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Solving a large heterogeneous-agent general equilibrium model with labour market frictions*†

by

Marek Antosiewicz

Warsaw School of Economics,
Al. Niepodległości 162, 02-554, Warszawa, Poland
ma37123@sgh.waw.pl

Abstract: In this paper we build and solve a heterogeneous-agent dynamic stochastic general equilibrium (DSGE) model with incomplete markets in the spirit of Krussel and Smith (1998). We expand this model to account for search and matching labour market frictions, using the classic Mortensen and Pissarides (1994) framework. The model, therefore, combines two important strands of economic modelling and presents a numerical challenge in terms of solving the model due to the inclusion of additional dimensions in the optimization problem. Despite the addition of additional state variables and higher dimensionality, we show that we are able to efficiently solve it numerically using value function iteration and we document basic properties of the model.

Keywords: heterogeneous–agent model, search and matching, DSGE, computational method

1. Introduction

The model here considered is an attempt to integrate two different strands of economic literature that have received a lot of interest in recent years. The first strand is constituted by the search and matching mechanism, pioneered by Mortensen and Pissarides (1994) and Pissarides (2000). It is commonly used as the basic mechanism of modelling labour market frictions and equilibrium unemployment, see Rogerson et al. (2005). The second strand of literature that we incorporate refers to the incomplete-markets, heterogenous-agent modelling literature, started by the articles such as Hugget (1993) or Krussel and Smith (1998). Work in this field is still being carried out and economists are trying to incorporate more real-life elements into heterogenous-agent models, while, at

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the same time, they are developing better, more computationally effective ways of solving such models, see Den Haan (2010). The starting point for the work we present here is a basic heterogeneous-agent model, where individuals can only insure themselves against fluctuations in income by means of capital holdings, which are subject to a lower bound inequality constraint. The representative firm in the model economy rents capital from household members and produces output using two factors of production, namely capital and labour. We extend this basic setup by modelling the labour market process according to the search and matching framework, where unemployed individuals send job offers with constant intensity, whereas firms post vacancies. Wages are set through Nash wage bargaining. The inclusion of both elements in a model is challenging from a computational perspective and is also interesting from an economic point of view.

The standard incomplete markets model of Krussel and Smith is already difficult to solve for two reasons. First of all, the initial guess for value function of agents and other equilibrium variables has to be close to their true value or else the value function iteration solution algorithm does not converge. Secondly, the algorithm requires significant computational time. Incorporating labour market frictions adds additional state variables to the agent's decision problem and increases the dimension, amplifying the problems that the original Krussel and Smith model is faced with. Furthermore, in this setup, additional value functions (such as the value of posting a vacancy for a firm) must be calculated simultaneously with remaining value functions. From an economic point of view, such a model allows for the study of labour market dynamics under a richer specification. It is, for example, possible to analyse how various policies affect the labour market, such as unemployment benefits in a model, which departs from the standard risk sharing assumption, made routinely in representative agent models. It is therefore possible to study welfare effects and the behaviour of consumption of different types of workers in deeper detail. Furthermore, this setup can help to better understand the cyclical behaviour of wages, since in this setup they will depend on individual asset holdings of agents.

2. Model setup and notation

In this section we discuss the setup of the model, notation and outline the solution method used to solve the model. All of the model assumptions made here are common for the economic literature, but they are discussed here in more detail for the more technical reader. Time in the model is discrete and denoted by $t = 0, 1, 2, ...$ Model variables and parameters are described when they are introduced in an equation. All variables, which pertain to individual household members (agents) are denoted by lowercase letters, while those pertaining to macroeconomic aggregates are denoted by uppercase letters.

