overline{L}(X,Y)$ is called *positive definite* if

$$\sum_{i,k=1}^{n}\big(K[z_i,z_k]x_i,x_k\big)\geq 0,\quad \forall n\in\mathbb{N},\ \forall x_1,\dots,x_n\in X,\ \forall z_1,\dots,z_n\in Z.$$

Denote by $\mathfrak{S}(X,Y)$ the family of all collections $\mathcal{S}$ of $\sigma(X,Y)$-bounded subsets of X which are directed (that is for every $S_1,S_2\in\mathcal{S}$ there is $S_3\in\mathcal{S}$ such that $S_1\cup S_2\subseteq S_3$) and such that the linear span of $\cup\mathcal{S}$ is $\sigma(X,Y)$-dense in X. Then, by Theorem 8.5.1 of [7], for every $\mathcal{S}\in\mathfrak{S}(X,Y)$, the sets of the form

$$[S,\varepsilon]:=\{y\in Y:|(y,x)|<\varepsilon \text{ for all } x\in S\},\ \text{where } S\in\mathcal{S} \text{ and } \varepsilon>0,$$

define a base at zero of a Hausdorff locally convex topology $\tau_{\mathcal{S}}$ on Y (all topological spaces in the article are assumed to be Hausdorff), and we set $Y_{\mathcal{S}}:=(Y,\tau_{\mathcal{S}})$. If X' is the topological dual space of X we set $\mathfrak{S}(X):=\mathfrak{S}(X,X')$. In the case $\mathcal{S}=\mathsf{Bo}$ is the family of all bounded subsets of a complex lcs X, the topology $\tau_{\mathsf{Bo}}=\beta(X',X)$ is the strong topology on X'. If E is a locally convex space, we denote by $L_{\mathcal{S}}(X,E)$ (respectively, $\overline{L}_{\mathcal{S}}(X,E)$) the family of all linear (respectively, antilinear) operators from X into E which are continuous on the elements of $\mathcal{S}$. In particular, if $\mathcal{S}=\mathsf{Fin}$ is the family of all finite subsets of X, then $L_{\mathsf{Fin}}(X,E)$ is the space $L(X,E)$ of all linear operators from X to E. We denote by $CL(X,E)$ and $C\overline{L}(X,E)$ the spaces of all continuous linear and continuous antilinear operators from X into E, respectively.

If H is a complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and $T\in CL(X,H)$, we define the adjoint operator $T^*\in\overline{L}(H,X')$ by the formula $(T^*f,x):=\langle Tx,f\rangle$ for $f\in H, x\in X$.

We can now describe the content of the paper. The paper consists of two sections besides the introduction. In Section 2 we prove factorization theorems, first for a general set Z (see Theorem 2.1) and then when Z is endowed with a topological structure, see Theorem 2.6. We consider topologies on Y defined by directed families of $\sigma(X,Y)$-bounded sets, and for such a topology $\tau_{\mathcal{S}}$ define the notion of spaces with the $\mathcal{S}$-factorization property, and introduce the new family of $\mathcal{S}$-barrelled spaces. In Section 3 we prove a dilation result (see Theorem 3.2) which generalizes both the Sz.-Nagy and Naimark dilation theorems. In the previous factorization and dilation results obtained in [1–3, 6, 8, 9] the authors considered only the cases when $Y=X'$ or $X=Y'$, $\mathcal{S}=\mathsf{Bo}$ and Z is an abstract set or a unital Banach algebra. So our results generalize the existent ones in two directions: (1) instead of the strong topology on X' we study the general case of polar topology $\tau_{\mathcal{S}}$ on X', and (2) we consider the case when Z is a topological space and operator functions and representations are strongly or weakly continuous.

2. FACTORIZATION THEOREMS FOR DUAL PAIRS OF LOCALLY CONVEX SPACES

The following theorem generalizes Theorem 2.7 of [2] and its proof essentially uses the original idea of A.N. Kolmogorov.

Theorem 2.1. *Let Z be a set, (X, Y) a dual pair of complex lcs and $\mathcal{S} \in \mathfrak{S}(X, Y)$. Assume that a positive definite kernel $K : Z \times Z \to \overline{L}(X, Y_{\mathcal{S}})$ satisfies the condition*

$$K[z,z] \in \overline{L}_{\mathcal{S}}(X, Y_{\mathcal{S}}) \textit{ for all } z \in Z. \tag{2.1}$$

Then there exists a Hilbert space H and an operator function $T : Z \to L_{\mathcal{S}}(X, H)$ such that

$$\big(K[u,v]x, y\big) = \big\langle T(v)y, T(u)x\big\rangle, \quad \forall u, v \in Z,\ \forall x, y \in X. \tag{2.2}$$

In the case $Y = X'$ and H is minimal, i.e., $H = \overline{\operatorname{span}}\big\{T(Z)X\big\}$, then H and T are unique up to unitary equivalence.

Proof. Consider the complex valued kernel $R(h, g)$ of two variables $h = (t, v)$ and $g = (x, u)$ from $X \times Z$ defined by

$$R(h, g) := \big(K[u, v]x, t\big). \tag{2.3}$$

Since K is an operator valued positive definite kernel, for each $x \in X$ and $u, v \in Z$, the operator $K[u, v](\cdot)$ is antilinear and $K[u, v]x \in Y$ is a continuous linear functional of X. So, for each $n \in \mathbb{N}$ and every $g_1 = (x_1, z_1), \dots, g_n = (x_n, z_n) \in X \times Z$ and $c_1, \dots, c_n \in \mathbb{C}$, we have

$$\sum_{i,k=1}^{n} R(g_i, g_k)c_i\overline{c_k} = \sum_{i,k=1}^{n} \big(K[z_k, z_i]x_k, x_i\big)c_i\overline{c_k} = \sum_{i,k=1}^{n} \big(K[z_k, z_i]c_k x_k, c_i x_i\big) \geq 0$$

and hence the scalar valued kernel $R(h, g)$ is positive definite. Also this shows that for every n points $g_1, \dots, g_n \in X \times Z$ the complex matrix $[R(g_i, g_k)]$ is positive definite. Therefore there exists an n-dimensional Gaussian probability measure $\mu^{g_1 \cdots g_n}$ with mean vector zero whose covariance matrix is $[R(g_i, g_k)]$. The family of probability measures $\mu^{g_1 \cdots g_n}$ is obviously consistent. Consider the Borel space Ω of all complex valued functions f on $X \times Z$ with the smallest σ-field relative to which every projection map $\pi_g : \Omega \to \mathbb{C}, \pi_g(f) := f(g)$, is measurable. By Kolmogorov's theorem there exists a probability measure μ on Ω such that the joint distribution of $\big(f(g_1), \dots, f(g_n)\big)$, $f \in \Omega$, is $\mu^{g_1 \cdots g_n}$ for every $g_1, \dots, g_n \in X \times Z$. If we consider the Hilbert space $L^2(\mu)$ and define $\xi(g)(f) := \pi_g(f) = f(g), f \in \Omega$, then $\xi(g) \in L^2(\mu)$ and

$$\big\langle \xi(h), \xi(g)\big\rangle = \int_{\Omega} f(h)\overline{f(g)}d\mu(f) = R(h, g). \tag{2.4}$$

Set $H := \overline{\operatorname{span}}\{\xi(g) : g \in X \times Z\}$. Then H is a Hilbert space and, by (2.3) and (2.4), we have

$$\big(K[u, v]x, y\big) = R(h, g) = \big\langle \xi(h), \xi(g)\big\rangle = \big\langle \xi(y, v), \xi(x, u)\big\rangle.$$

For every $u \in Z$ define an operator $T(u) : X \to H$ by the formula

$$T(u)(x) := \xi(x,u), \quad x \in X.$$

We check that $T(u)$ is a linear operator for every $u \in Z$. For this we show that $\xi(x,u)$ is linear by the first variable. For every $a,b \in \mathbb{C}$ and $x,y,z \in X$, we have

$$\begin{aligned}\big\langle \xi(z,v), \xi(ax+by,u)\big\rangle &= \big(K[u,v](ax+by), z\big) = \overline{a}\big(K[u,v]x,z\big) + \overline{b}\big(K[u,v]y,z\big)\\ &= \overline{a}\big\langle \xi(z,v),\xi(x,u)\big\rangle + \overline{b}\big\langle \xi(z,v),\xi(y,u)\big\rangle\\ &= \big\langle \xi(z,v), a\xi(x,u)+b\xi(y,u)\big\rangle,\end{aligned}$$

and since $\operatorname{span}\{\xi(z,v) : (z,v) \in X \times Z\}$ is dense in H, it follows that $\xi(ax+by,u) = a\xi(x,u) + b\xi(y,u)$. So $T(u)$ is a linear operator.

