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Vibrations of a Horizontal Elastic Band Plate Submerged in Fluid of Constant Depth

Kazimierz Szmidt, Benedykt Hedzielski

Institute of Hydro-Engineering, Polish Academy of Sciences, ul. Kościerska 7, 80-328 Gdańsk, Poland, e-mail: jks@ibwpan.gda.pl

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Abstract

The paper deals with free and forced vibrations of a horizontal thin elastic plate submerged in an infinite layer of fluid of constant depth. In free vibrations, the pressure load on the plate results from assumed displacements of the plate. In forced vibrations, the fluid pressure is mainly induced by water waves arriving at the plate. In both cases, we have a coupled problem of hydrodynamics in which the plate and fluid motions are coupled through boundary conditions at the plate surface. At the same time, the pressure load on the plate depends on the gap between the plate and the fluid bottom. The motion of the plate is accompanied by the fluid motion. This leads to the so-called co-vibrating mass of fluid, which strongly changes the eigenfrequencies of the plate. In formulation of this problem, a linear theory of small deflections of the plate is employed. In order to calculate the fluid pressure, a solution of Laplace's equation is constructed in the doubly connected infinite fluid domain. To this end, this infinite domain is divided into sub-domains of simple geometry, and the solution of the problem equation is constructed separately for each of these domains. Numerical experiments are conducted to illustrate the formulation developed in this paper.

Key words: elastic plate, free vibrations, forced vibrations, eigenfrequencies, co-vibrating mass of fluid

1. Introduction

In offshore engineering, we frequently deal with the problem of water flow-induced loads on structures. Hydrodynamic forces depend on fluid flows in the vicinity of the structure as well as on the structure size, shape, rigidity and foundation. Usually, such a structure consists of parts of simple geometry such as bars, pipes and plates, and therefore, in a theoretical description of the structure dynamics, it is reasonable to investigate a dynamic behaviour of individual elements. Among them, of primary importance are elastic plates submerged in fluid and loaded with forces induced by gravitational waves. An example is a horizontal plate foundation of a windmill installed in a sea coastal zone. Usually, hydrodynamic forces depend not only on the waves themselves, but also on the foundation of the plate and its orientation to the direction of wave propagation. For instance, wave forces on a plane plate perpendicular to the wave propagation direction are different from those for a plate whose surface is parallel to the wave direction. In general, these forces may also depend on the distance between the plate and the boundaries of the fluid domain. In cases of horizontal plates placed at a small distance from the sea bottom, one may expect a certain amplification of hydrodynamic forces loading these plates. This phenomenon is associated with changes in the velocities in flows on the upper and bottom surfaces of the plates. At the same time, a vibration of the plate submerged in fluid leads to the so-called co-vibrating mass of fluid, which strongly changes the eigenfrequencies of this plate.

With respect to the above, we focus our investigations on the coupled hydrodynamic problem of a horizontal plate vibrating in a layer of fluid of constant depth. In order to simplify our discussion, we confine our attention to a simply supported elastic band plate, which makes it possible to reduce the description of a physical three-dimensional problem to a two-dimensional one. In a formal way, the two-dimensional description model corresponds directly to a simply supported horizontal beam, submerged in fluid of constant depth. An additional simplification introduced into the description is that plate deflections are assumed infinitesimally small. In theoretical investigations, we resort to approximate modeling that can describe the main features of this phenomenon.

As regards vibrations of plates in contact with fluid, Solecki (1966) discussed the problem of an infinite plate floating on a water half-space. A similar problem of the deformation of floating ice plates was investigated by Kerr and Palmer (1972). As far as a finite fluid body is concerned, Sawicki (1975) discussed the problem of the dynamics of floating roofs of cylindrical tanks. A detailed discussion on the dynamics of an elastic band plate floating on a tank with a rectangular cross section is given in Sawicki (1976). In particular, general solutions for the problem of free and forced vibrations of the plate may be found in that paper. The problem discussed in the present paper corresponds in a sense to that of Sawicki, but it deals with an infinite fluid domain and a fully submerged plate. Our main goal is to calculate a set of the lowest eigenfrequencies of the plate, dependent on the width of the gap between the plate and the bottom, as well as to evaluate deflections of the plate loaded with surface gravitational waves.

2. Problem Formulation

Let us consider the two-dimensional problem of a thin elastic band plate submerged in fluid, as shown schematically in Fig. 1.

The motion of the plate is accompanied by the fluid motion, and thus we have the so-called coupled problem of hydrodynamics. This coupling takes place through the boundary conditions at the upper and bottom surfaces of the plate. The normal components of the fluid and plate velocities should be equal to each other. With respect

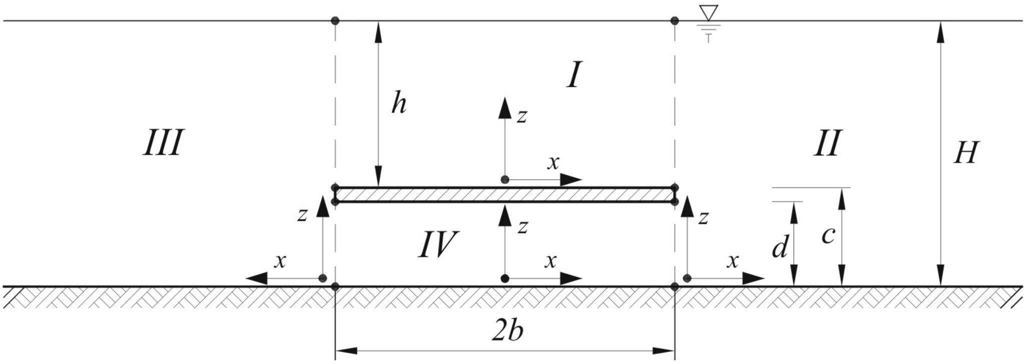

Fig. 1. Elastic plate submerged in a layer of fluid

to small deflections of the plate, its motion is governed by the following equation (Nowacki 1972):

$$m_{pl.}\frac{\partial^2 w}{\partial t^2} + D^* \frac{\partial^4 w}{\partial x^4} = p_{low.} - p_{upp.}, \tag{1}$$

where $m_{pl.}$ is the mass per unit width and length of the plate, $D^* = E\delta^3/12(1-\nu^2)$ is the flexural rigidity of the plate (δ is the plate thickness, and ν is Poisson's ratio), $p_{low.}$ and $p_{upp.}$ denote fluid pressure at the lower and upper surfaces of the plate. It should be stressed that the 'density' of the plate $m_{pl.} = (\rho_{pl.} - \rho)\delta$, where $\rho_{pl.}$ is the density of the plate material, and ρ is the fluid density. Assuming a potential velocity of the fluid motion, the associated fluid pressure is described by the formula

$$p = -\rho \frac{\partial \Phi}{\partial t}, \tag{2}$$

where $\Phi(x, z, t)$ is the velocity potential satisfying Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = \nabla^2 \Phi = 0 \tag{3}$$

within the fluid domain and appropriate boundary conditions at the fluid boundaries.

In discussing the problem of free vibrations of the plate and, in particular, the problem of the plate eigenfrequencies when the fluid flow is induced solely by the motion of the plate, it is justified to assume that the free surface of the fluid is flat over the entire range of time considered (fluid pressure is constant at $z = H$). With this assumption, the plate-fluid system is conservative, i.e. there is no damping of the plate motion. When vibrations of the plate are forced, for instance by gravitational waves arriving from infinity, the potential function within the fluid domain is affected by vibrations of the plate. In the cases of both free and forced vibrations of the plate, it is necessary to solve the coupled problem of the plate and fluid motion. Before doing that, however, it is convenient to consider, in the first step, the simplest case of free vibrations of the plate in air. For such a case, equation (1) reduces to the following one:

$$m_{pl.}\frac{\partial^2 w}{\partial t^2} + D^* \frac{\partial^4 w}{\partial x^4} = 0, \tag{4}$$

where $m_{pl.} = \rho_{pl.}\delta$.
For harmonic vibrations, the following relation holds:

$$w(x,t) = W(x)\exp(i\omega t). \tag{5}$$

Substitution of this relation into equation (4) gives

$$-\omega^2 m_{pl.} W + D^* \frac{\partial^4 W}{\partial x^4} = 0. \tag{6}$$

The deflection amplitude $W(x)$ is expressed in the form

$$W(x) = \sum_n A_n \sin r_n(x+b), \quad r_n = \frac{n\pi}{2b}, \ n = 1, 2, \cdots, \tag{7}$$

which satisfies boundary conditions at $x = \pm b$. From substitution of this solution into (6), the following is obtained:

$$\sum_n A_n \left[-m_{pl.}\omega^2 + D^*(r_n)^4\right] \sin r_n(x+b) = 0. \tag{8}$$

