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ON THE QUASI-UNIFORM CONVERGENCE

Robert Drozdowski[a], **Jacek Jędrzejewski**[b], **Agata Sochaczewska**[c]

[a] *Institute of Mathematics, Academia Pomeraniensis*
ul. Arciszewskiego 22b, 76-200 Słupsk, Poland
e-mail: r.drozdowski@wp.pl

[b] *Institute of Mathematics and Computer Science*
Jan Długosz University in Częstochowa
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
e-mail: jacek.m.jedrzejewski@gmail.com

[c] *Institute of Mathematics, Academia Pomeraniensis*
ul. Arciszewskiego 22b, 76-200 Słupsk, Poland
e-mail: agata_sochaczewska@wp.pl

Abstract. Arzelá [1] considered the weaker form of uniform convergence which is as good as uniform convergence of sequences of functions in respect to continuity of the limit of a sequence of continuous functions. Some generalization of such convergence can be found in [5]. Similar kinds of convergence of function sequences were considered in [3] and [4]. In our article we generalize those kinds of convergence for functions defined in a topological space with values in a topological space.

In the article we use terminology which is explained in Engelking's monograph "General Topology" [2]. Among others, we use the notion of a star with respect to an open cover. If X is a topological space and α is a cover of this space, then the star $\mathrm{St}(x,\alpha)$ of a point $x \in X$ with respect to the cover α is defined as the union of all the sets from α which contain the point x, i.e.

$$\mathrm{St}(x,\alpha) = \bigcup \{U : x \in U \wedge U \in \alpha\}$$

Definition 1 *Let X, Y be topological spaces and f, f_n, $n \in \mathbb{N}$, be functions defined in X with values in Y. It is said that $(f_n)_{n=1}^{\infty}$ is quasi-uniformly convergent to f if:*

(1) *$(f_n)_{n=1}^{\infty}$ converges pointwise to f;*

(2) *for each open cover α of Y and each $n \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $n_1, \ldots, n_k \geq n$ such that if $t \in X$, then*

$$f_{n_1}(t) \in \mathrm{St}(f(t), \alpha) \vee f_{n_2}(t) \in \mathrm{St}(f(t), \alpha) \vee \ldots \vee f_{n_k}(t) \in \mathrm{St}(f(t), \alpha).$$

Theorem 1 *Let X be a compact space and Y be an arbitrary topological space. If $f_n \colon X \to Y$, where $n \in \mathbb{N}$, $f \colon X \to Y$ are continuous functions and $(f_n)_{n=1}^{\infty}$ is pointwise convergent to f, then $(f_n)_{n=1}^{\infty}$ is quasi-uniform convergent to f.*

Proof. Let us take an arbitrary open cover α of Y. For an arbitrary, but fixed, positive integer m denote:

$$A_m = \{t \in X \colon f_m(t) \in \mathrm{St}(f(t), \alpha)\}.$$

Since $(f_n)_{n=1}^{\infty}$ is pointwise convergent, the equality

$$X = \bigcup_{m \geq n} A_m \tag{1}$$

is satisfied. Additionally, the set A_m is open for each $m \geq n$.

Indeed, if $t \in A_m$, then $f_m \in \mathrm{St}(f(t), \alpha)$, i.e. one can find $V \in \alpha$ such that $f(t) \in V$. Continuity of both f_m and f implies the existence of neighborhoods W_1, W_2 of t such that $f(W_1) \subset \mathrm{St}(f(t), \alpha)$ and $f_m(W_2) \subset \mathrm{St}(f(t), \alpha)$.

Now, let $W_0 = W_1 \cap W_2$. Of course, W_0 is a neighborhood of t and for each $t' \in W_0$ we have $f_m(t') \in \mathrm{St}(f(t'), \alpha)$, i.e. $W_0 \subset A_m$. We have shown that the set A_m is open for each $m \geq n$.

By the above, by the fact that X is compact and by (1), one can find positive integers k and $n_1, \ldots, n_k \geq n$ such that

$$X = A_{n_1} \cup \ldots \cup A_{n_k}.$$

Hence

$$f_{n_1}(t) \in \mathrm{St}(f(t), \alpha) \vee \ldots \vee f_{n_k}(t) \in \mathrm{St}(f(t), \alpha)$$

for $t \in X$. Finally, combining this with the pointwise convergence, we get that $(f_n)_{n=1}^{\infty}$ is quasi-uniformly convergent to f. $\square$

We shall use the symbol $\mathcal{C}(f)$ for the set of all points of continuity of a function f.