By (2.4), for every $x,y \in X$ and $u,v \in Z$, we have $\big(K[u,v]x,y\big) = \big\langle T(v)y, T(u)x\big\rangle$, which shows that (2.2) holds.

To show that $T(u) \in L_{\mathcal{S}}(X,H)$ fix $S \in \mathcal{S}$ and $x \in S$. Let $\varepsilon > 0$. By (2.1) choose a neighborhood W of x in S such that $K[u,u]y - K[u,u]x \in \big(\varepsilon^2/2\big)S^\circ$ for every $y \in W$, where S° is the polar of S. So

$$\big|\big(K[u,u](y-x),t\big)\big| < \frac{\varepsilon^2}{2}, \quad \text{for every } t \in S. \tag{2.5}$$

Then, for every $y \in W \subseteq S$, (2.5) implies

$$\begin{aligned}\|T(u)(y-x)\|^2 &= \big|\big\langle T(u)(y-x), T(u)(y-x)\big\rangle\big| = \big|\big(K[u,u](y-x), y-x\big)\big|\\ &\le \big|\big(K[u,u](y-x),y\big)\big| + \big|\big(K[u,u](y-x),x\big)\big| < \varepsilon^2\end{aligned}$$

and hence $\|T(u)(y) - T(u)(x)\| < \varepsilon$. Thus $T(u) \in L_{\mathcal{S}}(X,H)$.

Assume that $Y = X'$, H_1 and H_2 are minimal and let $T_i(\cdot) : X \to H_i$, where $H_i = \overline{\operatorname{span}}\big\{T_i(z)X : z \in Z\big\}$, for $i = 1,2$, be such that

$$\big\langle T_1(v)y, T_1(u)x\big\rangle = \big(K[u,v]x,y\big) = \big\langle T_2(v)y, T_2(u)x\big\rangle, \quad \text{for all } u,v \in Z. \tag{2.6}$$

Denote by U_0 a map defined on $\operatorname{span}\big\{T_1(z)X : z \in Z\big\}$ by

$$\sum_{k=1}^N T_1(u_k)x_k \to \sum_{k=1}^N T_2(u_k)x_k$$

Then U_0 is well-defined. Indeed, if $\sum_{k=1}^N T_1(u_k)x_k = \sum_{i=1}^M T_1(v_i)y_i$, then, by (2.6), we have

$$\left\|\sum_{k=1}^N T_2(u_k)x_k - \sum_{i=1}^M T_2(v_i)y_i\right\|^2 = \left\|\sum_{k=1}^N T_1(u_k)x_k - \sum_{i=1}^M T_1(v_i)y_i\right\|^2 = 0.$$

So $\sum_{k=1}^N T_2(u_k)x_k = \sum_{i=1}^M T_2(v_i)y_i$ and U_0 is well-defined. Also (2.6) shows that U_0 preserves the inner product. Therefore U_0 extends to the unitary operator $U : H_1 \to H_2$ such that $UT_1(u) = T_2(u)$ for every $u \in Z$. $\square$

In Theorem 2.1 we obtained that $T(u)$ is continuous only on the elements of the family $\mathcal{S}$. To obtain the continuity of $T(u)$ we need the following property.

Definition 2.2. Let (X,Y) be a dual pair of complex lcs and $\mathcal{S} \in \mathfrak{S}(X,Y)$. The space X is said to have the (*strong*) $\mathcal{S}_Y$*-factorization property* if for each positive operator $R \in C\overline{L}(X, Y_{\mathcal{S}})$ (respectively, $R \in \overline{L}_{\mathcal{S}}(X, Y_{\mathcal{S}})$) the function $x \mapsto (Rx)(x)$ is continuous. In the case $Y = X'$ we will say that X has the (*strong*) $\mathcal{S}$*-factorization property.*

Note that if $Y = X'$, then X has the Fin-factorization property if and only if X has the strong factorization property in the sense [2], and X has the Bo-factorization property if and only if X has the factorization property in the sense of Definition 2.11 in [2].

The following theorem generalizes Theorem 2.12 of [2].

Theorem 2.3. *Let Z be a set, (X,Y) a dual pair of complex lcs and $\mathcal{S} \in \mathfrak{S}(X,Y)$. Assume that X has the (strong) $\mathcal{S}_Y$-factorization property and a positive definite kernel $K : Z \times Z \to \overline{L}(X, Y_{\mathcal{S}})$ satisfies the condition*

$$K[z,z] \in C\overline{L}(X, Y_{\mathcal{S}}) \ (\textit{respectively, } K[z,z] \in \overline{L}_{\mathcal{S}}(X, Y_{\mathcal{S}})) \textit{ for all } z \in Z. \tag{2.7}$$

Then there exists a Hilbert space H and an operator function $T : Z \to CL(X,H)$ such that

$$\big(K[u,v]x, y\big) = \big\langle T(v)y, T(u)x\big\rangle, \quad \forall u,v \in Z, \ \forall x,y \in X. \tag{2.8}$$

If $Y = X'$, then $K[u,v] \in C\overline{L}(X, X'_{\mathcal{S}})$ and

$$K[u,v] = T^*(v)T(u), \quad \forall u,v \in Z. \tag{2.9}$$

Moreover, if $Y = X'$ and H is minimal, i.e., $H = \overline{\operatorname{span}}\big\{T(Z)X\big\}$, then H and T are unique up to unitary equivalence.

Proof. By Theorem 2.1 there exist a Hilbert space H and an operator function $T : Z \to L_{\mathcal{S}}(X,H)$ such that (2.8) holds and $\|T(u)x\|^2 = \big(K[u,u]x, x\big)$ for every $u \in Z$ and $x \in X$. We have to check that $T(u) \in CL(X,H)$ for every $u \in Z$. Since the operator $K[u,u]$ is positive, (2.7) and the (strong) $\mathcal{S}_Y$-factorization property imply that the function $\|T(u)x\|^2$ is continuous. Thus $T(u)$ is continuous.

Assume that $Y = X'$. Then (2.8) implies that $K[u,v] = T^*(v)T(u)$ for all $u,v \in Z$, so (2.9) holds. Fix $u,v \in Z$. We show that $K[u,v] \in C\overline{L}(X, X'_{\mathcal{S}})$. Let $S \in \mathcal{S}$ and $\varepsilon > 0$. Since $\sigma(X,X')$-bounded sets are bounded in X by Theorem 8.8.7 of [7], $T(v)S$ is a bounded subset of H. Therefore there is $C > 0$ such that $\|T(v)y\| \le C$ for every $y \in S$. As $T(u)$ is continuous there is a neighborhood W of zero in X such that $\|T(u)x\| < \varepsilon / C$ for every $x \in W$. Then for every $x \in W$ and $y \in S$ we have

$$\big|\big(K[u,v]x, y\big)\big| = \big|\big\langle T(v)y, T(u)x\big\rangle\big| \le \|T(v)y\| \cdot \|T(u)x\| < C \cdot \frac{\varepsilon}{C} = \varepsilon.$$

Hence $K[u,v]x \in [S,\varepsilon]$ for every $x \in W$. Thus $K[u,v] \in C\overline{L}(X, X'_{\mathcal{S}})$.

The last assertion follows from Theorem 2.1. $\square$

In Theorems 2.1 and 2.3 the set Z is abstract. However the case Z is a topological space is of importance. This motivates the following definition.

Definition 2.4. Let (Z,τ) be a topological space, (X,Y) a dual pair of complex lcs and $\mathcal{S}\in\mathfrak{S}(X,Y)$. A positive definite kernel $K: Z\times Z\to C\overline{L}(X,Y_{\mathcal{S}})$ is called *weakly continuous* if the function

$$\big(K[u,v]x,t\big) : Z\times Z\to\mathbb{C}$$

is separately continuous for every $x,t\in X$, and we say that K is strongly continuous if K is weakly continuous and the map

$$K[u,u]x : Z\to Y_{\mathcal{S}}$$

is continuous for every $x\in X$.

Analogously we define weakly (strongly) continuous operator functions.