It may be seen that the functions $\sin r_n(x+b)$ ($n = 1, 2, \cdots$) form the eigenfunction set of the problem. At the same time, equation (8) leads to the set of eigenfrequencies of the plate

$$\omega_n = \left(\frac{n\pi}{2b}\right)^2 \sqrt{\frac{D^*}{m_{pl.}}}, \quad n = 1, 2, \cdots. \tag{9}$$

In order to find an associated set of frequencies for the case of free vibrations of the plate in fluid, it is necessary to calculate the fluid pressure and to solve equation (1). The plate deflection for this case is expressed in the form of a series with respect to eigenfunctions $\sin r_n(x+b)$, inherent for vibrations in air. To calculate the fluid pressure, we have to find a solution of Laplace's equation in the doubly connected fluid domain, satisfying the following system of boundary conditions:

$$\begin{gathered} \Phi|_{x\to\pm\infty} = 0, \quad \left.\tfrac{\partial\Phi}{\partial x}\right|_{x\to\pm\infty} = 0, \quad \left.\tfrac{\partial\Phi}{\partial z}\right|_{z=0} = 0, \\ \frac{\partial\Phi}{\partial n} \simeq \left.\frac{\partial\Phi}{\partial z}\right|_{upp.bot.} = \frac{\partial w(x,t)}{\partial t} = \dot{w}(x,t), \end{gathered} \tag{10}$$

where '*upp.bot.*' means the upper and bottom surfaces of the plate.

For the thin plate considered, the normal velocity components of the fluid at two sides of the plate (upper and bottom surfaces) are assumed to be equal to the transverse velocity of the plate centre. A remark is needed. In addition to the boundary conditions given above, it is necessary to investigate the potential behaviour at the plate end points ($x = \pm b$). If the cross section of the plate is rectangular with two right angles at these ends, the fluid velocity field is singular at these corner points. It may be shown, however, that this fluid velocity field is an integrable function along an arbitrary path in the vicinity of the end points.

In order to find a solution of Laplace's equation in the doubly connected fluid domain, we divide this domain into four parts: *I*, *II*, *III* and *IV* (see Fig. 1). In descriptions of the potential functions within these domains, it is convenient to introduce local Cartesian coordinate systems. Thus, with respect to these coordinate systems, the general solution of Laplace's equation read:

Subdomain *III* ($0 \le x < \infty, \quad 0 \le z \le H$)

$$\phi(x,z,t) = -\sum_{j=1} C_j(t)\frac{1}{k_j}\exp(-k_j x)\cos k_j z, \quad k_j = \frac{2j-1}{2H}\pi, \quad j = 1,2,\cdots. \tag{11}$$

Subdomain *II* ($0 \le x < \infty, \quad 0 \le z \le H$)

$$\phi(x,z,t) = -\sum_{j=1} B_j(t)\frac{1}{k_j}\exp(-k_j x)\cos k_j z, \quad k_j = \frac{2j-1}{2H}\pi, \quad j = 1,2,\cdots. \tag{12}$$

Subdomain *I* ($-b \le x < +b, \quad 0 \le z \le h$)

$$\begin{aligned}\phi(x,z,t) = &-\sum_{n=1} \dot{A}_n(t)\frac{1}{r_n}\frac{1}{v_n}\left[\exp(-r_n z) - \exp r_n(z-2h)\right]\sin r_n(x+b)+ \\ &+\sum_{m=1}\left[D_m^1(t)\frac{\cosh(k_m^* x)}{\cosh(k_m^* b)} + D_m^2(t)\frac{\sinh(k_m^* x)}{\sinh(k_m^* b)}\right]\cos k_m^* z, \quad r_n = \frac{n\pi}{2b}, \\ &v_n = 1 + \exp(-2r_n h), \quad n = 1,2,\cdots, \quad k_m^* = \frac{2m-1}{2h}\pi, \quad m = 1,2,\cdots,\end{aligned} \tag{13}$$

where $\dot{A}_n(t) = dA_n/dt$.

Subdomain *IV* ($-b \le x \le +b, \quad 0 \le z \le d$)

$$\begin{aligned}\varphi(x,z,t) = &\sum_{n=1} \dot{A}_n(t)\frac{1}{r_n}\frac{\cosh r_n z}{\sinh r_n d}\sin r_n(x+b)+ \\ &+ E_0(t) + \sum_{m=1}\left[E_m^1(t)\frac{\cosh k_m x}{\cosh k_m b} + E_m^2(t)\frac{\sinh k_m x}{\sinh k_m b}\right]\cos k_m z, \\ &r_n = \frac{n\pi}{2b}, \quad n = 1,2,\cdots, \qquad k_m = \frac{m\pi}{d}, \quad m = 1,2,\cdots.\end{aligned} \tag{14}$$

One can see that the series in the infinite domains (equations 11 and 12) quickly decay as $x \to \infty$. In practical calculations, it is justified to neglect the series for $x \ge L$, where L may be specified for a particular fluid motion considered. In this way, the solution in the finite fluid domain ($0 \le x \le L$) will be practically equal to that valid in the infinite domain ($0 \le x \le \infty$). To save space, hereinafter we omit the time character t in description of the functions $\dot{A}(t), \cdots, E(t)$, i.e. all functions are named constants. In accordance with the linear problem considered, all constants, $B_j, C_j, \quad j = 1, 2, \cdots$, $D^1_m, D^2_m, \quad m = 1, 2, \cdots$ and E^0, and $E^1_m, E^2_m, m = 1, 2, \cdots$, may be expressed in terms of the constants $\dot{A}_n, \quad n = 1, 2, \cdots$. It means that, for an arbitrary deflection of the plate, it is possible to find appropriate solutions within the corresponding fluid domains. To this end, we match the solutions at common boundaries of the subdomains. Thus, let us assume that, in advance, the solutions corresponding to B_j and C_j ($j = 1, 2, \cdots$) are known. The potential $\varphi(x = b, z)$ below the plate should be equal to that of the right-hand side domain, i.e. to $\phi(x = 0, z)$ at the common boundary. This condition gives

$$E_0 + \sum_{m=1} \left(E^1_m + E^2_m\right) \cos k_m z = -\sum_{j=1} B_j \frac{1}{k_j} \cos k_j z. \tag{15}$$

In a similar way, at the boundary ($x = -b, z$) we have

$$E_0 + \sum_{m=1} \left(E^1_m - E^2_m\right) \cos k_m z = -\sum_{j=1} C_j \frac{1}{k_j} \cos k_j z. \tag{16}$$

Multiplication of equation (17) in succession by $\cos k_m z$ ($m = 1, 2, \cdots$) and then integration in the range ($0 \le z \le d$) leads to the following formulae:

$$\begin{aligned} &E_0 = -\frac{1}{d} \sum_{j=1} B_j \frac{1}{(k_j)^2} \sin k_j d, \quad k_j = \frac{2j-1}{2H}\pi, \quad k_m = \frac{m\pi}{d}, \\ &E^1_m + E^2_m = -\frac{2}{d} \sum_{j=1} B_j \left[\begin{array}{ll} \dfrac{(-1)^m \sin k_j d}{(k_j)^2 - (k_m)^2} & \text{for} \quad k_j \ne k_m \\ \dfrac{d}{2k_j} & \text{for} \quad k_j = k_m \end{array} \right. , \quad j, m = 1, 2, \cdots \end{aligned} \tag{17}$$

Similar results hold for the left boundary:

$$\begin{aligned} &E_0 = -\frac{1}{d} \sum_{j=1} C_j \frac{1}{(k_j)^2} \sin k_j d, \\ &E^1_m - E^2_m = -\frac{2}{d} \sum_{j=1} C_j \left[\begin{array}{ll} \dfrac{(-1)^m \sin k_j d}{(k_j)^2 - (k_m)^2} & \text{for} \quad k_j \ne k_m \\ \dfrac{d}{2k_j} & \text{for} \quad k_j = k_m, \quad j, m = 1, 2, \cdots. \end{array} \right. \end{aligned} \tag{18}$$

