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THE PULLEY SYSTEM WITH THREE DEGREES OF FREEDOM

This paper discusses the dynamic system solution with three degree of freedom consist a weightless cable thrown over two fixed side pulleys and the third, free pulley moving along the rope. The cable is stretched with weights attached to its ends and the weight of the free pulley causes the cable dip. Passive resistance on the pulleys stops them from moving. The mathematical description uses a system of differential equations combined with complex kinematic relationships and a description of the passive resistances. The set of equations is converted to a format suitable for numerical solution and a simulation model is created in the Matlab –Simulink program. The results are presented in the form of various relations of kinematic and dynamic system variables.

1. INTRODUCTION

Mathematical modeling and simulation of dynamic systems has become an integral part of machine design and construction. The elaboration of a functional and plausible simulation model is subject to the preparation of a mechanical scheme including, if possible, all influences affecting the behavior of the system, creation of a system of equations of motion (i.e. a mathematical model featuring a description of active elements as well as passive resistances and mechanical bonds) and a preparation of a simulation computer model providing for the design of a mathematical model for specified system parameters and initial conditions. The mathematical model is elaborated using methods of vector dynamics (the release method in combination with the Newton or d´Alembert principle) as well as analytic mechanics (various modifications of Lagrange’s equations), or a combination of both these methods.

This paper focuses on the process of designing a dynamic system with more degrees of freedom, consisting of an ideal rope slung over two fixed end pulleys on which is a third, free moving pulley. The ends of the rope are strung using ballast weights and the weight of the free pulley causes the rope to curve. The pulleys are considered to have a passive

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resistance which decelerates their movement. The mathematical description leads to a system of differential equations complemented with complex kinematic couplings and a description of the passive resistance effect. The system of equations is adjusted so that its form would be suitable for a numerical solution, and a simulation model is set up using the Matlab –Simulink program.

2. DEMONSTRATION OF A PROBLEM SOLUTION

A mechanical scheme of the described system is presented in Fig. 1. The mathematical model (system equations of motion) has been derived using a free body method which specifically introduces internal coupling forces displayed in the scheme. According to the d`Alembert principle it is possible to derive the following motion equations.
For sliding elements 1 and 2 it applies that

$$m_1 g - S_A - m_1 \ddot{y}_1 = 0, \quad (1)$$

$$m_2 g - S_B - m_2 \ddot{y}_2 = 0, \quad (2)$$

where the dots over the value signify derivation according to time.

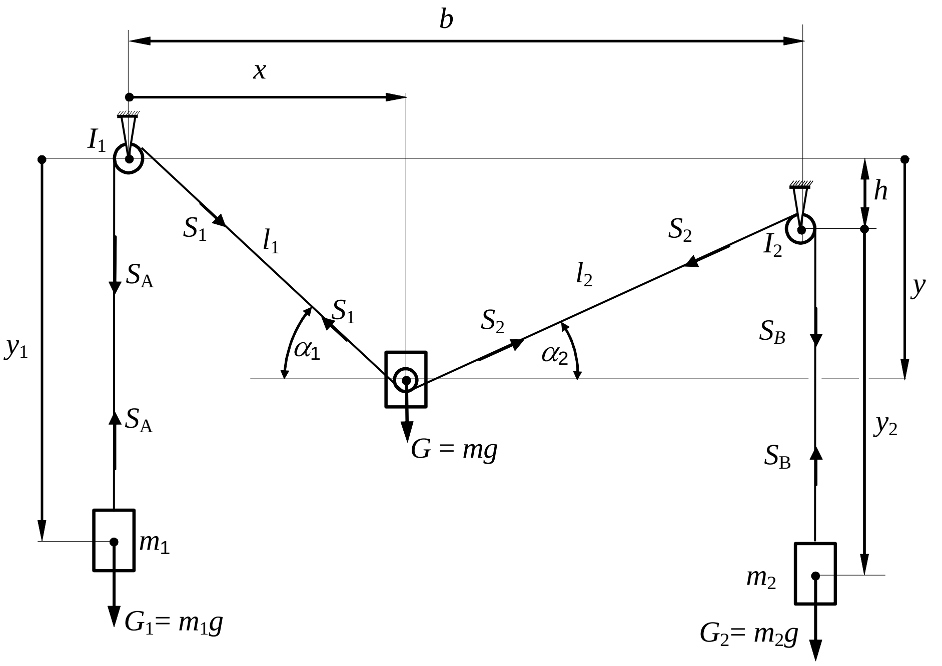

Fig. 1. Mechanical scheme of the system

In the case of the fixed pulley 1 with respect to the braking moments as displayed in Fig. 2 it applies that