Definition 2.5. Let (Z,τ) be a topological space and let X and E be complex lcs. An operator function $T: Z\to CL(X,E)$ is called *weakly continuous* if the function

$$\big(t,T(u)x\big) : Z\to\mathbb{C}$$

is continuous for every $x\in X$ and $t\in E'$, and we say that T is strongly continuous if the map

$$T(u)x : Z\to E$$

is continuous for every $x\in X$.

The following theorem is a topological version of Theorems 2.1 and 2.3.

Theorem 2.6. *Let Z be a topological space, (X,Y) a dual pair of complex lcs and $\mathcal{S}\in\mathfrak{S}(X,Y)$. Assume that X has the $\mathcal{S}_Y$-factorization property and a positive definite kernel $K: Z\times Z\to C\overline{L}(X,Y_{\mathcal{S}})$ is weakly (strongly) continuous (and $\bigcup\mathcal{S}=X$, respectively). Then there exist a Hilbert space H and a weakly (strongly) continuous operator function $T: Z\to CL(X,H)$ such that*

$$\big(K[u,v]x,y\big)=\big\langle T(v)y,T(u)x\big\rangle,\quad \forall u,v\in Z,\ \forall x,y\in X. \tag{2.10}$$

If $Y=X'$, then

$$K[u,v]=T^*(v)T(u),\quad \forall u,v\in Z, \tag{2.11}$$

and if H is minimal, i.e., $H=\overline{\operatorname{span}}\big\{T(Z)X\big\}$, then H and T are unique up to unitary equivalence.

Proof. By Theorem 2.3 there exists a Hilbert space H and an operator function $T: Z\to CL(X,H)$ such that (2.10) and (2.11) hold, and if H is minimal, then H and T are unique up to unitary equivalence. So we shall assume that H is minimal. We have to check only that T is weakly (strongly) continuous.

Case 1. Assume that K is weakly continuous. Fix $u_0 \in Z$, $x \in X$, $t \in H$ and $\varepsilon > 0$. As X has the $\mathcal{S}_Y$-factorization property, the function $\|T(u)x\|^2 = (K[u,u]x,x)$ is continuous by x. So we can choose a neighborhood W of u_0 such that $\|T(u)x\| \le C$ for some $C > 0$ and every $u \in W$. Now we choose

$$h := \sum_{i=1}^{n} T(v_i)x_i \in H$$

such that

$$\|t - h\| < \frac{\varepsilon}{3C+3}. \tag{2.12}$$

For every $1 \le i \le n$ choose a neighborhood W_i of u_0 such that

$$\big|(K[v_i,u]x_i,x) - (K[v_i,u_0]x_i,x)\big| < \frac{\varepsilon}{3n}, \text{ for every } u \in W_i. \tag{2.13}$$

Set $U := W \cap \bigcap_{i=1}^{n} W_i$. So U is a neighborhood of u_0 such that for every $u \in U$, by (2.10), (2.12) and (2.13), we have

$$\begin{aligned}
\big|\langle T(u)x,t\rangle - \langle T(u_0)x,t\rangle\big| &\le \big|\langle T(u)x,t\rangle - \langle T(u)x,h\rangle\big| + \big|\langle T(u)x,h\rangle - \langle T(u_0)x,h\rangle\big| \\
&\quad + \big|\langle T(u_0)x,h\rangle - \langle T(u_0)x,t\rangle\big| \\
&\le \|T(u)x\| \cdot \|t-h\| + \sum_{i=1}^{n} \big|(K[v_i,u]x_i,x) - (K[v_i,u_0]x_i,x)\big| \\
&\quad + \|T(u_0)x\| \cdot \|t-h\| < \varepsilon.
\end{aligned}$$

So $\langle T(u)x,t\rangle$ is continuous at u_0.

Case 2. Assume that K is strongly continuous. Fix $u_0 \in Z$, $x \in X$ and a neighborhood $[S',\varepsilon]$ of zero in $Y_{\mathcal{S}}$, where $S' \in \mathcal{S}$ and $\varepsilon > 0$. Choose $S \in \mathcal{S}$ such that $\{x\} \cup S' \subseteq S$ (this is possible since $\mathcal{S}$ is directed and $\bigcup \mathcal{S} = X$). Since $K[u,u]x$ is continuous at u_0 and $(K[u,u]x,x)$ is separately continuous at (u_0,u_0), we can choose a neighborhood U of u_0 in Z such that

$$K[u,u]x - K[u_0,u_0]x \in \big[S, \varepsilon^2/3\big], \tag{2.14}$$

and

$$\big|(K[u_0,u_0]x,x) - (K[u_0,u]x,x)\big| < \frac{\varepsilon^2}{3}, \ \big|(K[u_0,u_0]x,x) - (K[u,u_0]x,x)\big| < \frac{\varepsilon^2}{3}, \tag{2.15}$$

for every $u \in U$. Then, for every $u \in U$, (2.10), (2.14) and (2.15) imply (recall that $x \in S$)

$$\begin{aligned}
\|T(u)x - T(u_0)x\|^2 &= \langle T(u)x,T(u)x\rangle - \langle T(u)x,T(u_0)x\rangle - \langle T(u_0)x,T(u)x\rangle \\
&\quad + \langle T(u_0)x,T(u_0)x\rangle \\
&= \Big|\big((K[u,u]x,x) - (K[u_0,u_0]x,x)\big) + \big((K[u_0,u_0]x,x) - (K[u_0,u]x,x)\big) \\
&\quad + \big((K[u_0,u_0]x,x) - (K[u,u_0]x,x)\big)\Big| \le \frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} = \varepsilon^2.
\end{aligned}$$

So $\|T(u)x - T(u_0)x\| \le \varepsilon$ for every $u \in U$, and hence $T(u)x$ is continuous at u_0. $\square$

Theorems 2.3 and 2.6 emphasize the importance of the $\mathcal{S}$-factorization property. In the next theorem (see also Theorem 3.9 in the next section) we give several characterizations of spaces with the $\mathcal{S}$-factorization property and generalize Proposition 2.13 of [2].

Theorem 2.7. *Let X be a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. Then the following assertions are equivalent.*

(i) *X has the $\mathcal{S}$-factorization property.*

(ii) *For each positive definite kernel $K : Z \times Z \to \overline{L}(X, X'_{\mathcal{S}})$ satisfying the condition*
$$K[z,z] \in C\overline{L}(X, X'_{\mathcal{S}}) \text{ for all } z \in Z,$$
there exists a Hilbert space H and an operator function $T : Z \to CL(X,H)$ such that $K[u,v] = T^*(v)T(u)$ for all $u, v \in Z$. Moreover, if H is minimal, i.e., $H = \overline{\operatorname{span}}\{T(Z)X\}$, then H and T are unique up to unitary equivalence.

(iii) *For each positive operator $R \in C\overline{L}(X, X'_{\mathcal{S}})$ there exist a Hilbert space H and $T \in CL(X,H)$ such that $R = T^*T$. Moreover, if H is minimal, i.e., $H = \overline{\operatorname{span}}\{T(Z)X\}$, then H and T are unique up to unitary equivalence.*

(iv) *For each inner product $\langle \cdot, \cdot \rangle$ defined on X satisfying the condition*
$$\begin{gathered} \text{for every } S \in \mathcal{S},\ p_S(x) := \sup_{y \in S} |\langle x, y\rangle| < \infty \\ \text{and the seminorm } p_S \text{ is continuous,} \end{gathered} \tag{2.16}$$
the function $\langle x, x\rangle$ is continuous.

Proof. (i)⇒(ii) follows from Theorem 2.3, and (ii)⇒(iii) follows from Theorem 2.3 if we put Z is a singleton.

(iii)⇒(iv) Let $\langle \cdot, \cdot \rangle$ be an inner product in X for which (2.16) holds. Define an operator $R : X \to X'$ by the formula
$$(Rx)(y) := \langle y, x\rangle, \quad \forall x, y \in X.$$
Then R is positive and antilinear. To prove that $R \in C\overline{L}(X, X'_{\mathcal{S}})$ fix $S \in \mathcal{S}$ and $\varepsilon > 0$. By (2.16) take a neighborhood W of zero such that $p_S(x) < \varepsilon$ for every $x \in W$. Then, for every $x \in W$, we obtain
$$|(Rx)(y)| < \varepsilon, \quad \forall y \in S,$$
which means that $Rx \in [S, \varepsilon]$. Hence $R \in C\overline{L}(X, X'_{\mathcal{S}})$. By (iii) there is a Hilbert space H and $T \in CL(X,H)$ such that $R = T^*T$. So $\langle x, x\rangle = (Rx)(x) = \|Tx\|^2$ is continuous.