The same procedure is employed for the upper fluid. Simple manipulations give

$$\begin{aligned} &\sum_{m=1} \left(D^1_m + D^2_m\right) \cos k^*_m z = -\sum_j B_j \frac{1}{k_j} \cos k_j (z + c), \\ &\sum_{m=1} \left(D^1_m - D^2_m\right) \cos k^*_m z = -\sum_j C_j \frac{1}{k_j} \cos k_j (z + c), \end{aligned} \tag{19}$$

and, finally

$$D_m^1 + D_m^2 = -\frac{2}{h}\sum_{j=1} B_j \left[\begin{array}{ll} \dfrac{\sin k_j c}{(k_m^*)^2 - (k_j)^2} & \text{for} \quad k_j \neq k_m^*, \\ \dfrac{1}{2(k_j)^2}\left(k_j h \cdot \cos k_j c - \sin k_j c\right) & \text{for} \quad k_j = k_m^*, \end{array}\right. \tag{20}$$
$$k_m^* = \frac{(2m-1)}{2h}\pi, \qquad j, m = 1, 2, \cdots,$$

and

$$D_m^1 - D_m^2 = -\frac{2}{h}\sum_{j=1} C_j \left[\begin{array}{ll} \dfrac{\sin k_j c}{(k_m^*)^2 - (k_j)^2} & \text{for} \quad k_j \neq k_m^*, \\ \dfrac{1}{2(k_j)^2}\left(k_j h \cdot \cos k_j c - \sin k_j c\right) & \text{for} \quad k_j = k_m^*, \end{array}\right. \tag{21}$$
$$j, m = 1, 2, \cdots.$$

Equations (17–21) result from comparison of the potential functions at common boundaries. These equations ensure equal pressure at the boundaries formed by neighbouring fluid domains. With the relations derived, the description of the problem has been reduced to fewer unknown constants, i.e. to B_j and $C_j (j = 1, 2, \cdots)$. Obviously, the fluid velocities at the common boundaries of matching domains must be the same. To this end, not only the fluid pressure, but also the normal components of the velocity field at the common boundaries should be uniquely defined. This condition makes it possible to express all constants in terms of the constants $\dot{A}_n$ that enter the description of the plate deflection. Thus, with respect to the above, it is necessary to calculate the horizontal velocity components. For the fluid below the plate $(-b \leq x \leq +b,\ 0 \leq z \leq d)$, one obtains

$$\begin{aligned} u = \frac{\partial \varphi}{\partial x} &= \sum_{n=1} \dot{A}_n \frac{\cosh r_n z}{\sinh r_n d} \cos r_n(x+b) + \\ &+ \sum_{m=1} k_m \left(E_m^1 \frac{\sinh k_m x}{\cosh k_m b} + E_m^2 \frac{\cosh k_m x}{\sinh k_m b}\right) \cos k_m z, \quad r_n = \frac{n\pi}{2b}. \end{aligned} \tag{22}$$

For the upper fluid domain $(-b \leq x \leq +b,\ 0 \leq z \leq h)$, we have a similar relation:

$$\begin{aligned} u = \frac{\partial \phi}{\partial x} &= -\sum_{n=1} \dot{A}_n \frac{1}{v_n} \left[\exp(-r_n z) - \exp r_n(z - 2h)\right] \cos r_n(x+b) + \\ &+ \sum_{m=1} k_m^* \left(D_m^1 \frac{\sinh k_m^* x}{\cosh k_m^* b} + D_m^2 \frac{\cosh k_m^* x}{\sinh k_m^* b}\right) \cos k_m z. \end{aligned} \tag{23}$$

It should be stressed that the vertical coordinate z in equation (23) differs from that in equation (22) (they are local with respect to corresponding fluid domains). With equations (22) and (23) it is possible to calculate horizontal velocities at the boundaries $x = \pm b$. For the right boundary $(x = +b)$, equations (22) and (23) give

$$
\begin{aligned}
&0 \le z \le d \\
&u = \left.\frac{\partial \varphi}{\partial x}\right|_{x=b} = \\
&= \sum_{n=1} \dot{A}_n (-1)^n \frac{\cosh r_n z}{\sinh r_n d} + \sum_{m=1} k_m \left(E_m^1 \tanh k_m b + E_m^2 \frac{1}{\tanh k_m b} \right) \cos k_m z = \\
&\qquad = \sum_j B_j \cos k_j z \ ,
\end{aligned} \tag{24}
$$

and

$$
\begin{aligned}
&c \le z \le H \\
&u = \left.\frac{\partial \phi}{\partial x}\right|_{x=b} = -\sum_{n=1} \dot{A}_n (-1)^n \frac{1}{\upsilon_n} \left[\exp - r_n (z-c) - \exp - r_n (z-c-2h) \right] + \\
&\qquad + \sum_{m=1} k_m^* \left(D_m^1 \tanh k_m^* b + D_m^2 \frac{1}{\tanh k_m^* b} \right) \cos k_m^* (z-c) = \\
&\qquad = \sum_j B_j \cos k_j z \ .
\end{aligned} \tag{25}
$$

In a similar way, for the boundary at $(x = -b)$, the following relations hold:

$$
\begin{aligned}
&0 \le z \le d \\
&u = \left.\frac{\partial \varphi}{\partial x}\right|_{x=b} = \\
&\sum_{n=1} \dot{A}_n \frac{\cosh r_n z}{\sinh r_n d} + \sum_{m=1} k_m \left(-E_m^1 \tanh k_m b + E_m^2 \frac{1}{\tanh k_m b} \right) \cos k_m z = \\
&\qquad = -\sum_j C_j \cos k_j z \ ,
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
&c \le z \le H \\
&u = \left.\frac{\partial \phi}{\partial x}\right|_{x=b} = -\sum_{n=1} \dot{A}_n \frac{1}{\upsilon_n} \left[\exp - r_n (z-c) - \exp - r_n (z-c-2h) \right] + \\
&\qquad + \sum_{m=1} k_m^* \left(-D_m^1 \tanh k_m^* b + D_m^2 \frac{1}{\tanh k_m^* b} \right) \cos k_m^* (z-c) = \\
&\qquad = -\sum_j C_j \cos k_j z \ .
\end{aligned} \tag{27}
$$

The difference $(c - d)$ in the relations equals the plate thickness. Knowing that the constants $E_m^1, E_m^2,$ and D_n^1, D_n^1 $(m, n = 1, 2, \cdots)$ in the above relations depend on B_j and C_j $(j = 1, 2, \cdots)$, one can use equations (22–27) to express the latter constants in terms of $\dot{A}_n$ $(n = 1, 2, \cdots)$. In order to find the desired relations, equations (24–27) are multiplied in succession by $\cos k_j z$ $(j = 1, 2, \cdots)$ and then integrated within the range $(0 \le z \le H)$. Such a procedure leads to two systems of equations:

$$
\begin{aligned}
&x = b\\
&\sum_{n=1} \dot{A}_n(-1)^n \frac{1}{\sinh r_n d} \int_0^d \cosh r_n z \cdot \cos k_j z \, dz +\\
&\quad + \sum_{m=1} k_m \left[E_m^1 \tanh k_m b + E_m^2 \frac{1}{\tanh k_m b} \right] \int_0^d \cos k_m z \cdot \cos k_j z \, dz +\\
&\quad - \sum_{n=1} \dot{A}_n(-1)^n \frac{1}{v_n} \int_c^H \left[e^{-r_n(z-c)} - e^{-r_n(z-c-2h)} \right] \cdot \cos k_j z \, dz +\\
&\quad + \sum_{m=1} k_m^* \left[D_m^1 \tanh k_m^* b + D_m^2 \frac{1}{\tanh k_m^* b} \right] \int_c^H \cos k_m^*(z-c) \cdot \cos k_j z \, dz =\\
&\quad = B_j \frac{H}{2},
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
&x = -b\\
&\sum_{n=1} \dot{A}_n \frac{1}{\sinh r_n d} \int_0^d \cosh r_n z \cdot \cos k_j z \, dz +\\
&\quad + \sum_{m=1} k_m \left[-E_m^1 \tanh k_m b + E_m^2 \frac{1}{\tanh k_m b} \right] \int_0^d \cos k_m z \cdot \cos k_j z \, dz +\\
&\quad - \sum_{n=1} \dot{A}_n \frac{1}{v_n} \int_c^H \left[e^{-r_n(z-c)} - e^{-r_n(z-c-2h)} \right] \cdot \cos k_j z \, dz +\\
&\quad + \sum_{m=1} k_m^* \left[-D_m^1 \tanh k_m^* b + D_m^2 \frac{1}{\tanh k_m^* b} \right] \int_c^H \cos k_m^*(z-c) \cdot \cos k_j z \, dz =\\
&\quad = -C_j \frac{H}{2}.
\end{aligned}
\tag{29}
$$