$$r_1 S_A - r_1 S_1 - M_{b1} - I_1 \ddot{\varphi}_1 = 0. \quad (3)$$

The orientation of the braking moment depends on the sign of the angular velocity $\omega_1 = \dot{\varphi}_1$ which can be respected using a relation of $M_{b1} = |M_{b1}| \operatorname{sgn} \dot{\varphi}_1$. With respect to the kinematic coupling $\dot{y}_1 = r_1 \dot{\varphi}_1$ we will adjust the equation (3) to achieve the form of

$$S_A - S_1 - |F_{b1}| \operatorname{sgn} \dot{y}_1 - \frac{I_1}{r_1^2} \ddot{y}_1 = 0, \tag{4}$$

where $|F_{b1}| = |M_{b1}| / r_1$ signifies the braking force on the circumference of the pulley replacing the braking moment.

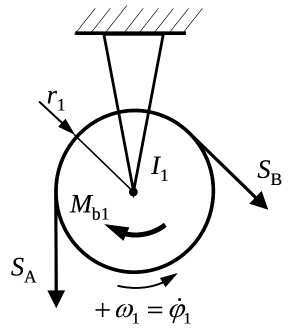

Fig. 2. Coupling forces and a braking moment on a fixed pulley

By analogy we can derive an equation of motion for the second fixed pulley

$$S_B - S_2 - |F_{b2}| \operatorname{sgn} \dot{y}_2 - \frac{I_2}{r_2^2} \ddot{y}_2 = 0 , \quad \text{kde} \quad |F_{b2}| = |M_{b2}| / r_2 . \tag{5}$$

By excluding force S_A from equations (1) and (4), or force S_B from equations (2) and (5) we will obtain

$$m_{1red} \ddot{y}_1 = m_1 g - S_1 - |F_{b1}| \operatorname{sgn} \dot{y}_1 , \tag{6}$$

$$m_{2red} \ddot{y}_2 = m_2 g - S_2 - |F_{b2}| \operatorname{sgn} \dot{y}_2 , \tag{7}$$

where the reduced weights

$$m_{1red} = m_1 + \frac{I_1}{r_1^2}, \qquad m_{2red} = m_2 + \frac{I_2}{r_2^2} , \tag{8}$$

associate inertial effects of the sliding elements and of the rotating pulleys.

The free pulley has two degrees of freedom determined by coordinates x and y. Its equations of motion are

$$S_2 \cos\alpha_2 - S_1 \cos\alpha_1 - m\ddot{x} = 0, \tag{9}$$

$$mg - S_2 \sin \alpha_2 - S_1 \sin \alpha_1 - m\ddot{y} = 0 . \tag{10}$$

The relation between forces S_1 and S_2 will ensue from the balance equation of moments when the inertia moment is neglected. With regard to the consideration of the braking moment (see Fig. 3) it will apply that

$$S_1 r - S_2 r - |M_b| \operatorname{sgn} \omega = 0 . \tag{11}$$

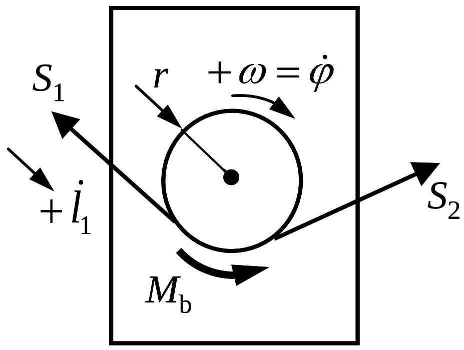

Fig. 3. Coupling forces and a braking moment on a free pulley

The sign of angular velocity ω is obviously related to the time change of the length l_1 of the first section of the rope, i.e. $\operatorname{sgn}\omega = \operatorname{sgn}\dot{l}_1$. If we introduce the braking force it will apply that

$$S_2 = S_1 - |F_b| \operatorname{sgn} \dot{l}_1 = 0. \tag{12}$$

Internal forces S_1 and S_2 can then be derived from equations (6) and (12) in this manner

$$S_1 = m_1 g - m_{1red}\ddot{y}_1 - |F_{b1}| \operatorname{sgn} \dot{y}_1, \tag{13}$$

$$S_2 = m_1 g - m_{1red}\ddot{y}_1 - |F_{b1}| \operatorname{sgn} \dot{y}_1 - |F_b| \operatorname{sgn} \dot{l}_1. \tag{14}$$