(iv)⇒(i) For a positive operator $R \in C\overline{L}(X, X'_{\mathcal{S}})$ define an inner product in X by the formula $\langle y, x\rangle := (Rx)(y)$ for $x, y \in X$. We show that this inner product satisfies (2.16). Fix $S \in \mathcal{S}$. Then, for every $x \in X$, $p_S(x)$ is finite because the continuous functional Rx is bounded on S. Let us show that p_S is continuous. Let $\varepsilon > 0$. As $R \in C\overline{L}(X, X'_{\mathcal{S}})$ choose a neighborhood W of zero in X such that $Rx \in [S, \varepsilon]$ for every $x \in W$. Then
$$p_S(x) = \sup_{y \in S} |(Rx)(y)| \le \varepsilon \ \text{ for every } x \in W.$$
Hence p_S is continuous. Now (iv) implies that the function $(Rx)(x) = \langle x, x\rangle$ is continuous. Thus X has the $\mathcal{S}$-factorization property. □

Below we define a class of complex locally convex spaces which have the $\mathcal{S}$-factorization property.

Definition 2.8. Let X be a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. We say that X is $\mathcal{S}$-barrelled if each lower semicontinuous seminorm p on X which is continuous on every $S \in \mathcal{S}$ is continuous.

Note that a space X is Bo-barrelled if and only if it is pseudo-barrelled in the sense of Definition 2.1 in [1]. The next proposition generalizes Proposition 3.1 of [2] and has a simpler proof.

Proposition 2.9. *Each $\mathcal{S}$-barrelled space has the $\mathcal{S}$-factorization property.*

Proof. Let R be a positive operator from $C\overline{L}(X, X'_{\mathcal{S}})$. Then, by Theorem 2.1, there is a minimal Hilbert space H and an operator $T \in L_{\mathcal{S}}(X, H)$ such that $R = T^*T$, so $T(X)$ is dense in H by the minimality of H. For every $h \in T(X) \cap B_H$, where B_H is the closed unit ball of H, define a seminorm p_h on X by the equality

$$p_h(x) := |\langle h, T(x)\rangle|, \quad x \in X.$$

Note that each p_h is continuous. Indeed, if $h = T(y)$ for some $y \in T^{-1}\big(T(X) \cap B_H\big)$ and a net $\{x_\alpha\}$ converges to $x \in X$, then $R(y) \in X'$ and

$$\begin{aligned} p_h(x_\alpha) &= \big|\big\langle h, T(x_\alpha)\big\rangle\big| = \big|\big\langle T(y), T(x_\alpha)\big\rangle\big| \\ &= |R(y)(x_\alpha)| \to |R(y)(x)| = \big|\big\langle h, T(x)\big\rangle\big| = p_h(x). \end{aligned}$$

Hence, the new seminorm p on X defined by

$$p(x) := \sup\left\{p_h(x) : h \in T(X) \cap B_H\right\}, \quad x \in X,$$

is lower semicontinuous as the supremum of continuous seminorms. Since $p_h(x) \le \|T(x)\|$ we obtain $p(x) \le \|T(x)\|$. On the other hand, for $h = T(x)/\|T(x)\| \in T(X) \cap B_H$, we have $p_h(x) = \|T(x)\|$. Therefore $p(x) = \|T(x)\|$ for every $x \in X$. As $T \in L_{\mathcal{S}}(X, H)$ we obtain that $p(x) = \|T(x)\|$ is also continuous on the elements of $\mathcal{S}$. So $\mathcal{S}$-barrelledness of X implies that p is continuous. Hence $(Rx)(x) = \|T(x)\|^2$ is continuous. Thus X has the $\mathcal{S}$-factorization property. □

Let (X, Y) be a dual pair of complex lcs and $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{S}(X, Y)$. We say that $\mathcal{S}_1 \preceq \mathcal{S}_2$ if for every $S_1 \in \mathcal{S}_1$ there is $S_2 \in \mathcal{S}_2$ such that $S_1 \subseteq S_2$. Clearly, $\mathsf{Fin} \preceq \mathsf{Bo}$. For future references we note the following simple assertion whose proof is straightforward.

Lemma 2.10. *Let (X, Y) be a dual pair of complex lcs and $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{S}(X, Y)$. If $\mathcal{S}_1 \preceq \mathcal{S}_2$, then*

(i) $\tau_{\mathcal{S}_1} \subseteq \tau_{\mathcal{S}_2}$;
(ii) $\overline{L}_{\mathcal{S}_2}(X, Y_{\mathcal{S}_2}) \subseteq \overline{L}_{\mathcal{S}_1}(X, Y_{\mathcal{S}_1})$ *and* $C\overline{L}(X, Y_{\mathcal{S}_2}) \subseteq C\overline{L}(X, Y_{\mathcal{S}_1})$;
(iii) *if X has the (strong) $(\mathcal{S}_1)_Y$-factorization property, then X has the (strong) $(\mathcal{S}_2)_Y$-factorization property.*

Corollary 2.11. *Let X be a complex lcs. Then X is* Fin*-barrelled if and only if it is barrelled. In particular, a barrelled lcs has the $\mathcal{S}$-factorization property for every $\mathcal{S} \in \mathfrak{S}(X)$.*

Proof. The first assertion follows from [7, Theorem 11.4.3]. The second one follows from the first assertion, Proposition 2.9 and Lemma 2.10 (iii). □

So Proposition 3.5 of [2] follows from Proposition 2.9.

Remark 2.12. Let X be a complex lcs and let $\mathcal{S} \in \mathfrak{S}(X)$ (in particular, $\mathcal{S} = \mathsf{KD}$ is the family of all compact discs of X). Then Lemma 2.10 shows that if X has the (strong) $\mathcal{S}$-factorization property, then X has the (strong) Bo-factorization property. So there are complex lcs X without the (strong) $\mathcal{S}$-factorization property by Example 1.4 of [1].

Remark 2.13. Let X be a complex Banach space and let $\mathcal{N}$ be the family of all closed balls in X centered at zero. Clearly, $\mathcal{N} \in \mathfrak{S}(X)$ and $\tau_{\mathcal{N}}$ is the Banach space topology on X'. So $\overline{L}_{\mathcal{N}}(X, X'_{\mathcal{N}}) = C\overline{L}(X, X'_{\mathcal{N}})$, and hence X has the strong $\mathcal{N}$-factorization property if and only if it has the $\mathcal{N}$-factorization property.

We do not know whether there exists a complex lcs X with the $\mathcal{S}$-factorization property but without the strong $\mathcal{S}$-factorization property.

3. DILATION THEORY

Let G be a multiplicative (unital) semigroup and let H be a Hilbert space. The operator valued map $\pi : G \to CL(H)$ is called a (*unital*) *representation of* G if $\pi(uv) = \pi(u)\pi(v)$ for every $u, v \in G$ (and, respectively, $\pi(e) = I_H$, where e is a unit of G and I_H is the identity operator in H).

The next definition modifies and generalizes Definition 4.1 of [2].

Definition 3.1. Let X be a complex lcs, $\mathcal{S} \in \mathfrak{S}(X)$, G a semigroup and let $K : G \times G \to \overline{L}(X, X'_{\mathcal{S}})$ be a positive kernel. An $\mathcal{S}$-*dilation* of the kernel K is a triple (H, π, R) consisting of a Hilbert space H, a representation $\pi : G \to CL(H)$ of G in H and an operator $R \in CL(X, H)$ such that

$$K[u, v] = R^* \pi(v)^* \pi(u) R, \quad \forall u, v \in G. \tag{3.1}$$

$\mathcal{S}$-dilations (H, π, R) and (H_1, π_1, R_1) of K are *unitary equivalent* if there is a unitary operator $U : H \to H_1$ such that

$$U\pi(g)R = \pi_1(g)R_1, \quad \forall g \in G. \tag{3.2}$$

Note that if G is unital and π and π_1 are unital representations, then

$$U\pi(e)R = UR = \pi_1(e)R_1 = R_1.$$

So we can write (3.2) as follows

$$U\pi(g)R = \pi_1(g)UR, \quad \forall g \in G.$$

It is easy to show (see the proof of Theorem 3.2 below) that (3.1) implies that the kernel K satisfies the *boundedness condition*, i.e., there is a function $\rho : G \to [0,\infty)$ such that

$$\sum_{i,k=1}^{n} (K[gu_i, gu_k]x_i, x_k) \leq \rho(g) \sum_{i,k=1}^{n} (K[u_i, u_k]x_i, x_k), \tag{3.3}$$

for all $n \in \mathbb{N}$, $u_1, \ldots, u_n, g \in G$ and $x_1, \ldots, x_n \in X$. If G is also a topological space we say that K satisfies the *locally boundedness condition* if the function $\rho(g)$ in (3.3) is locally bounded, i.e., for every $g \in G$ there is a neighborhood U of g such that $\rho(U)$ is a bounded subset of $\mathbb{R}$.