The formulae written above have a complicated structure. In order to make our further discussion clear and to simplify the description of the problem, we confine our attention to a finite number of terms in the infinite series entering all the above relations. Thus, let us assume that na denotes the number of constants $\dot{A}_n$ taken into account. And, similarly, nd, ne and nj denote the numbers of constants D_m, E_m and $B_j(C_j)$, respectively. With respect to these finite numbers of terms in the series, it is convenient to make the following substitutions:

$$
\begin{aligned}
JA_j^n &= \frac{2}{H}(-1)^n \left[\frac{1}{\sinh r_n d} \int\limits_0^d \cosh r_n z \cdot \cos k_j z \, dz + \right. \\
&\left. - \frac{1}{v_n} \int\limits_c^H \left(e^{-r_n(z-c)} - e^{-r_n(z-c-2h)} \right) \cdot \cos k_j z \, dz \right], \\
JD_j^m &= \frac{2}{H} k_m^* \tanh k_m^* b \int\limits_c^H \cos k_m^*(z-c) \cdot \cos k_j z \, dz, \\
JE_j^m &= \frac{2}{H} k_m \tanh k_m b \int\limits_0^d \cos k_m z \cdot \cos k_j z \, dz,
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
KA_j^n &= \frac{2}{H} \frac{1}{\sinh r_n d} \int\limits_0^d \cosh r_n z \cdot \cos k_j z \, dz + \\
&- \frac{1}{v_n} \int\limits_c^H \left[e^{-r_n(z-c)} - e^{-r_n(z-c-2h)} \right] \cdot \cos k_j z \, dz, \\
KD_j^m &= \frac{2}{H} k_m^* \frac{1}{\tanh k_m^* b} \int\limits_c^H \cos k_m^*(z-c) \cdot \cos k_j z \, dz, \\
KE_j^m &= \frac{2}{H} k_m \frac{1}{\tanh k_m b} \int\limits_0^d \cos k_m z \cdot \cos k_j z \, dz.
\end{aligned}
\tag{31}
$$

From substitution of (30) and (31) into relations (28) and (29), we obtain

$$
\begin{aligned}
&x = b \\
&\sum_{n=1}^{na} \dot{A}_n JA_j^n + \sum_{m=1}^{ne} \left(E_m^1 JE_j^m + E_m^2 KE_j^m \right) + \sum_{m=1}^{nd} \left(D_m^1 JD_j^m + D_m^2 KD_j^m \right) = B_j,
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
&x = -b \\
&\sum_{n=1}^{na} \dot{A}_n KA_j^n + \sum_{m=1}^{ne} \left(-E_m^1 JE_j^m + E_m^2 KE_j^m \right) + \\
&+ \sum_{m=1}^{nd} \left(-D_m^1 JD_j^m + D_m^2 KD_j^m \right) = -C_j.
\end{aligned}
\tag{33}
$$

Equations (32) and (33) are written for $j = 1, 2, \cdots, nj$. The system of equations obtained in this way may be written in a more compact, matrix form:

$$\begin{aligned}(\boldsymbol{B}) &= [\boldsymbol{JA}]\left(\dot{\boldsymbol{A}}\right) + [\boldsymbol{JE}]\,(\boldsymbol{E}1) + [\boldsymbol{KE}]\,(\boldsymbol{E}2) + [\boldsymbol{JD}]\,(\boldsymbol{D}1) + [\boldsymbol{KD}]\,(\boldsymbol{D}2),\\ (\boldsymbol{C}) &= -\left\{[\boldsymbol{KA}]\left(\dot{\boldsymbol{A}}\right) - [\boldsymbol{JE}]\,(\boldsymbol{E}1) + [\boldsymbol{KE}]\,(\boldsymbol{E}2) - [\boldsymbol{JD}]\,(\boldsymbol{D}1) + [\boldsymbol{KD}]\,(\boldsymbol{D}2)\right\}.\end{aligned} \tag{34}$$

The vector matrices ($\boldsymbol{B}$) and ($\boldsymbol{C}$) in these equations have nj elements. The dimensions of the square matrices correspond to the number of terms taken into account in the associated series. With the notation presented above, the dimensions of the associated matrices are

$$\boldsymbol{JA}, \boldsymbol{KA} \rightarrow (nj \times na), \;\; \boldsymbol{JE}, \boldsymbol{KE} \rightarrow (nj \times ne), \;\; \boldsymbol{JD}, \boldsymbol{KD} \rightarrow (nj \times nd). \tag{35}$$

In accordance with the finite matrix description, equations (17–21) are also replaced by a finite system of equations with a finite number of terms. These equations are written in the following matrix form:

$$\begin{aligned}E_0 &= [\boldsymbol{EOB}]\,(\boldsymbol{B} + \boldsymbol{C}),\\ (\boldsymbol{E}1 + \boldsymbol{E}2) &= [\boldsymbol{EB}]\,(\boldsymbol{B}),\\ (\boldsymbol{E}1 - \boldsymbol{E}2) &= [\boldsymbol{EB}]\,(\boldsymbol{C}),\\ (\boldsymbol{D}1 + \boldsymbol{D}2) &= [\boldsymbol{DB}]\,(\boldsymbol{B}),\\ (\boldsymbol{D}1 - \boldsymbol{D}2) &= [\boldsymbol{DB}]\,(\boldsymbol{C}),\end{aligned} \tag{36}$$

where $(1 \times nj)$ matrix $\boldsymbol{EOB}$ reads

$$\boldsymbol{EOB} = \left[\ldots\ldots, -\frac{1}{2d}\frac{1}{(k_j)^2}\sin k_j d, \ldots\ldots..\right]. \tag{37}$$

At the same time, equations (18) and (19) lead to the $(ne \times nj)$ matrix $\boldsymbol{EB}$:

$$\boldsymbol{EB}_{m,j} = -\frac{2}{d}\begin{cases}\dfrac{(-1)^m \sin k_j d}{(k_j)^2 - (k_m)^2} & \text{for} \quad k_j \neq k_m\\[2ex] \dfrac{d}{2k_j} \quad \text{for} \quad k_j = k_m.\end{cases} \tag{38}$$

Finally, the $(nd \times nj)$ matrix $\boldsymbol{DB}$ is defined by the formula

$$\boldsymbol{DB}_{m,j} = -\frac{2}{h}\begin{cases}\dfrac{\sin k_j c}{(k_m^*)^2 - (k_j)^2} \quad \text{for} \quad k_j \neq k_m^*\\[2ex] \dfrac{1}{2(k_j)^2}\left(k_j h \cdot \cos k_j c - \sin k_j c\right) \quad \text{for} \quad k_j = k_m^*.\end{cases} \tag{39}$$

The sum and difference of the two equations (34) lead to the following relations

$$\begin{aligned}(\boldsymbol{X}) &= (\boldsymbol{B} + \boldsymbol{C}) = [\boldsymbol{JA} - \boldsymbol{KA}]\,(\dot{\boldsymbol{A}}) + 2\,[\boldsymbol{JE}]\,(\boldsymbol{E}1) + 2\,[\boldsymbol{JD}]\,(\boldsymbol{D}1),\\ (\boldsymbol{Y}) &= (\boldsymbol{B} - \boldsymbol{C}) = [\boldsymbol{JA} + \boldsymbol{KA}]\,(\dot{\boldsymbol{A}}) + 2\,[\boldsymbol{KE}]\,(\boldsymbol{E}2) + 2\,[\boldsymbol{KD}]\,(\boldsymbol{D}2).\end{aligned} \tag{40}$$

On the other hand, equations (36) give

$$\begin{aligned}
2(\boldsymbol{E}1) &= [\boldsymbol{EB}]\,(\boldsymbol{B}+\boldsymbol{C}) = [\boldsymbol{EB}]\,(\boldsymbol{X}),\\
2(\boldsymbol{E}2) &= [\boldsymbol{EB}]\,(\boldsymbol{B}-\boldsymbol{C}) = [\boldsymbol{EB}]\,(\boldsymbol{Y}),\\
2(\boldsymbol{D}1) &= [\boldsymbol{DB}]\,(\boldsymbol{B}+\boldsymbol{C}) = [\boldsymbol{DB}]\,(\boldsymbol{X}),\\
2(\boldsymbol{D}2) &= [\boldsymbol{DB}]\,(\boldsymbol{B}-\boldsymbol{C}) = [\boldsymbol{DB}]\,(\boldsymbol{Y}).
\end{aligned} \tag{41}$$