2.1. The household

As is common for the economic literature, we assume that the household is composed of an infinite number of infinitely lived members, whom we index by the upper index i. For simplicity, we assume that the index belongs to the interval $[0, 1]$. Each household member seeks to maximize expected lifetime utility from individual consumption c_t^i:

$$E \sum_{t=0}^{\infty} \beta^t u(c_t^i), \tag{1}$$

where parameter $\beta < 1$ is the discount rate and the utility function is assumed to be of the commonly used constant relative risk aversion (CRRA) type:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}. \tag{2}$$

In each period agent i can be either employed or unemployed. The employment status is denoted by $\epsilon_t^i \in \{0, 1\}$ and when employed ($\epsilon_t^i = 1$) the agent supplies one unit of labour and receives wage less income tax $w_t^i(1-\tau_W)$, with τ_W defining the labour income tax rate. When unemployed ($\epsilon_t^i = 0$), the agent receives unemployment benefits b_t^i. As is standard for economic models, we also assume that agents are the owners of capital k_t^i which they rent out to the firm and receive income given by the rate of return on capital R_t. Furthermore, capital holdings is the only storage of wealth for agents and it is assumed that agents face an asset constraint $k_t^i \geq \underline{k}$. This is the crucial assumption, which introduces heterogeneity among household members. In order to understand this, it is important to observe two things. Household members, who are unemployed for a significant proportion of the time periods, will not be able to sustain a high level of consumption, because they will not be able to borrow below the limit set by the capital constraint and repay the debt in the future, when they become employed again. Furthermore, we impose that the household members cannot insure themselves against unemployment with remaining household members in order to smooth consumption. In the end, since each household member will have a different employment history and will make different decisions on how much to consume and how much to invest in capital in each period, the result will be a distribution of capital among them. Finally, we assume that each member of the household owns an equal share of the firm, so that each member of the household receives the same share of the profit of the firm denoted by Π_t.

Using the above, we can write down the budget constraint, facing the i-th household member:

$$c_t^i + k_{t+1}^i = (1 - \delta + R_t)k_t^i + (1 - \tau_W)\epsilon_t^i w_t^i + (1 - \epsilon_t^i)b_t^i + \Pi_t \tag{3}$$

$$k_{t+1}^i \geq \underline{k}, \tag{4}$$

where parameter δ is the capital depreciation rate. The decision problem of a household member is specified and solved in a recursive form. We, therefore, define value functions of an employed $V_E()$ and unemployed $V_U()$ household member, who has given capital holding, which are then maximized. It is important to remember that the value functions also depend on the aggregate state of the economy, given by the aggregate capital holdings, level of technology, aggregate employment and the rate of return on capital (these elements are described in the following paragraphs). Since we are using the recursive form, we omit the time index and use the symbol ' to specify the next period value of a variable. Given the above, an employed member of the household, who possesses capital holdings equal to k, chooses next-period capital holding k' in order to maximize the value from employment given by:

$$V^E(k) = \max_c u(c) + \beta E\left((1-\rho)V^E(k')) + \rho V^U(k')\right), \tag{5}$$

where parameter ρ is the exogenous probability that the household member will become unemployed. On the other hand, an unemployed member of the household maximizes the following value function:

$$V^U(k) = \max_c u(c) + \beta E\left((\Phi V^E(k')) + (1-\Phi)V^U(k')\right) \tag{6}$$

where Φ is a variable, which is the probability of finding a job in the next period (and it is discussed in the paragraph, concerning the labour market). The result of the optimization of the value functions is constituted by the decision rules regarding consumption and capital savings for employed and unemployed members of the household. We denote these by $\phi^e(k)$ and $\phi^u(k)$, respectively.

2.2. The firm

We assume that there is one representative firm, which is owned by the household. In each period, the firm maximizes expected discounted profits. In order to produce output Y_t, the firm operates a standard production function with parameter α, using aggregate capital K_t and labour N_t as its inputs. The functional form is the following:

$$Y_t(K_t, N_t) = A_t N_t^{1-\alpha} K_t^{\alpha}, \tag{7}$$

where A_t is the level of aggregate technology. In order to hire new workers the firm must post vacancies V_t, whose unit cost is equal to parameter ϖ. Altogether, the firm faces the following budget constraint:

$$Y_t = R_t K_t + W_t N_t + V_t \varpi + \Pi_t, \tag{8}$$

where Π_t is profits and the wage W_t is the average wage over all persons employed in the economy and is given by:

$$W_t = \int_0^1 w_t^i di. \tag{9}$$

The clearing condition for capital must also hold:

$$K_t = \int_0^1 k_t^i di. \tag{10}$$

The firm also needs to take into consideration the constraint, regarding the dynamics of the labour market while deciding on the amount of vacancies that it posts:

$$N_{t+1} = (1-\rho)N_t + \Phi_t V_t \tag{11}$$

where Φ_t is the probability of filling a vacancy, which is discussed in more detail in the section devoted to the labour market. The optimization problem of the firm can be formulated recursively as maximizing the value of $\tilde{\Pi}_t$:

$$\tilde{\Pi}_t(N_t) = \max_{V_t, K_t} \Pi_t + \beta E\left(\tilde{\Pi}_{t+1}(N_{t+1})\right) \tag{12}$$

subject to the above constraints. The result of this optimization is a decision rule of the firm regarding the amount of posted vacancies. Let $\phi^f(N)$ denote this decision rule. In order to calculate the wage, we need to specify the value for a firm $V^J(k)$ of employing a worker, whose capital holdings are k. Again, we define this problem recursively, omitting the time subscripts.

$$V^J(k) = Y_N'(K, N) - w(k) + \beta\left((1-\rho)V^J(k')\right). \tag{13}$$

The value of posting a vacancy V^V is given by:

$$V^V = -\varpi + \beta E\left(\int_{\underline{k}}^{\infty} \Phi V^J(k) dk\right). \tag{14}$$

2.3. The government

We assume that the government collects a single wage tax with tax rate τ_W and spends all its proceeds on benefits for the unemployed. Total government revenue T_t is given by:

$$T_t = \tau_W N_t W_t. \tag{15}$$

The proceeds from the tax are distributed as equal benefits for the unemployed. Dividing tax revenue by the number of unemployed $1 - N_t$, we arrive at the amount of benefit:

$$b_t = \frac{T_t}{1-N_t}. \tag{16}$$

2.4. The labour market

The basis of the labour market is the standard search and matching mechanism, which is the most commonly used setup for the labour market in economic models. According to this mechanism, firms post vacancies, whereas unemployed members of the household look for jobs. The resulting number of the unemployed, who find a job (number of new job matches), M_t, is a function of the job seeking effort by the unemployed $1 - N_t$ and the number of vacancies V_t:

$$M_t = \Upsilon(1 - N_t)^{\eta} V_t^{1-\eta} \tag{17}$$

where parameter Υ sets the efficiency of the matching process and η sets the elasticities of the function. Furthermore, we assume that in each period a fraction ρ of employment relationships are exogenously severed and these workers become unemployed. The aggregate employment evolves according to:

$$N_{t+1} = (1 - \delta)N_t + M_t. \tag{18}$$

We can therefore specify the probability for an unemployed person to find a job Ψ_t and the probability of filling a vacancy Φ_t as (these variables are used in the firm and household description):

$$\Psi_t = \frac{M_t}{1 - N_t} \qquad \Phi_t = \frac{M_t}{V_t}. \tag{19}$$

To summarize, the employment transition matrix, denoted by $\pi(\epsilon'|\epsilon)$ has the following form:

$$\pi(\epsilon'|\epsilon) = \begin{pmatrix} 1 - \Psi_t & \Psi_t \\ \rho & 1 - \rho \end{pmatrix}. \tag{20}$$

We assume that wages are set according to the standard Nash wage bargaining procedure, whose goal is to maximize the weighted surplus of the employee and the firm. The value functions depend only on the value of the individual capital holding of individual i:

$$w(k) = \arg\max_{w} (V^E(k) - V^U(k))^{\xi} (V^J(k) - V^V)^{1-\xi} \tag{21}$$

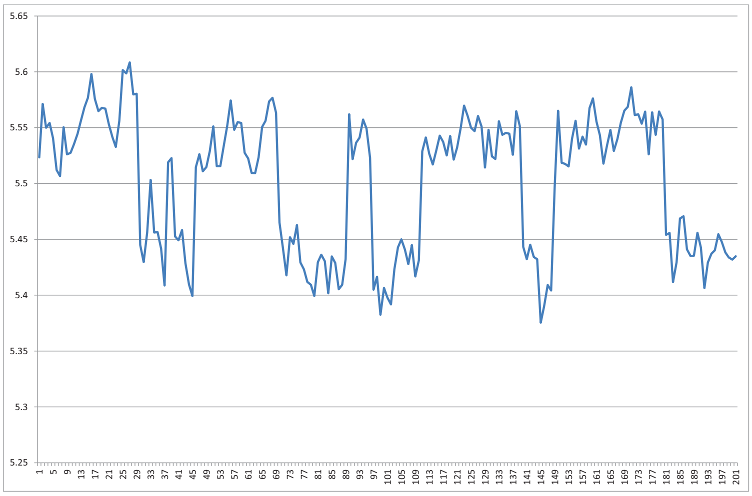
where $V^E(k)$ is the value for agent of capital holding k of being employed, $V^U(k)$ is the value of being unemployed, V^V is the value of posting a vacancy and $V^J(k)$ is the value for a firm from employing agent with capital holding k. These functions are defined in the previous sections.