The following theorem generalizes Theorem 2.12 of [2].

Theorem 3.2 (Generalized dilation theorem)**.** *Let G be a unital semigroup, X a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. Assume that X has the (strong) $\mathcal{S}$-factorization property and $K : G \times G \to \overline{L}(X, X'_{\mathcal{S}})$ is a positive definite kernel satisfying the condition*

$$K[g,g] \in C\overline{L}(X, X'_{\mathcal{S}}) \ (\textit{respectively, } K[g,g] \in \overline{L}_{\mathcal{S}}(X, X'_{\mathcal{S}})) \textit{ for all } g \in G.$$

Then K satisfies the boundedness condition (3.3) *if and only if K has an $\mathcal{S}$-dilation (H, π, R), where π is a unital representation of G in the Hilbert space H. The minimality condition $H = \overline{\operatorname{span}}\{\pi(G)RH\}$ determines (H, π, R) uniquely up to unitary equivalence.*

Proof. Assume that K satisfies the boundedness condition (3.3). By Theorem 2.3, $K[u,v] \in C\overline{L}(X, X'_{\mathcal{S}})$ for all $u, v \in G$, and there exists a Hilbert space H and an operator function $T : G \to CL(X, H)$ such that

$$K[u,v] = T^*(v)T(u), \quad \forall u, v \in G.$$

Moreover, if H is minimal, i.e., $\operatorname{span}\{T(G)X\}$ is dense in H, then H and T are unique up to unitary equivalence. In what follows we shall assume that H is minimal.

If $\sum_{i=1}^{n} T(u_i)x_i = 0$ for some $n \in \mathbb{N}$, $u_1, \ldots, u_n \in G$ and $x_1, \ldots, x_n \in X$, then the boundedness condition (3.3) implies

$$\begin{aligned} 0 &\leq \left\langle \sum_{i=1}^{n} T(gu_i)x_i, \sum_{i=1}^{n} T(gu_i)x_i \right\rangle = \sum_{i,k=1}^{n} \big(K[gu_i, gu_k]x_i, x_k\big) \\ &\leq \rho(g) \sum_{i,k=1}^{n} \big(K[u_i, u_k]x_i, x_k\big) = \rho(g) \left\langle \sum_{i=1}^{n} T(u_i)x_i, \sum_{i=1}^{n} T(u_i)x_i \right\rangle = 0, \end{aligned}$$

for every $g \in G$. So for every $g \in G$ we can define an operator $\pi(g)$ on the dense linear subspace $\operatorname{span}\{T(g)X : g \in G\}$ of H by the formula

$$\pi(g)\left(\sum_{i=1}^{n} T(u_i)x_i\right) := \sum_{i=1}^{n} T(gu_i)x_i. \tag{3.4}$$

The boundedness condition (3.3) implies that $\pi(g)$ extends uniquely to a continuous operator $\pi(g) \in CL(H)$. Since

$$\pi(gh)\big(T(u)x\big) = T(ghu)x = \pi(g)\big(T(hu)x\big) = \pi(g)\big(\pi(h)\big(T(u)x\big)\big)$$

for all $g, h, u \in G$ and $x \in X$, and

$$\pi(e)\big(T(u)x\big) = T(eu)x = T(u)x = I_H\big(T(u)x\big),$$

we obtain that π is a unital representation of G on H. As

$$K[ug, vh] = T^*(vh)T(ug) = T^*(h)\pi^*(v)\pi(u)T(g), \quad \forall u, v, g, h \in G,$$

for $g = h = e$ and $R := T(e)$ we obtain

$$K[u, v] = R^*\pi^*(v)\pi(u)R.$$

Thus (H, π, R) is an $\mathcal{S}$-dilation of K.

Let us show that (H, π, R) is unique up to unitary equivalence. First we recall that by (3.4)

$$T(g) = T(g \cdot e) = \pi(g)T(e) = \pi(g)R.$$

Now let (H_1, π_1, R_1) be an $\mathcal{S}$-dilation of K such that $H_1 = \overline{\operatorname{span}}\{\pi_1(G)R_1H_1\}$. For every $g \in G$, set $T_1(g) := \pi_1(g)R_1 \in CL(X, H_1)$. Then $K[u, v] = T_1^*(v)T_1(u)$ for every $u, v \in G$ and $H_1 = \overline{\operatorname{span}}\{T_1(G)H_1\}$. So, by Theorem 2.3, there exists a unitary operator $U : H \to H_1$ such that $UT(g) = T_1(g)$ for every $g \in G$. This exactly means that $U\pi(g)R = \pi_1(g)R_1$ for every $g \in G$. Thus (H, π, R) and (H_1, π_1, R_1) are unitary equivalent.

Conversely, assume that K has an $\mathcal{S}$-dilation (H, π, R). Then, for all $n \in \mathbb{N}$, $u_1, \dots, u_n, g \in G$ and $x_1, \dots, x_n \in X$, the equality (3.1) implies

$$\begin{aligned}
\sum_{i,k=1}^{n} (K[gu_i, gu_k]x_i, x_k) &= \sum_{i,k=1}^{n} \big(R^*\pi(gu_k)^*\pi(gu_i)Rx_i, x_k\big) \\
&= \sum_{i,k=1}^{n} \big\langle \pi(g)\pi(u_i)Rx_i, \pi(g)\pi(u_k)Rx_k \big\rangle \\
&\le \|\pi(g)\|^2 \sum_{i,k=1}^{n} \big\langle \pi(u_i)Rx_i, \pi(u_k)Rx_k \big\rangle \\
&= \|\pi(g)\|^2 \sum_{i,k=1}^{n} (K[u_i, u_k]x_i, x_k).
\end{aligned}$$

So K satisfies the boundedness condition (3.3) with $\rho(g) = \|\pi(g)\|^2$. $\square$

Recall that a (unital) semigroup G is a $*$-*semigroup* if there is an involution $*$ on G, i.e., a unary operation $*$ on G such that $(gh)^* = h^*g^*$ and $g^{**} = g$ (and $e^* = e$, respectively).

Definition 3.3. Let G be a $*$-semigroup, X a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. A map $B : G \to \overline{L}(X, X'_{\mathcal{S}})$ is called *positive definite* if the function $K[u,v] : G \times G \to \overline{L}(X, X'_{\mathcal{S}})$ defined by $K[u,v] := B(v^*u)$ is positive definite. A positive definite function B is said to be (*strongly*) *weakly continuous* if the associated map $K[u,v]$ is (strongly) weakly continuous, and B has an $\mathcal{S}$-dilation if $K[u,v]$ has an $\mathcal{S}$-dilation.

A representation π of a $*$-semigroup G in a Hilbert space H is called a $*$-*representation* if $\pi(g^*) = \pi^*(g)$ for every $g \in G$. Theorem 3.2 implies the following generalization of the Sz.-Nagy theorem.

Corollary 3.4 (Sz.-Nagy dilation theorem)**.** *Let G be a unital $*$-semigroup, X a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. Assume that X has the (strong) $\mathcal{S}$-factorization property and let $B : G \to \overline{L}(X, X'_{\mathcal{S}})$ be a positive definite function such that*

$$B(g^*g) \in C\overline{L}(X, X'_{\mathcal{S}}) \quad (\textit{respectively, } B(g^*g) \in \overline{L}_{\mathcal{S}}(X, X'_{\mathcal{S}})) \textit{ for all } g \in G.$$

If the function $B(v^*u)$ satisfies the boundedness condition (3.3)*, then B has an $\mathcal{S}$-dilation (H, π, R), where π is a unital $*$-representation of G on a Hilbert space H. The minimality condition $H = \overline{\operatorname{span}}\{\pi(G)RH\}$ determines (H, π, R) uniquely up to unitary equivalence.*

Proof. We have to show only that the representation π built in the proof of Theorem 3.2 is a $*$-representation.