From substitution of equations (41) into relations (40), the following system of equations is obtained:

$$\begin{aligned}
[\boldsymbol{RA}]\,(\boldsymbol{X}) &= [\boldsymbol{JA}-\boldsymbol{KA}]\left(\dot{\boldsymbol{A}}\right),\\
[\boldsymbol{RB}]\,(\boldsymbol{Y}) &= [\boldsymbol{JA}+\boldsymbol{KA}]\left(\dot{\boldsymbol{A}}\right),
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
[\boldsymbol{RA}] &= [\boldsymbol{I}] - [\boldsymbol{JE}]\,[\boldsymbol{EB}] - [\boldsymbol{JD}]\,[\boldsymbol{DB}],\\
[\boldsymbol{RB}] &= [\boldsymbol{I}] - [\boldsymbol{KE}]\,[\boldsymbol{EB}] - [\boldsymbol{KD}]\,[\boldsymbol{DB}].
\end{aligned} \tag{43}$$

The matrix $[\boldsymbol{I}]$ in these relations is the $(nj \times nj)$ unit diagonal matrix. From equations (44), the following solutions are obtained:

$$\begin{aligned}
(\boldsymbol{X}) &= [\boldsymbol{RA}]^{-1}\,[\boldsymbol{JA}-\boldsymbol{KA}]\left(\dot{\boldsymbol{A}}\right) = [\boldsymbol{XA}]\left(\dot{\boldsymbol{A}}\right),\\
(\boldsymbol{Y}) &= [\boldsymbol{RB}]^{-1}\,[\boldsymbol{JA}+\boldsymbol{KA}]\left(\dot{\boldsymbol{A}}\right) = [\boldsymbol{YA}]\left(\dot{\boldsymbol{A}}\right).
\end{aligned} \tag{44}$$

Substitution of these relations into equations (36) and (41) gives

$$\begin{aligned}
&E_0 = [\boldsymbol{EOB}]\,[\boldsymbol{XA}]\left(\dot{\boldsymbol{A}}\right)\\
&(\boldsymbol{E}1) = \frac{1}{2}\,[\boldsymbol{EB}]\,[\boldsymbol{XA}]\left(\dot{\boldsymbol{A}}\right),\ (\boldsymbol{E}2) = \frac{1}{2}\,[\boldsymbol{EB}]\,[\boldsymbol{YA}]\left(\dot{\boldsymbol{A}}\right),\\
&(\boldsymbol{D}1) = \frac{1}{2}\,[\boldsymbol{DB}]\,[\boldsymbol{XA}]\left(\dot{\boldsymbol{A}}\right),\ (\boldsymbol{D}2) = \frac{1}{2}\,[\boldsymbol{DB}]\,[\boldsymbol{YA}]\left(\dot{\boldsymbol{A}}\right).
\end{aligned} \tag{45}$$

With these relations, all unknown constants (parameters) of the problem considered are expressed in terms of independent parameters that correspond to the plate deflection. It is important to note that none of the matrices in these equations, i.e. $[\boldsymbol{EOB}]$, $[\boldsymbol{EB}]$, $[\boldsymbol{DB}]$, $[\boldsymbol{XA}]$ or $[\boldsymbol{YA}]$, depends on time.

3. Free Vibrations of the Plate and Co-vibrating Mass of Fluid

The solution presented in the preceding section corresponds to a general case of standing waves (the pressure at the surface $z = H$ is zero) and arbitrary motion of the plate. In order to find a fundamental set of natural frequencies of the band plate submerged in a layer of fluid, both the deflection of the plate and potential functions are expressed in the following forms:

$$\begin{aligned} w(x,t) &= W(x)\exp(i\omega t),\\ \phi^*(x,z,t) &= i\omega\phi(x,z)\exp(i\omega t),\\ \varphi^*(x,z,t) &= i\omega\varphi(x,z)\exp(i\omega t). \end{aligned} \tag{46}$$

Substitution of these relations into equation (1) gives the fundamental equation for the free vibrations of the plate submerged in fluid:

$$-m_{pl}\omega^2\left[W(x) + \frac{\rho}{m_{pl}}(\varphi - \phi)_{pl}\right] + D^*\frac{\partial^4 W(x)}{\partial x^4} = 0. \tag{47}$$

The frequency ω in this relation is different from that corresponding to free vibrations of the plate in air. The potential functions ϕ and φ in (47) are described by the general formulae given in the previous section. Vibrations of the plate in fluid are accompanied by the so-called co-vibrating mass of fluid, which leads to the above-mentioned differences in natural frequencies. This co-vibrating mass of fluid leads to a shift of the natural frequencies of the coupled vibrations to smaller values as compared to frequencies in air. At the same time, as in the case of vibrations in air, the plate deflection and its space derivatives are described by the formulae

$$\begin{aligned} W(x) &= \sum_1^{na} A_n \sin r_n(x+b),\\ \frac{\partial^4 W(x)}{\partial x^4} &= \sum_1^{na} A_n (r_n)^4 \sin r_n(x+b), \quad r_n = \frac{n\pi}{2b}. \end{aligned} \tag{48}$$

From substitution of these relations into equation (47), one obtains

$$\begin{aligned} &-m_{pl}\omega^2 \sum_{n=1}^{NA} A_n \sin r_n(x+b) +\\ &+D^* \sum_{n=1}^{NA} A_n (r_n)^4 \sin r_n(x+b) - m_{pl}\omega^2 \frac{\rho}{m_{pl}}(\varphi - \phi)_{pl} = 0. \end{aligned} \tag{49}$$

Multiplication of this equation in succession by $\sin r_n(x+b)$ $(n = 1, 2, \cdot\cdot\cdot, nj)$ and integration of the result within the range $(-b \le x \le +b)$, leads to the system of equations:

$$\frac{b}{(r_n)^4}A_n + \frac{1}{(r_n)^4}\frac{\rho}{m_{pl}}\int_{-b}^{+b}(\varphi - \phi)_{pl}\sin r_n(x+b)dx - \frac{D^* b}{m_{pl}\omega^2}A_n = 0, \tag{50}$$
$$n = 1, 2, \cdot\cdot\cdot.$$

The term $\rho(\varphi - \phi)_{pl}$ in equation (49) corresponds to the mass of fluid vibrating together with the plate. From equations (31) and (32), it follows that

$$
\begin{aligned}
(\varphi - \phi)_{pl} = {} & \sum_{n=1}^{NA} A_n \frac{1}{r_n}\left(\frac{1}{\tanh r_n d} + \frac{1-e^{-2r_n h}}{1+e^{-2r_n h}}\right)\sin r_n(x+b) + \\
& + E_0 + \sum_{m=1}^{NE}(-1)^m\left(E_m^1 \frac{\cosh k_m x}{\cosh k_m b} + E_m^2 \frac{\sinh k_m x}{\sinh k_m b}\right) + \\
& - \sum_{m=1}^{ND}\left(D_m^1 \frac{\cosh k_m^* x}{\cosh k_m^* b} + D_m^2 \frac{\sinh k_m^* x}{\sinh k_m^* b}\right), \\
r_n = \frac{n\pi}{2b}, \quad & k_m = \frac{m\pi}{d}, \quad k_m^* = \frac{2m-1}{2h}\pi.
\end{aligned}
\tag{51}
$$

Substitution of this equation into (50) gives

$$
\begin{aligned}
& \frac{b}{(r_n)^4}\left[1 + \frac{\rho}{m_{pl}}\frac{1}{r_n}\left(\frac{1}{\tanh r_n d} + \frac{1-e^{-2r_n h}}{1+e^{-2r_n h}}\right)\right]A_n - \frac{D^* b}{m_{pl}\omega^2}A_n + \\
& + \frac{\rho}{m_{pl}(r_n)^4}\left[E_0 \int_{-b}^{+b} \sin r_n(x+b)dx + \right. \\
& + \sum_{m=1}^{NE}(-1)^m \int_{-b}^{+b}\left(E_m^1 \frac{\cosh k_m x}{\cosh k_m b} + E_m^2 \frac{\sinh k_m x}{\sinh k_m b}\right)\sin r_n(x+b)dx + \\
& \left. - \sum_{m=1}^{ND}\int_{-b}^{+b}\left(D_m^1 \frac{\cosh k_m^* x}{\cosh k_m^* b} + D_m^2 \frac{\sinh k_m^* x}{\sinh k_m^* b}\right)\sin r_n(x+b)dx\right] = 0, \; n = 1, 2, \cdots
\end{aligned}
\tag{52}
$$