Definition 2 *Let $\mathfrak{I}$ be a σ-ideal of subsets of X. A function f from a topological space X to a topological space Y is said to be $\mathfrak{I}$-continuous if $X \setminus \mathcal{C}(f)$ belongs to $\mathfrak{I}$.*

Theorem 2 *Let X be a topological space, Y be a regular one and f_n, f be functions from X to Y for each positive integer n. Let $\mathfrak{I}$ be a σ-ideal of subsets of X such that $X \notin \mathfrak{I}$. If $(f_n)_{n=1}^{\infty}$ is $\mathfrak{I}$-continuous for each $n \in \mathbb{N}$ and $(f_n)_{n=1}^{\infty}$ is quasi-uniformly convergent to f, then f is $\mathfrak{I}$-continuous as well.*

Proof. Let

$$E = \bigcap_{n=1}^{\infty} \mathcal{C}(f_n), \tag{2}$$

where $\mathcal{C}(f_n)$ denotes the set of points of continuity for f_n. We will show that the set $\mathcal{C}(f)$ of points of continuity for f contains the set E.

Let $x \in E$ and U be an arbitrary neighborhood of $f(x)$. By regularity of the space Y, there exists an open set V such that

$$f(x) \in V \subset \mathrm{cl}(V) \subset U. \tag{3}$$

Consider the family $\alpha = \{U, X \setminus \mathrm{cl}(V)\}$ which forms an open cover of Y. By pointwise convergence of $(f_n)_{n=1}^{\infty}$ to f, there exists n_0 such that

$$f_n(x) \in V \tag{4}$$

for each $n \geq n_0$. By the second condition of quasi-uniform convergence, one can find indexes $n_1 \geq n_0, \ldots, n_k \geq n_0$ such that

$$f_{n_1}(t) \in \mathrm{St}(f(t), \alpha) \vee f_{n_2}(t) \in \mathrm{St}(f(t), \alpha) \vee \ldots \vee f_{n_k}(t) \in \mathrm{St}(f(t), \alpha)$$

for each $t \in X$. By the fact that $f(x) \in V$, $f_{n_i}(x) \in V$ and f_{n_i} is continuous, we infer that there exists a neighborhood W_i of x such that $f(W_i) \subset V$ for each $i = 1, \ldots, k$.

Let $W_0 = \bigcap_{i=1}^{k} W_i$. Thus, W_0 is a neighborhood of x and if $t \in W_0$, then there exists n_j such that $f_{n_j}(t) \in V$ and $f_{n_j}(t) \in \mathrm{St}(f(t), \alpha)$. This means that $f(t) \in U$ for each $t \in W_0$, i.e. the inclusion $f(W_0) \subset U$ holds.

In the consequence, f is continuous at each point $x \in E$. Now, conditions $X \setminus \mathcal{C}(f) \subset E$ and $E \in \mathfrak{I}$ imply that $X \setminus \mathcal{C}(f) \in \mathfrak{I}$, i.e. f is $\mathfrak{I}$-continuous and the proof is complete. $\square$

Definition 3 *Let X, Y be topological spaces and f, f_n, $n \in \mathbb{N}$, be functions defined on X with values in Y. A sequence $(f_n)_{n=1}^{\infty}$ is called* St*-monotonically convergent to f if for every $x \in X$, for every $n \in \mathbb{N}$ and for every open cover of Y the implication*

$$f_n(t) \in \mathrm{St}(f(t), \alpha) \Longrightarrow f_{n+1}(t) \in \mathrm{St}(f(t), \alpha)$$

holds.

Theorem 3 *If X and Y are topological spaces, f_n, f are functions from X into Y, then $(f_n)_{n=1}^{\infty}$ converges uniformly to f if and only if $(f_n)_{n=1}^{\infty}$ is quasi-uniform and* St*-monotonically convergent to f.*

Proof. It is known that uniform convergence implies quasi-uniform convergence. It is not difficult to see that uniform convergence also implies St-monotonic convergence.

Now, assume that $(f_n)_{n=1}^{\infty}$ is quasi-uniformly and St-monotonically convergent to f. Let us choose an arbitrary open cover α of Y. By quasi-uniform convergence, one can find $k \in \mathbb{N}$ and $n_1, \ldots, n_k \geq n$ such that

$$f_{n_1}(t) \in \mathrm{St}(f(t), \alpha) \vee f_{n_2}(t) \in \mathrm{St}(f(t), \alpha) \vee \ldots \vee f_{n_k}(t) \in \mathrm{St}(f(t), \alpha) \quad (5)$$

for each $t \in X$. By (5), we infer that for $t \in X$ there exists n_i such that $f_{n_i}(t) \in \mathrm{St}(f(t), \alpha)$, whence by St-monotonic convergence for each $p \in \mathbb{N}$ we have:

$$f_{n_i+p}(t) \in \mathrm{St}(f(t), \alpha).$$

Let $n_0 = \max\{n_1, \ldots, n_k\}$. Obviously, if $t \in X$ and $n \geq n_0$, then

$$f_n(t) \in \mathrm{St}(f(t), \alpha),$$

which proves that $(f_n)_{n=1}^{\infty}$ is uniformly convergent to f. □

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