Note that for every $u, v \in G$ and $x, y \in X$, the definition of $K[u,v]$ and (2.2) imply

$$\begin{aligned}\langle T(v)y, \pi(g^*)T(u)x\rangle &= \langle T(v)y, T(g^*u)x\rangle = \big(K[g^*u, v]x, y\big) = \big(B(v^*g^*u)x, y\big) \\ &= \big(B\big((gv)^*u\big)x, y\big) = \big(K[u, gv]x, y\big) = \langle T(gv)y, T(u)x\rangle \\ &= \langle \pi(g)T(v)y, T(u)x\rangle = \langle T(v)y, \pi^*(g)T(u)x\rangle.\end{aligned} \tag{3.5}$$

Taking into account that $H = \overline{\operatorname{span}}\{T(u)X : u \in G\}$, (3.5) implies that $\pi(g^*) = \pi^*(g)$. Thus π is a $*$-representation. $\square$

Let (Ω, Σ) be a measurable space. Then the σ-algebra Σ can be considered as a unital $*$-semigroup with the unit $e := \Omega$ and operations

$$\Delta_1 \cdot \Delta_2 := \Delta_1 \cap \Delta_2 \quad \text{and} \quad \Delta^* := \Delta.$$

This remark motivates the following definition.

Definition 3.5. Let (Ω, Σ) be a measurable space, X a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. A map $E : \Sigma \to C\overline{L}(X, X'_{\mathcal{S}})$ is called a *positive $C\overline{L}(X, X'_{\mathcal{S}})$-valued measure* on (Ω, Σ) if the function

$$K[\Delta_1, \Delta_2] : \Sigma \times \Sigma \to C\overline{L}(X, X'_{\mathcal{S}}), \quad K[\Delta_1, \Delta_2] := E(\Delta_2^*\Delta_1) = E(\Delta_1 \cap \Delta_2),$$

is positive definite and

$$\big(E(\Delta)x, x\big)$$

is a positive measure on (Ω, Σ) for every $x \in X$.

Lemma 3.6. *$K[\Delta_1,\Delta_2]$ satisfies the boundedness condition* (3.3) *with* $\rho(\Delta)\equiv 1$.

Proof. For every $\Delta,\Delta'\in\Sigma$ and each $x,x'\in X$ we have

$$\begin{aligned}&\big(E(\Delta\Delta')x,x'\big)+\big(E(\Delta'\Delta)x',x\big)\\&=\big(E(\Delta\Delta')(x+x'),x+x'\big)-\big(E(\Delta\Delta')x,x\big)-\big(E(\Delta\Delta')x',x'\big).\end{aligned}$$

Using this equality and the definition of K, for every $\Delta_1,\dots,\Delta_n\in\Sigma$ and each $x_1,\dots,x_n\in X$ we obtain

$$\begin{aligned}\sum_{i,k=1}^{n}\big(K[\Delta_i,\Delta_k]x_i,x_k\big)&=\sum_{i,k=1}^{n}\big(E(\Delta_i\Delta_k)x_i,x_k\big)\\&=\sum_{i=1}^{n}\big(E(\Delta_i)x_i,x_i\big)\\&\quad+\sum_{1\le i<k\le n}\big[\big(E(\Delta_i\Delta_k)x_i,x_k\big)+\big(E(\Delta_i\Delta_k)x_k,x_i\big)\big]\\&=\sum_{i=1}^{n}\big(E(\Delta_i)x_i,x_i\big)+\sum_{1\le i<k\le n}\big(E(\Delta_i\Delta_k)(x_i+x_k),x_i+x_k\big)\\&\quad-\sum_{1\le i<k\le n}\big[\big(E(\Delta_i\Delta_k)x_i,x_i\big)+\big(E(\Delta_i\Delta_k)x_k,x_k\big)\big]\\&=\sum_{1\le i<k\le n}\big(E(\Delta_i\Delta_k)(x_i+x_k),x_i+x_k\big)\\&\quad+\sum_{i=1}^{n}\left[\big(E(\Delta_i)x_i,x_i\big)-\sum_{k=1,k\neq i}^{n}\big(E(\Delta_i\Delta_k)x_i,x_i\big)\right],\end{aligned}$$

and hence (recall that $(E(\Delta)x,x)$ is a positive measure)

$$\begin{aligned}&\sum_{i,k=1}^{n}\big(K[\Delta_i,\Delta_k]x_i,x_k\big)\\&=\sum_{1\le i<k\le n}\big(E(\Delta_i\Delta_k)(x_i+x_k),x_i+x_k\big)+\sum_{i=1}^{n}\left(E\left(\Delta_i\setminus\bigcup_{k=1,k\neq i}^{n}\Delta_k\right)x_i,x_i\right).\end{aligned}$$

This representation easily implies the desired. □

Recall also the following definition.

Definition 3.7. Let (Ω,Σ) be a measurable space and H a complex Hilbert space. A map $E:(\Omega,\Sigma)\to CL(H)$ is called a *spectral measure* if, for every $\Delta\in\Sigma$, the operator $E(\Delta)$ is symmetric (i.e., $\langle E(\Delta)x,y\rangle=\langle x,E(\Delta)y\rangle$ for all $x,y\in H$), idempotent (i.e., $E(\Delta)^2=E(\Delta)$), and $E(\Omega)=I_H$.

The next corollary generalizes Corollary 4.6 of [2].

Corollary 3.8 (Naimark dilation theorem)**.** *Let X be a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. Assume that X has the $\mathcal{S}$-factorization property and let E be a positive $C\overline{L}(X, X'_{\mathcal{S}})$-valued measure on a measurable space (Ω, Σ). Then E has an $\mathcal{S}$-dilation (H, E_H, R), where E_H is a spectral measure on (Ω, Σ). The minimality condition $H = \overline{\operatorname{span}}\{E_H(\Sigma)RH\}$ determines (H, E_H, R) uniquely up to unitary equivalence.*

Proof. By Lemma 3.6 we can apply Corollary 3.4 to find an $\mathcal{S}$-dilation (H, E_H, R), where $E_H : (\Omega, \Sigma) \to CL(H)$ is a unital $*$-representation of Σ on a Hilbert space H. The minimality condition determines (H, E_H, R) uniquely up to unitary equivalence. Since

$$\langle E_H(\Delta)x, y\rangle = \langle x, E_H(\Delta)^* y\rangle = \langle x, E_H(\Delta^*)y\rangle = \langle x, E_H(\Delta)y\rangle$$

the operator $E_H(\Delta)$ is symmetric, it is also idempotent since

$$E_H(\Delta)^2 = E_H(\Delta \cdot \Delta) = E_H(\Delta).$$

As E_H is unital, we have $E_H(\Omega) = I_H$. Thus E_H is a spectral measure. □

Theorem 3.9. *Let X be a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. Then the following assertions are equivalent to* (i)-(iv) *of Theorem 2.7.*

(v) *For each unital semigroup G and every positive definite kernel $K : G \times G \to \overline{L}(X, X'_{\mathcal{S}})$ satisfying the boundedness condition* (3.3) *and such that $K[g, g] \in C\overline{L}(X, X'_{\mathcal{S}})$ for all $g \in G$, K has an $\mathcal{S}$-dilation (H, π, R), where π is a unital representation of G. The minimality condition $H = \overline{\operatorname{span}}\{\pi(G)RX\}$ determines (H, π, R) uniquely up to unitary equivalence.*

(vi) *For each unital $*$-semigroup G and every positive definite function $B : G \to \overline{L}(X, X'_{\mathcal{S}})$ for which $B(g^*g) \in C\overline{L}(X, X'_{\mathcal{S}})$ for all $g \in G$ and $B(v^*u)$ satisfies the boundedness condition* (3.3)*, B has an $\mathcal{S}$-dilation (H, π, R), where π is a unital $*$-representation of G on the Hilbert space H. The minimality condition $H = \overline{\operatorname{span}}\{\pi(G)RX\}$ determines (H, π, R) uniquely up to unitary equivalence.*

(vii) *For each positive $C\overline{L}(X, X'_{\mathcal{S}})$-valued measure on a measurable space (Ω, Σ), E has an $\mathcal{S}$-dilation (H, E_H, R), where $E_H : (\Omega, \Sigma) \to CL(H)$ is a spectral measure on (Ω, Σ). The minimality condition $H = \overline{\operatorname{span}}\{E_H(\Sigma)RH\}$ determines (H, E_H, R) uniquely up to unitary equivalence.*

Proof. (i)⇒(v) is Theorem 3.2. (v) implies (vi) by the proof of Corollary 3.4, where we show that π is a $*$-representation. (vi)⇒(vii) follows from Corollary 3.8. Finally, (vii) implies (iii) for the singleton space $\big(\{z\}, \Sigma\big)$ and the positive $C\overline{L}(X, X'_{\mathcal{S}})$-valued measure E defined by $E(\emptyset) := 0$ and $E(\{z\}) := R$. □

Below we consider a topological analogue of Theorem 3.2. Recall that a ($*$-semigroup) semigroup G is called *semitopological* if G is a topological space such that the multiplication is separately continuous (and, additionally, the involution $*$ is continuous).