Obviously, all constants E and D in these equations depend on A_n $(n = 1, 2, \cdots)$, and therefore, the final system of equations will uniquely depend on these latter constants. From these equations it follows that, in the limit $\rho \to 0$, we arrive at equations corresponding to free vibration of the plate in air. The integrals in equations (52) are expressed in exact form as

$$
\begin{aligned}
&LA_n^0 = \int\limits_{-b}^{+b} \sin r_n(x+b)dx = \frac{1}{r_n}\left[1-(-1)^n\right],\\
&LA_n^m = \frac{(-1)^m}{\cosh k_m b}\int\limits_{-b}^{+b} \cosh k_m x\cdot \sin r_n(x+b)dx =\\
&= \frac{(-1)^m r_n}{(k_m)^2+(r_n)^2}\left[1-(-1)^n\right],\\
&LB_n^m = \frac{(-1)^m}{\sinh k_m b}\int\limits_{-b}^{+b} \sinh k_m x\cdot \sin r_n(x+b)dx =\\
&-\frac{(-1)^m r_n}{(k_m)^2+(r_n)^2}\left[1+(-1)^n\right],\\
&KA_n^m = \frac{1}{\cosh k_m^* b}\int\limits_{-b}^{+b} \cosh k_m^* x\cdot \sin r_n(x+b)dx =\\
&\frac{r_n}{(k_m^*)^2+(r_n)^2}\left[1-(-1)^n\right],\\
&KB_n^m = \frac{1}{\sinh k_m^* b}\int\limits_{-b}^{+b} \sinh k_m^* x\cdot \sin r_n(x+b)dx =\\
&-\frac{r_n}{(k_m^*)^2+(r_n)^2}\left[1+(-1)^n\right].
\end{aligned}
\tag{53}
$$

With these results, equations (52) may be written in the following form:

$$
\begin{aligned}
\frac{b}{(r_n)^4}\left[1+\frac{\rho}{m_{pl}}\frac{1}{r_n}\left(\frac{1}{\tanh r_n d}+\frac{1-e^{-2r_n h}}{1+e^{-2r_n h}}\right)\right]A_n-\frac{D^* b}{m_{pl}\omega^2}A_n+\\
+\frac{\rho}{m_{pl}(r_n)^4}\left[LA_n^0 E_0+\sum_{m=1}^{NE}\left(LA_n^m E_m^1+LB_n^m E_m^2\right)+\right.\\
\left.-\sum_{m=1}^{ND}\left(KA_n^m D_m^1+KB_n^m D_m^2\right)\right]=0\ ,\ \ n=1,2,\cdots
\end{aligned}
\tag{54}
$$

Finally, all these equations are written in the matrix form

$$
\begin{aligned}
[\boldsymbol{AB}]\,(\boldsymbol{A})+(\boldsymbol{EO})E_0+[\boldsymbol{LA}]\,(\boldsymbol{E1})+\\
[\boldsymbol{LB}]\,(\boldsymbol{E2})-[\boldsymbol{KA}]\,(\boldsymbol{D1})-[\boldsymbol{KB}]\,(\boldsymbol{E2})-\lambda\,[\boldsymbol{I}]\,(\boldsymbol{A})=\boldsymbol{0},
\end{aligned}
\tag{55}
$$

where

$$
\lambda = \frac{bD^*}{m_{pl}\omega^2}. \tag{56}
$$

The elements of the diagonal matrix $[\boldsymbol{AB}]$ and vector matrix $(\boldsymbol{E0})$ are described by the formulae

$$\begin{aligned} AB_n^n &= \frac{b}{(r_n)^4}\left[1+\frac{\rho}{m_{pl}}\frac{1}{r_n}\left(\frac{1}{\tanh r_n d}+\frac{1-e^{-2r_n h}}{1+e^{-2r_n h}}\right)\right],\\ EO_n &= \frac{\rho}{m_{pl}(r_n)^4}\frac{1-(-1)^n}{r_n}. \end{aligned} \tag{57}$$

Substituting equations (57), (53) and (45) into (55) and making simple manipulations, one obtains the final system of equations

$$([\boldsymbol{AA}]-\lambda\,[\boldsymbol{I}])\,(\boldsymbol{A})=\boldsymbol{0}. \tag{58}$$

The matrix $[\boldsymbol{AA}]$ in this equation is a square matrix. With this matrix, dependent on the plate parameters and a gap between the plate and the fluid bottom, it is possible to calculate the associated set of eigenfrequencies of the plate.

Table 1. Steel plate, $\delta = 4$ mm, $D = 1207.38$ kg$\cdot$ m$^3 \cdot$ s^{-2}, $H = 0.60$ m

	Eigenfrequencies of the plate in air				
	ω_1	ω_2	ω_3	ω_4	ω_5
	61.201	244.803	550.807	979.212	1530.019
c	Eigenfrequencies of the plate in water				
[cm]	ω_1	ω_2	ω_3	ω_4	ω_5
1	2.510	20.309	66.427	158.082	299.403
2	3.946	32.074	101.374	239.614	440.465
3	4.870	39.564	121.811	285.119	513.626
4	5.567	45.081	135.855	314.741	558.640
5	6.127	49.385	146.166	335.256	588.357
6	6.595	52.849	154.023	349.959	608.760
7	6.995	55.694	160.154	360.725	623.125
8	7.343	58.063	165.021	368.724	633.413
9	7.650	60.056	168.933	374.727	640.876
10	7.924	61.746	172.108	379.263	646.343
11	8.170	63.190	174.706	382.706	650.381
12	8.393	64.430	176.845	385.327	653.383
13	8.596	65.500	178.616	387.325	655.629
14	8.782	66.427	180.090	388.848	657.320
15	8.954	67.235	181.323	390.005	658.600
16	9.114	67.941	182.359	390.882	659.575
17	9.263	68.561	183.232	391.543	660.323
18	9.402	69.107	183.972	392.037	660.899
19	9.532	69.592	184.601	392.401	661.347
20	9.656	70.023	185.140	392.667	661.698

The solution of the problem presented above is illustrated by numerical examples below. Two plates are considered. The first one is a steel plate of thickness $\delta = 4$ mm, and the second one of thickness $\delta = 20$ mm is made of reinforced concrete. In order to investigate the influence of the gap between the plates and the fluid bottom on the eigenfrequencies of these plates, a chosen set of the gap widths is taken into account. Some of the results obtained in numerical calculations are drawn up in tables 1 and 2.

Table 2. Reinforced concrete plate, $\delta = 2$ mm, $D = 2.8964 \cdot 10^4$ kg$\cdot$ m$^3 \cdot$ s^{-2}, $H = 0.60$ m

	Eigenfrequencies of the plate in air				
	ω_1	ω_2	ω_3	ω_4	ω_5
	237.546	950.184	2137.915	3800.739	5938.655
c	Eigenfrequencies of the plate in water				
[cm]	ω_1	ω_2	ω_3	ω_4	ω_5
3	14.838	121.016	387.919	925.757	1722.011
4	20.212	165.480	513.935	1217.468	2207.365
5	23.972	196.111	594.404	1394.321	2484.107
6	26.896	219.299	651.606	1512.898	2660.854
7	29.288	237.646	694.424	1596.309	2780.070
8	31.307	252.544	727.481	1656.660	2863.118
9	33.049	264.851	753.535	1701.131	2922.216
10	34.575	275.147	774.379	1734.308	2964.911
11	35.928	283.841	791.243	1759.273	2996.108
12	37.140	291.240	805.008	1778.167	3019.111
13	38.236	297.575	816.328	1792.520	3036.201
14	39.233	303.031	825.693	1803.443	3048.985
15	40.146	307.752	833.484	1811.758	3058.608
16	40.988	311.857	839.996	1818.079	3065.895
17	41.767	315.441	845.466	1822.870	3071.447
18	42.493	318.585	850.081	1826.484	3075.705
19	43.172	321.357	853.992	1829.190	3078.992
20	43.810	323.812	857.323	1831.196	3081.550

Table 1 contains a set of five lowest eigenfrequencies of the steel plate. Similar results for the reinforced concrete plate are shown in table 2. In order to obtain a better insight into the solutions obtained, the lowest eigenfrequencies are illustrated in Fig. 2. From the data collected in these tables and from the plots in this figure, it may be

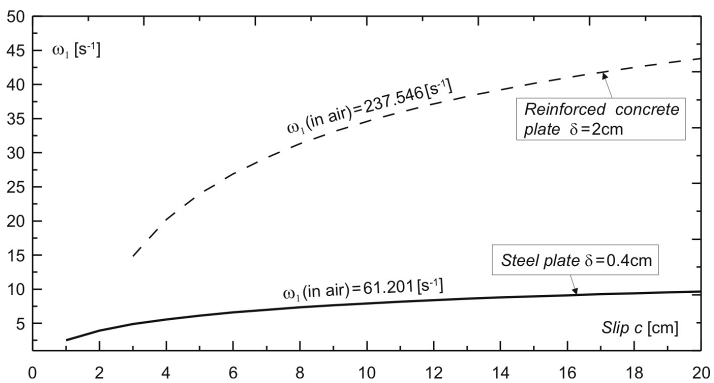

Fig. 2. Eigenfrequencies of steel and reinforced plates versus slip width

seen that the lowest frequencies correspond to the lowest gaps. It is important for practical reasons that the formulation presented above makes it possible to calculate the smallest plate eigenfrequency for a given distance between the horizontal plate and the fluid (sea) bottom.