By incorporating S_1 and S_2 according to (13) and (14) into the equations of motion (9) and (10) of the free pulley and then by deducing the equations (6) and (7) with respect to the equation (12), and after adjustment, we will obtain a system of three equations of motion

$$\begin{aligned} &m_1 g(\cos\alpha_2 - \cos\alpha_1) - m_{1red}(\cos\alpha_2 - \cos\alpha_1)\ddot{y}_1 - m\ddot{x} - \\ &\qquad - |F_{b1}| \operatorname{sgn} \dot{y}_1(\cos\alpha_2 - \cos\alpha_1) - |F_b| \operatorname{sgn} \dot{l}_1 \cos\alpha_2 = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &m\ddot{y} - m_{2red}(\sin\alpha_2 + \sin\alpha_1)\ddot{y}_1 - mg + m_1 g(\sin\alpha_2 + \sin\alpha_1) + \\ &\qquad + |F_{b1}| \operatorname{sgn} \dot{y}_1(\sin\alpha_2 + \sin\alpha_1) + |F_b| \operatorname{sgn} \dot{l}_1 \sin\alpha_2 = 0, \end{aligned} \tag{16}$$

$$m_{1red}\ddot{y}_1 - m_{2red}\ddot{y}_2 - (m_1 - m_2)g + |F_{b1}| \operatorname{sgn} \dot{y}_1 - |F_{b2}| \operatorname{sgn} \dot{y}_2 + |F_b| \operatorname{sgn} \dot{l}_1 = 0. \tag{17}$$

The goniometric function in equations (15) and (16) will be determined according to relations of

$$\cos\alpha_1 = \frac{x}{l_1}, \quad \sin\alpha_1 = \frac{y}{l_1}, \quad \cos\alpha_2 = \frac{b-x}{l_2}, \quad \sin\alpha_2 = \frac{y-h}{l_2}, \tag{18}$$

where l_1 and l_2 are the lengths of the rope sections in front of and behind the free pulley

$$l_1 = \sqrt{x^2 + y^2}, \quad l_2 = \sqrt{(b-x)^2 + (y-h)^2}. \tag{19}$$

Equations of motion contain four length coordinates x, y, y_1 and y_2 mutually joint through a relation for the whole rope length

$$l_0 = y_1 + y_2 + l_1 + l_2 \,. \tag{20}$$

By incorporating equation (19) into equation (20) we can express the unknown coordinate

$$y_2 = l_0 - y_1 - l_1 - l_2 = l_0 - y_1 - \sqrt{x^2 + y^2} - \sqrt{(b-x)^2 + (y-h)^2} \,, \tag{21}$$

through the derivation of which according to the time value we obtain

$$\dot{y}_2 = -\dot{y}_1 - l_1^{-1}(x\dot{x} + y\dot{y}) + l_2^{-1}[(b-x)\dot{x} - (y-h)\dot{y}] \tag{22}$$

$$\begin{aligned} \ddot{y}_2 = -\ddot{y}_1 - l_1^{-3}(x\dot{x} + y\dot{y})^2 - l_1^{-1}(x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2) + l_2^{-3}[(b-x)\dot{x} - (y-h)\dot{y}]^2 + \\ + l_2^{-1}[(b-x)\ddot{x} - \dot{x}^2 - (y-h)\ddot{y} - \dot{y}^2] \,, \end{aligned} \tag{23}$$

or after adjustment

$$\begin{aligned} \ddot{y}_2 = -\ddot{y}_1 - [l_1^{-1}x - l_2^{-1}(b-x)]\ddot{x} - [l_1^{-1}y + l_2^{-1}(y-h)]\ddot{y} - \\ - l_1^{-3}(x\dot{x} + y\dot{y})^2 + l_2^{-3}[(b-x)\dot{x} - (y-h)\dot{y}]^2 - (l_1^{-1} + l_2^{-1})(\dot{x}^2 + \dot{y}^2) \,. \end{aligned} \tag{24}$$

This expression will be integrated into the equation (17) and thus we will eliminate the redundant coordinate of y_2. After the adjustment has been performed, we can modify the system of equations of motion into the matrix form of