2.5. Recursive stationary equilibrium and the outline for the computational method

The stationary equilibrium of the model is given by the following elements: value functions: $V^E(k)$, $V^U(k)$, $V^J(k)$, V^V, wage function: $w(k)$, decision rules of household agents and firm: $\phi^e(k)$, $\phi^u(k)$, $\phi^f(N)$, values for vacancies V, new labour matches: M, matching probabilities: Ψ_t, Φ_t, aggregate capital and labour and their prices: K_t, and N_t, R_t and W_t and the distribution of capital among household members, which specify the following conditions:

- Decision rules $\phi^e(k)$, $\phi^u(k)$ solve the consumer optimization problem given by equations (1) and (3).
- Decision rule $\phi^f(N)$ solves the firm optimization problem given by (12), while the number of vacancies is set according to the free entry condition $V^V = 0$.

Figure 1. Sample trajectory of product Y_t.

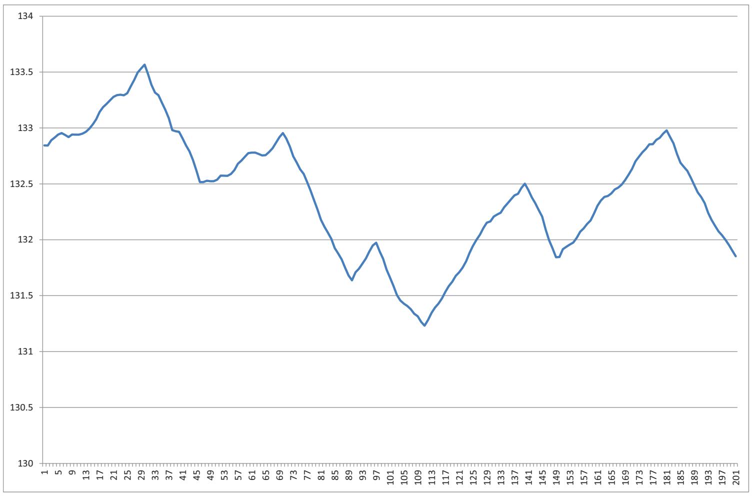


In order to solve the model, we have to make an assumption about the form of the technological progress. For simplicity, we assume that aggregate technology can be equal to either a good state ($A = 1.01$) and a bad state with ($A = 0.99$) with a constant transition matrix between these two states. The computational algorithm, which solves the model, works as follows:

1. Assume a functional form of the prediction rule for the future aggregate state of capital and unemployment.
2. Perform value function iteration of consumer value function in order to arrive at the decision rule regarding consumption and saving.
3. Perform value function iteration of the firm value function to arrive at the decision rule regarding investment and vacancy posting.
4. Perform Nash wage bargaining.
5. Simulate the economy for 3000 iterations using obtained the decision rules and using the obtained time series update the prediction rule from step 1. If the update of the rule is sufficiently small, terminate the algorithm, if

not, repeat the procedure, starting from step 1.

Figure 2. Sample trajectory of capital K_t



3. Results

3.1. Simulation procedure

Before we analyze model properties, we first discuss the simulation procedure. It is important to highlight that the solution of the model is in fact obtained from a simulation model, which is driven by the decision rules calculated as described in the previous paragraph. In order to sample model properties, we simulate an economy populated with 5000 agents for 4000 time steps. All agents are endowed with an initial capital holding equal to the steady state capital level in the economy and are randomly assigned labour market status (employed or unemployed) with probability given by the average level of employment. In each time step, all agents are activated sequentially and they decide on their level of consumption and investment in capital according to the decision rules. Iterating the economy for a significant number of time steps results in a distribution of capital holdings among agents, which is a consequence of individual agent histories of being employed or unemployed (employed agent increase their capital holdings, whereas unemployed agents decrease investment in order to smooth consumption). Simulation experiments show that after approximately 1000 time steps, the distribution of capital among agents reaches a steady state, therefore in order to assess model properties we discard these time steps.