4. Flow-induced Vibrations of the Plate

Results of the previous sections are employed here to investigate forced vibrations of the horizontal plate induced by gravitational waves. In the case considered, the plate is submerged in a semi-infinite layer of fluid, as shown schematically in Fig. 3.

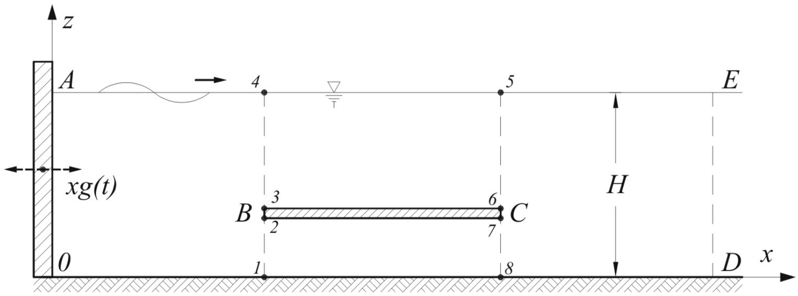

Fig. 3. Simply supported elastic plate *BC* submerged in a semi-infinite layer of fluid

Water waves are generated by a piston-type generator (rigid vertical wall *OA*) placed at the beginning of the layer. The generator motion is described by the formula

$$xg(t) = xg \exp(i\omega t), \tag{59}$$

where xg is the generation amplitude.

The potential function for the fluid domain (the fluid domain except for the fluid below the horizontal plate) consists of two parts, i.e. $\Phi^* = \Phi(x, z, t) + \phi(x, z, t)$. The first part corresponds to wave generation, and the second part is associated with vibrations of the plate. With respect to the harmonic generation, both Φ and ϕ are written in a form similar to that in equations (46), namely

$$\begin{aligned} \Phi(x, z, t) &= i\omega\ \Phi(x, z) \exp(i\omega t), \\ \phi(x, z, t) &= i\omega\ \phi(x, z) \exp(i\omega t). \end{aligned} \tag{60}$$

At the same time, the boundary conditions at the upper fluid surface and the surfaces of the plate read

$$\begin{aligned} -\omega^2 \Phi(x, z) + g \frac{\partial \Phi(x, z)}{\partial z} + g \frac{\partial \phi(x, z)}{\partial z}\bigg|_{z=H} &= 0, \\ \frac{\partial}{\partial z} \left[\Phi(x, z) + \phi(x, z)\right]\bigg|_{plate} = \frac{\partial \varphi(x, z)}{\partial z}\bigg|_{plate} &= W(x). \end{aligned} \tag{61}$$

In order to obtain the first part of the solution, i.e. $\Phi(x, z)$, we resort to a discrete formulation of the problem by means of the finite difference method (FDM). With this method, however, only a finite fluid domain may be considered. And therefore, instead of the infinite fluid layer in Fig. 3, a finite part of it, with a boundary at $x = L$, far off the generator-plate system, is taken into account. With such an approach, however, it is necessary to formulate transmitting boundary conditions at this boundary. Thus, for the steady state, harmonic motion and a sufficiently large distance L from the plate, it is justified to consider only a progressive wave, for which we have

$$\frac{\partial \Phi}{\partial t} + \frac{\omega}{k_0}\frac{\partial \Phi}{\partial x} = 0, \tag{62}$$

where

$$\omega^2 = g k_0 \tanh k_0 h. \tag{63}$$

Equation (62) describes a local boundary condition at the artificial boundary at $x = L$. Since we are dealing with a discrete formulation, it is reasonable to consider a non-local boundary condition, which is more convenient in the discrete method applied. One can show that for the FDM formulation, with vertical spacing of nodal points equal to a, the transmitting boundary condition for the velocity potential reads

$$\Phi|_{x=L+a} = -\,\Phi|_{x=L-a} + 2\,\Phi|_{x=L} \cos k_0 a. \tag{64}$$

This relation is employed for the difference analogue of Laplace's equation, written for all nodal points at $x = L$. In this way, the problem considered is reduced to unknown nodal values of the potential at points of the finite fluid domain (interior and boundary points of the fluid domain). A solution of the system of FDM equations for the discrete values of the potential $\Phi(x_i, z_i)$, where i means the number of a nodal point, depends not only on the generator amplitude, but also on the unknown potential $\phi(x, z)$. These two potentials are coupled through the boundary conditions at the free surface and the surface of the plate (equations 61). In order to find a solution to this problem, a two-step procedure is employed. In the first step, the solution to the potential $\Phi(x, z)$ is expressed in the form of a linear combination of solutions corresponding to unknown parameters A_n ($n = 1, 2, \cdots, na$). And then, in the second step, these parameters are obtained by a solution of the plate equation. This, in turn, makes it possible to calculate the deflection of the plate, as well as the pressure field and the amplitude of the free surface elevation.

With respect to the procedure described above, it is necessary to formulate associated boundary conditions for the potential $\Phi^*(x, z)$. Thus, at the boundary (1–2) in Fig. 3, one obtains

$$\begin{aligned}\left.\frac{\partial \Phi^*}{\partial x}\right|_{1-2} &= \frac{\partial \Phi}{\partial x} + \left.\frac{\partial \phi}{\partial x}\right|_{1-2} = \left.\frac{\partial \phi}{\partial x}\right|_{1-2} = \\ &= \sum_{n=1} A_n \frac{\cosh r_n z}{\sinh r_n d} + \sum_{m=1} k_m \left[-E_m^1 \tanh k_m b + E_m^1 \frac{1}{\tanh k_m b}\right] \cos k_m z.\end{aligned} \tag{65}$$

In a similar way,

$$\left.\frac{\partial \Phi^*}{\partial x}\right|_{2-3} = \frac{\partial \Phi}{\partial x} + \left.\frac{\partial \phi}{\partial x}\right|_{2-3} = 0, \quad \rightarrow \quad \left.\frac{\partial \Phi}{\partial x}\right|_{2-3} = \left.\frac{\partial \phi}{\partial x}\right|_{2-3} = 0. \tag{66}$$

At the right boundary at (5-6-7-8), the following relations hold:

$$\begin{aligned} &\left.\frac{\partial \Phi^*}{\partial x}\right|_{8-7} = \frac{\partial \Phi}{\partial x} + \left.\frac{\partial \phi}{\partial x}\right|_{8-7} = \left.\frac{\partial \phi}{\partial x}\right|_{8-7} = \\ &= \sum_{n=1} A_n (-1)^n \frac{\cosh r_n z}{\sinh r_n d} + \sum_{m=1} k_m \left[E_m^1 \tanh k_m b + E_m^1 \frac{1}{\tanh k_m b} \right] \cos k_m z \end{aligned} \tag{67}$$

and

$$\left.\frac{\partial \Phi^*}{\partial x}\right|_{7-6} = \frac{\partial \Phi}{\partial x} + \left.\frac{\partial \phi}{\partial x}\right|_{7-6} = 0, \quad \rightarrow \quad \left.\frac{\partial \Phi}{\partial x}\right|_{7-6} = \left.\frac{\partial \phi}{\partial x}\right|_{7-6} = 0. \tag{68}$$

At the upper surface of the plate (segment 3–6 in Fig. 3), we have

$$\left.\frac{\partial \Phi^*}{\partial z}\right|_{3-6} = \frac{\partial \Phi}{\partial z} + \left.\frac{\partial \phi}{\partial z}\right|_{3-6} = \left.\frac{\partial \phi}{\partial z}\right|_{3-6} = \sum_{n=1} A_n \sin r_n (x+b) \cos k_m z. \tag{69}$$

The boundary condition at the free surface of the fluid is described by the first equation (61), which is rewritten in the form

$$\begin{aligned} &-\omega^2 \Phi(x,z) + g\frac{\partial \Phi(x,z)}{\partial z} = -g \left.\frac{\partial \phi(x,z)}{\partial z}\right|_{z=H} = \\ &= -g \begin{cases} \sum_{j=1} C_j (-1)^{j+1} \exp(-k_j x), \quad \text{at} \;\; (A-4), \quad x - \text{local coordinate} \\ 2\sum_{n=1} A_n \dfrac{e^{-r_n h}}{1+e^{-2r_n h}} \sin r_n (x+b) + \\ + \sum_{m=1} k_m (-1)^{m+1} \left[D_m^1 \dfrac{\cosh k_m x}{\cosh k_m b} + D_m^2 \dfrac{\sinh k_m x}{\sinh k_m b} \right] \Bigg|_{(4-5)} \\ \sum_{j=1} B_j (-1)^{j+1} \exp(-k_j x), \quad \text{at} \;\; (5-E). \end{cases} \end{aligned} \tag{70}$$

The boundary condition at $x = L$ $(D - E)$ describes equation (64). Obviously, at the boundaries $(0 - 1)$ and $(6 - D)$, the normal derivative $(\partial\Phi/\partial z)$ is equal to zero. These boundary conditions depend on the constants (B, C, D, E) that can be expressed in terms of the parameters A_n $(n = 1, 2, \cdots, na)$. Following the procedure applied, the right-hand side of the discrete system of equations for the potential $\Phi(x_i, z_i)$ will depend not only on the generator amplitude, but also on the set of A_n $(n = 1, 2, \cdots, na)$. These latter parameters will be obtained from equation (1) describing the plate motion.