Theorem 3.10. *Let G be a unital semitopological semigroup, X a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. Assume that X has the $\mathcal{S}$-factorization property and let $K : G\times G \to C\overline{L}(X, X'_{\mathcal{S}})$ be a weakly (strongly) continuous positive definite kernel satisfying the boundedness condition (3.3) with a locally finite function $\rho(g)$ (and $\bigcup \mathcal{S} = X$, respectively). Then K has an $\mathcal{S}$-dilation (H, π, R), where π is a unital weakly (strongly) continuous representation of G. The minimality condition $H = \overline{\mathrm{span}}\{\pi(G)RX\}$ determines (H, π, R) uniquely up to unitary equivalence.*

Proof. By Theorem 2.6, there is a minimal Hilbert space H and a weakly (strongly) continuous operator function $T : Z \to CL(X, H)$ such that

$$K[u, v] = T^*(v)T(u), \quad \forall u, v \in G.$$

As in the proof of Theorem 3.2, there is a unital representation π of G in $CL(H)$ which is defined on the dense linear subspace $\mathrm{span}\{T(g)X : g \in G\}$ of H by the formula

$$\pi(g)\left(\sum_{i=1}^{n} T(u_i)x_i\right) := \sum_{i=1}^{n} T(gu_i)x_i$$

and satisfies the condition (where $R = T(e) \in CL(X, H)$)

$$K[u, v] = R^*\pi^*(v)\pi(u)R, \quad \forall u, v \in G.$$

So (H, π, R) is an $\mathcal{S}$-dilation of K which is unique up to unitary equivalence by Theorem 3.2. We have to show only that π is weakly (strongly) continuous. Note that the definition of $\pi(g)$ and the boundedness condition (3.3) imply that

$$\|\pi(g)\|^2 \le \rho(g), \quad \forall g \in G. \tag{3.6}$$

We distinguish between two cases.

Case 1. Assume that K is weakly continuous. Fix $g_0 \in G$, $s, t \in H$ and let $\varepsilon > 0$. Since $\rho(g)$ is locally bounded, by (3.6), we can choose a neighborhood W of g_0 and $C > 0$ such that

$$\|\pi(g)\| \le C, \quad \forall g \in W. \tag{3.7}$$

Choose

$$h := \sum_{i=1}^{n} T(u_i)x_i \in H \ \text{ and } \ z := \sum_{j=1}^{m} T(v_j)y_j \in H$$

such that

$$\|h - s\| < \frac{\varepsilon}{4C(\|s\| + 1)},\ \|z - t\| < \frac{\varepsilon}{4C(\|t\| + 1)} \ \text{ and } \ \big|\langle \pi(g_0)h, z\rangle - \langle \pi(g_0)s, t\rangle\big| < \frac{\varepsilon}{4}. \tag{3.8}$$

Since T is weakly continuous, choose a neighborhood $U \subseteq W$ of g_0 such that

$$\big|\langle T(gu_i)x_i, T(v_j)y_j\rangle - \langle T(g_0u_i)x_i, T(v_j)y_j\rangle\big| < \frac{\varepsilon}{4mn}, \tag{3.9}$$

for every $g \in U$ and all $1 \le i \le n$ and $1 \le j \le m$.

Then, for every $g \in U$, the inequalities (3.7)–(3.9) imply

$$\begin{aligned}
&\big|\langle \pi(g)s, t\rangle - \langle \pi(g_0)s, t\rangle\big| \\
&\le \big|\langle \pi(g)s, t\rangle - \langle \pi(g)s, z\rangle\big| \\
&\quad + \big|\langle \pi(g)s, z\rangle - \langle \pi(g)h, z\rangle\big| + \big|\langle \pi(g)h, z\rangle - \langle \pi(g_0)h, z\rangle\big| + \big|\langle \pi(g_0)h, z\rangle - \langle \pi(g_0)s, t\rangle\big| \\
&\le \|\pi(g)\| \cdot \|s\| \cdot \|t - z\| + \|\pi(g)\| \cdot \|h - s\| \cdot \|z\| \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{m} \big|\langle T(gu_i)x_i, T(v_j)y_j\rangle - \langle T(g_0u_i)x_i, T(v_j)y_j\rangle\big| + \frac{\varepsilon}{4} < \varepsilon.
\end{aligned}$$

Hence the function $\langle \pi(g)s, t\rangle$ is continuous by g. Thus π is weakly continuous.

Case 2. Assume that K is strongly continuous. Fix $g_0 \in G$, $s \in H$ and let $\varepsilon > 0$. Since $\rho(g)$ is locally bounded, by (3.6), we can choose a neighborhood W of g_0 and $C > 0$ such that (3.7) holds true. Choose $h := \sum_{i=1}^{n} T(u_i)x_i \in H$ such that

$$\|h - s\| < \frac{\varepsilon}{3C}. \tag{3.10}$$

Since T is strongly continuous choose a neighborhood $U \subseteq W$ of g_0 such that

$$\big\|T(gu_i)x_i - T(g_0u_i)x_i\big\| < \frac{\varepsilon}{3n}, \tag{3.11}$$

for every $g \in U$ and all $1 \le i \le n$. Now for every $g \in U$ the inequalities (3.7), (3.10) and (3.11) imply

$$\begin{aligned}
\|\pi(g)s - \pi(g_0)s\| &\le \|\pi(g)s - \pi(g)h\| + \|\pi(g)h - \pi(g_0)h\| + \|\pi(g_0)h - \pi(g_0)s\| \\
&\le \|\pi(g)\| \cdot \|s - h\| + \sum_{i=1}^{n} \big\|T(gu_i)x_i - T(g_0u_i)x_i\big\| \\
&\quad + \|\pi(g_0)\| \cdot \|h - s\| < \varepsilon.
\end{aligned}$$

Hence $\pi(g)s$ is continuous at g_0. Thus $\pi(g)s$ is continuous. □

The following result is a topological version of Corollary 3.4.

Corollary 3.11. *Let G be a unital semitopological $*$-semigroup, X a complex lcs and $\mathcal{S} \in \mathfrak{S}(X)$. Assume that X has the $\mathcal{S}$-factorization property and let $B : G \to C\overline{L}(X, X'_{\mathcal{S}})$ be a weakly (strongly) continuous positive definite function (and $\bigcup \mathcal{S} = X$, respectively). If the function $K[u, v] := B(v^*u)$ satisfies the boundedness condition (3.3) with a locally finite function $\rho(g)$, then B has an $\mathcal{S}$-dilation (H, π, R), where π is a unital weakly (strongly) continuous $*$-representation of G on a Hilbert space H. The minimality condition $H = \overline{\operatorname{span}}\{\pi(G)RX\}$ determines (H, π, R) uniquely up to unitary equivalence.*

We finish the article with the existence and uniqueness of propagators for operator functions. The theory of propagators is of importance, for explanations and historical remarks see §3 of [6].

Let G be a (unital) semigroup and Z be a set. We say that G *acts* on Z if there is a map $p : G \times Z \to Z$, $p(g,z) := g \cdot z$, such that $(gh) \cdot z = g \cdot (h \cdot z)$ (and $e \cdot z = z$, respectively) for every $g, h \in G$ and $z \in T$.