Figure 3. Sample trajectory of employment N_t

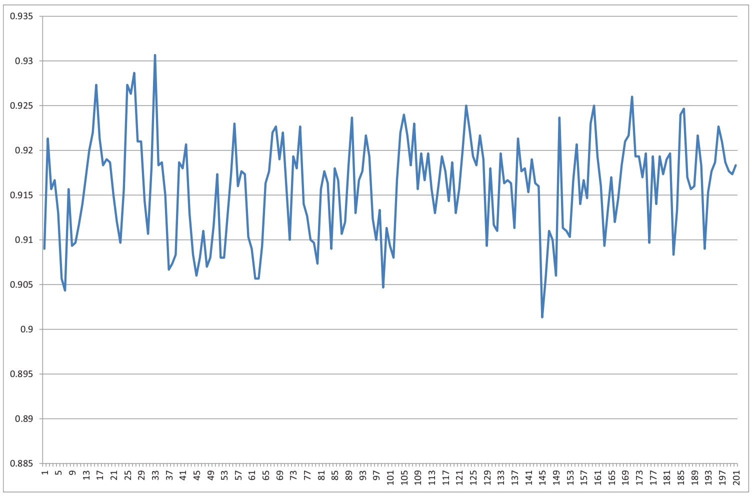
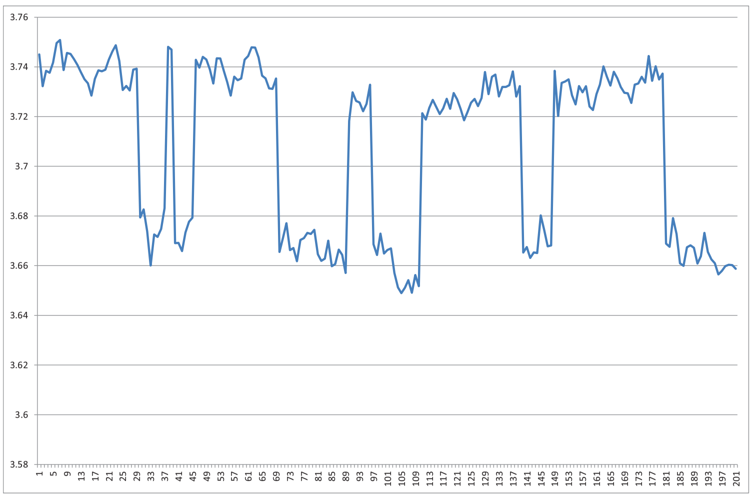


Figure 4. Sample trajectory of average wage W_t



3.2. Simulation results

We now show and discuss the initial results, shown as sample simulation trajectories for the main model variables: product, capital, employment and wages. The results of the model are as this can be expected of a model, which is a mix of a standard incomplete markets RBC model and a search and matching labour market model. The two variables that react the most strongly and in line with changes in technology are the wage and the product. Regarding employment, which is shown in Fig. 3, the time series also displays a clear difference between times of good and bad aggregate technology, however, during these spells there is much more volatility than for other time series. This volatility originates from the fact that when simulating the economy for a certain number of agents and using the endogenously calculated probability of transition between labour market states, the share of agents that switch labour market states does not have to be equal to these probabilities due to random number generation. As can be seen from Fig. 2, the adjustments of capital take much more time - in each period of good technology the stock of capital increases only slightly, which is in line with standard Real Business Cycle models.

We have successfully been able to solve a heterogeneous-agent general equilibrium model (find optimal decision rules for the household and firm with respect to the objective function) with search and matching frictions. This step significantly increased the computational time needed to find the model solution from approximately 2 hours for a standard Krusell and Smith (1998) model to 30 hours. Due to the rich representation of the labour market, the model can be used to test, for example, different schemes for unemployment benefits and their impact on wages and distribution of savings of household members. Basic model properties have been analyzed, but the future work, which includes the analysis of the distribution of capital and wages remains to be done.

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