With the boundary conditions in mind, the final system of FDM equations for the potential $\Phi(x_i, z_i)$ may be written in the following form:

$$[\boldsymbol{AA}]\,(\Phi) = (\boldsymbol{P})\,, \tag{71}$$

where $(\boldsymbol{P})$ depends on the generator amplitude and the potential $\phi(x, z)$. Non-zero components of $(\boldsymbol{P})$, which correspond to nodal points of the free surface, are

$$P_r = -\frac{2a^2}{b^*}\begin{cases} \sum\limits_{j=1} C_j(-1)^{j+1}\exp(-k_j x_r), \quad \text{at} \quad (A-4), \quad x_r - \text{local coordinate} \\ 2\sum\limits_{n=1} A_n \dfrac{e^{-r_n h}}{1+e^{-2r_n h}} \sin r_n(x_r+b) + \\ +\sum\limits_{m=1} k_m(-1)^{m+1}\left[D_m^1 \dfrac{\cosh k_m x_r}{\cosh k_m b} + D_m^2 \dfrac{\sinh k_m x_r}{\sinh k_m b}\right]\Bigg|_{(4-5)} \\ \sum\limits_{j=1} B_j(-1)^{j+1}\exp(-k_j x_r), \quad \text{at} \quad (5-E). \end{cases} \tag{72}$$

The parameters a and b^* in this equation denote the horizontal and vertical spacing of nodal points. It is worth adding here that all the constants $(B_j, C_{j,} \cdots, E_m^1, E_m^2)$ in this equation are uniquely expressed in terms of A_n $(n = 1, 2, \cdots, na)$. Finally, in order to find the parameters A_n $(n = 1, 2, \cdots, na)$, it is necessary to solve the system of equations describing the plate motion:

$$\begin{aligned} &A_n b\left\{\left[1+\frac{\rho}{m_{pl}}\frac{1}{r_n}\left(\frac{1}{\tanh r_n d}+\frac{1-e^{-2r_n h}}{1+e^{-2r_n h}}\right)\right]-\frac{D^*(r_n)^4}{m_{pl}\omega^2}\right\}+ \\ &+\frac{\rho}{m_{pl}}\left[E_0\int_{-b}^{+b}\sin r_n(x+b)dx+\right. \\ &\left.+\sum_{m=1}^{NE}(-1)^m\int_{-b}^{+b}\left(E_m^1\frac{\cosh k_m x}{\cosh k_m b}+E_m^2\frac{\sinh k_m x}{\sinh k_m b}\right)\sin r_n(x+b)dx\right]+ \\ &-\frac{\rho}{m_{pl}}\left[\sum_{m=1}^{ND}\int_{-b}^{+b}\left(D_m^1\frac{\cosh k_m^* x}{\cosh k_m^* b}+D_m^2\frac{\sinh k_m^* x}{\sinh k_m^* b}\right)\sin r_n(x+b)dx\right]+ \\ &-\frac{\rho}{m_{pl}}\left[\cdots\int_{-b}^{+b}\Phi(x, z=H)\sin r_n(x+b)dx\cdots\right]=0, \quad n=1,2,\cdots,na \end{aligned} \tag{73}$$

The integrals entering these equations are defined by equations (53). Simple, though tedious, manipulations allow us to calculate the set of independent variables A_n $(n = 1, 2, \cdots, na)$. This solution is illustrated in subsequent Figures 4 and 5.

Plots in Fig. 4 show amplitudes of the free-surface elevation and deflection of the plate. Figure 5 shows the distribution of the maximum deflection of the plate versus

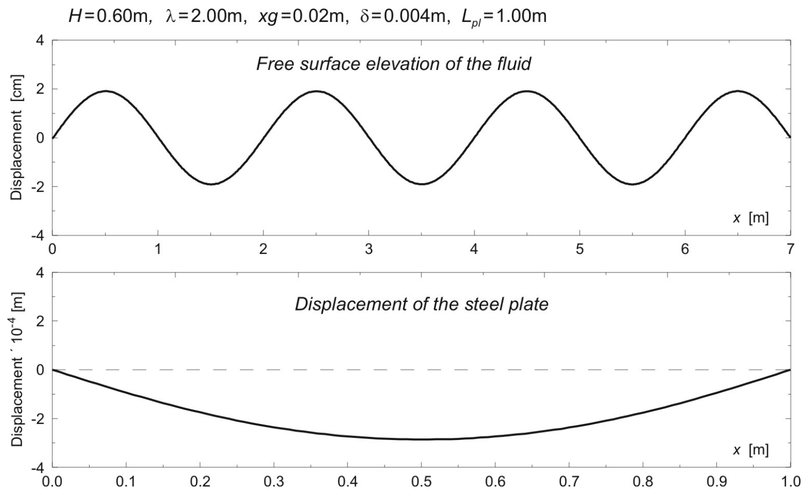

Fig. 4. Free-surface elevation of the fluid and deflection of the steel plate

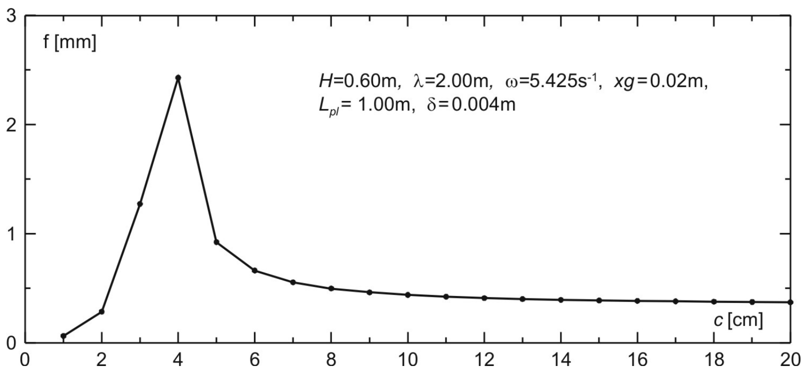

Fig. 5. Maximum deflection of the steel plate versus the gap width

the width of the gap between the plate and the fluid bottom. It should be stressed that this deflection depends on the wave length (associated with the generator frequency), the amplitude of the wave maker, as well as the distance between the plate and the piston generator. Therefore, in practical applications, one should be aware of a certain ambiguity in calculating the plate amplitude, which results from the fundamental assumption of the steady state harmonic motion of the system considered.

5. Concluding Remarks

The formulation developed in this paper makes it possible to calculate the co-vibrating mass of fluid and a set of eigenfrequencies of a horizontal thin elastic plate submerged in fluid of constant depth. As compared to vibrations of the plate in air, the most important result of these investigations is an assessment of the reduction in the plate eigenfrequencies due to the co-vibrating mass of fluid. At the same time, the approximate theory makes it possible to assess the influence of the gap width on this reduction. It is important to note that the lowest eigenfrequency of the plate vibrations may fall into the range inherent for surface gravitational waves. In such a case, one should be aware of the possibility of a resonance phenomenon that may lead to increased deflection of the plate. Obviously, under natural conditions, one may expect a certain damping of the plate vibrations. Nevertheless, the theoretical result of the possible resonance of waves and plate vibrations is important in the construction of such plates as foundations for offshore structures. At the same time, the numerical experiments conducted for forced vibrations of the plate reveal that, for a certain gap width, one should expect a maximum deflection of the plate. From investigations conducted above, it follows that for a safe operation of such a structure under natural conditions, it may be reasonable to place such a plate foundation at a relatively greater distance from the sea bottom.

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