Definition 3.12. Let Z be a set, G a semigroup acting on Z, X a complex lcs, H a Hilbert space and $T : Z \to CL(X,H)$ an operator function on Z. Set $H_T := \overline{\operatorname{span}}\{T(Z)X\}$ and let $\widetilde{T}$ be the corestriction of T to H_T. An operator function $P : G \to CL(H_T)$ is called a propagator or *controller of* T if

$$P(g) \circ \widetilde{T}(z) = \widetilde{T}(g \cdot z), \qquad \text{for every } z \in Z,\ g \in G. \tag{3.12}$$

Let Z be a set, G a semigroup acting on Z, X a complex lcs and H a Hilbert space. We say that an operator function $T : Z \to CL(X,H)$ satisfies the *boundedness condition* if there is a function $\rho : G \to [0,\infty)$ such that

$$\left\| \sum_{i=1}^{n} T\big(g \cdot z_i\big) x_i \right\| \le \rho(g) \left\| \sum_{i=1}^{n} T\big(z_i\big) x_i \right\|, \tag{3.13}$$

for all $n \in \mathbb{N}$, $g \in G$, $z_1, \dots, z_n \in Z$ and $x_1, \dots, x_n \in X$.

The next theorem generalizes Theorem 3.4 of [6].

Theorem 3.13. *Let Z be a set, G a* (*unital*) *semigroup acting on Z, X a complex lcs, H a Hilbert space and $T : Z \to CL(X,H)$ an operator function on Z. Then T satisfies the boundedness condition* (3.13) *if and only if T has a unique propagator P which is a* (*unital*) *representation of G in the Hilbert space H_T.*

Proof. We essentially repeat the proof of Theorem 3.2. Assume that T satisfies the boundedness condition (3.13). Considering H_T and $\widetilde{T}$ instead of H and T respectively, without loss of generality we can assume that $H = H_T$ and $T = \widetilde{T}$. If $\sum_{i=1}^{n} T(z_i)x_i = 0$ for some $n \in \mathbb{N}$, $z_1, \dots, z_n \in Z$ and $x_1, \dots, x_n \in X$, then the boundedness condition (3.13) implies

$$0 \le \left\| \sum_{i=1}^{n} T\big(g \cdot z_i\big) x_i \right\| \le \rho(g) \left\| \sum_{i=1}^{n} T\big(z_i\big) x_i \right\| = 0,$$

for every $g \in G$. So for every $g \in G$ we can define an operator $P(g)$ on the dense linear subspace $\operatorname{span}\{T(z)X : z \in Z\}$ of H by the formula

$$P(g)\left(\sum_{i=1}^{n} T(z_i)x_i \right) := \sum_{i=1}^{n} T(g \cdot z_i)x_i.$$

The boundedness condition (3.13) implies that $P(g)$ extends uniquely to a continuous operator $P(g) \in CL(H)$. In particular, for every $z \in Z$, $g \in G$ and $x \in X$ we have

$$P(g)\big(T(z)x\big) = T(g \cdot z)x, \text{ and hence } P(g) \circ T(z) = T(g \cdot z),$$

that means P is a propagator of T. Moreover, the equality (3.12) defines P uniquely.

Since

$$P(gh)\big(T(z)x\big) = T\big((gh)\cdot z\big)x = P(g)\big(T(h\cdot z)x\big) = P(g)\big(P(h)\big(T(z)x\big)\big)$$

for all $z \in Z$, $g, h \in G$ and $x \in X$, (and in the case G is unital $P(e)\big(T(z)x\big) = T(e\cdot z)x = T(z)x = I_H\big(T(z)x\big)$) we obtain that P is a (unital) representation of G on H.

Conversely, assume that T has a unique propagator P which is a (unital) representation of G in the Hilbert space H_T. Then for every $n \in \mathbb{N}$, $g \in G$, $z_1, \ldots, z_n \in Z$ and $x_1, \ldots, x_n \in X$ the equality (3.12) implies

$$\left\|\sum_{i=1}^{n} T\big(g\cdot z_i\big)x_i\right\| = \left\|P(g)\left(\sum_{i=1}^{n} T\big(z_i\big)x_i\right)\right\| \leq \|P(g)\| \left\|\sum_{i=1}^{n} T\big(z_i\big)x_i\right\|.$$

So the boundedness condition (3.13) holds with $\rho(g) := \|P(g)\|$. □

We say that a semitopological semigroup G acts *continuously* on a topological space Z if the valuation map $(g, z) \mapsto g\cdot z$ from $G\times Z$ to Z is continuous. Below we consider a topological version of Theorem 3.13.

Theorem 3.14. *Let Z be a topological space, G a* (*unital*) *semitopological semigroup acting continuously on Z, X a complex lcs, H a Hilbert space and $T : Z \to CL(X, H)$ a weakly* (*strongly*) *continuous operator function on Z. If T satisfies the boundedness condition* (3.13) *with a locally finite function $\rho(g)$, then T has a unique propagator P which is a* (*unital*) *weakly* (*strongly*) *continuous representation of G in the Hilbert space H_T.*

Proof. Considering H_T and $\widetilde{T}$ instead of H and T respectively, without loss of generality we can assume that $H = H_T$ and $T = \widetilde{T}$. By Theorem 3.13, T has a unique propagator $P : G \to CL(H)$, which is a (unital) representation of G such that

$$P(g)\left(\sum_{i=1}^{n} T(z_i)x_i\right) := \sum_{i=1}^{n} T(g\cdot z_i)x_i, \tag{3.14}$$

for every $n \in \mathbb{N}$, $g \in G$, $z_1, \ldots, z_n \in Z$ and $x_1, \ldots, x_n \in X$. Since $\operatorname{span}\{T(z)X : z \in Z\}$ is dense in H, (3.13) and (3.14) imply

$$\|P(g)\| \leq \rho(g), \quad g \in G. \tag{3.15}$$

Now replacing (3.6) by (3.15) and $\pi(g)$ by $P(g)$ and repeating word for word the proofs of Cases 1 and 2 in Theorem 3.10 we obtain that P is weakly (strongly) continuous. □

Acknowledgements

Daniel Alpay thanks the Foster G. and Mary McGaw Professorship in Mathematical Sciences, which supported this research.

REFERENCES

[1] J. Górniak, *Locally convex spaces with factorization property*, Colloq. Math. **48** (1984) 1, 69–79.

[2] J. Górniak, A. Weron, *Aronszajn-Kolmogorov type theorems for positive definite kernels in locally convex spaces*, Studia Math. **69** (1980/81) 3, 235–246.

[3] M.A. Lifshits, *Gaussian Random Functions*, Kluwer Academic Publishers, 1995.

[4] A. Makagon, *Remark on the Extrapolation of Banach Space Valued Stationary Processes*, Probability Theory on Vector Spaces, II (Proc. Second Internat. Conf., Blazejewko, 1979), Lecture Notes in Math., vol. 828, Springer, Berlin, 1980, 196–207.

[5] A. Makagon, H. Salehi, *Notes on Infinite-Dimensional Stationary Sequences*, Probability theory on vector spaces, IV (Łańcut, 1987), Lecture Notes in Math., vol. 1391, Springer, Berlin, 1989, 200–238.

[6] P. Masani, *Dilations as propagators of Hilbertian varieties*, SIAM J. Math. Anal. **9** (1978), 414–456.

[7] L. Narici, E. Beckenstein, *Topological Vector Spaces*, second ed., Pure and Applied Mathematics (Boca Raton), vol. 296, CRC Press, Boca Raton, FL, 2011.

[8] H. Niemi, A. Weron, *Dilation theorems for positive definite operator kernels having majorants*, J. Funct. Anal. **40** (1981) 1, 54–65.

[9] G. Pedrick, *Theory of reproducing kernels for Hilbert spaces of vector-valued functions*, Univ. of Kansas Tech. Rep. 19, Lawrence, 1957.

[10] N.N. Vakhania, *Probability Distributions on Linear Spaces*, North-Holland Publishing Co., New York, 1981, Translated from the Russian by I.I. Kotlarski, North-Holland Series in Probability and Applied Mathematics.

Daniel Alpay
alpay@chapman.edu

Department of Mathematics
Chapman University
One University Drive Orange, California 92866, USA

Saak Gabriyelyan
saak@math.bgu.ac.il

Department of Mathematics
Ben Gurion University of the Negev
P.O.B. 653, Be'er Sheva 84105, Israel

Received: March 28, 2017.
Accepted: August 23, 2017.