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{y}_1 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \text{where} \tag{25}$$

$$\left.\begin{aligned} & a_{11} = m_{2red}(\cos\alpha_1 - \cos\alpha_2) \qquad a_{23} = m_{1red}(\cos\alpha_2 - \cos\alpha_1) \\ & a_{12} = m_{2red}(\sin\alpha_2 + \sin\alpha_1) \qquad a_{31} = 0 \\ & a_{13} = m_{1red} + m_{2red} \qquad a_{32} = m \\ & a_{21} = m \qquad a_{33} = -m_{1red}(\sin\alpha_1 + \sin\alpha_2) \\ & a_{22} = 0 \\ & b_1 = m_{2red}\left\{ l_2^{-3}[(b-x)\dot{x} - (y-h)\dot{y}]^2 - (l_1^{-1} + l_2^{-1})(\dot{x}^2 + \dot{y}^2) + l_1^{-3}(x\dot{x} + y\dot{y})^2 \right\} + \\ & \qquad + (m_1 - m_2)g - |F_{b1}|\operatorname{sgn}\dot{y}_1 + |F_{b2}|\operatorname{sgn}\dot{y}_2 - |F_b|\operatorname{sgn}\dot{l}_1 \\ & b_2 = (m_1 g - |F_{b1}|\operatorname{sgn}\dot{y}_1)(\cos\alpha_2 - \cos\alpha_1) - |F_b|\operatorname{sgn}\dot{l}_1 \cos\alpha_2 \\ & b_3 = mg - (m_1 g + |F_{b1}|\operatorname{sgn}\dot{y}_1)(\sin\alpha_2 + \sin\alpha_1) + |F_b|\operatorname{sgn}\dot{l}_1 \sin\alpha_2 \end{aligned}\right\} \tag{26}$$

3. SIMULATION MODEL

Equations of motion were simulated in the Matlab – Simulink program. For the purposes of the numerical solution it is necessary to modify the system (25) so that only one value would be present in its second derivation in each of the equations. Let us mark the system determinant (25) with D and determinants (i = 1,2,3), which were derived from D by substituting the i- column with the right-side vector, with D_i; thereafter according to the Cramer's rule the second derivations can be expressed in the form of

$$\ddot{x} = \frac{D_1}{D}, \qquad \ddot{y} = \frac{D_2}{D}, \qquad \ddot{y}_1 = \frac{D_3}{D} \ . \tag{27}$$

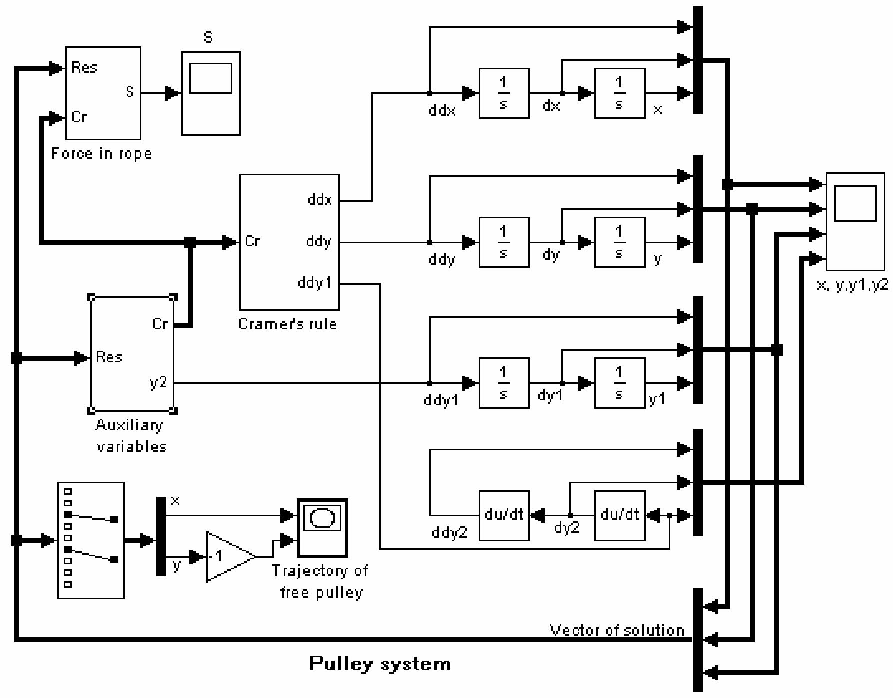

Fig. 4. Simulation model block scheme

The whole block scheme of the simulation model is displayed in Fig. 4. The coordinates of x, y, y_1 which correspond to the selected degrees of freedom are calculated using double integration of the second derivations $\ddot{x}, \ddot{y}, \ddot{y}_1$.

The remaining coordinate y_2 is calculated using an auxiliary subsystem and its derivations are, for the sake of completeness, calculated numerically.

Fig. 5 displays the detail of the subsystem in which rope section lengths l_1, l_2 in front of and behind the free pulley are calculated according to equation (18), and goniometric angle functions α_1, α_2 which these sections form with the horizontal axis (19). Based on the relation (20) the length of the rope section y_2 is calculated and the orientation of the passive resistances is determined. These variables are calculated together with kinematics values of $x, \dot{x}, y, \dot{y}$ passed over to the following block.

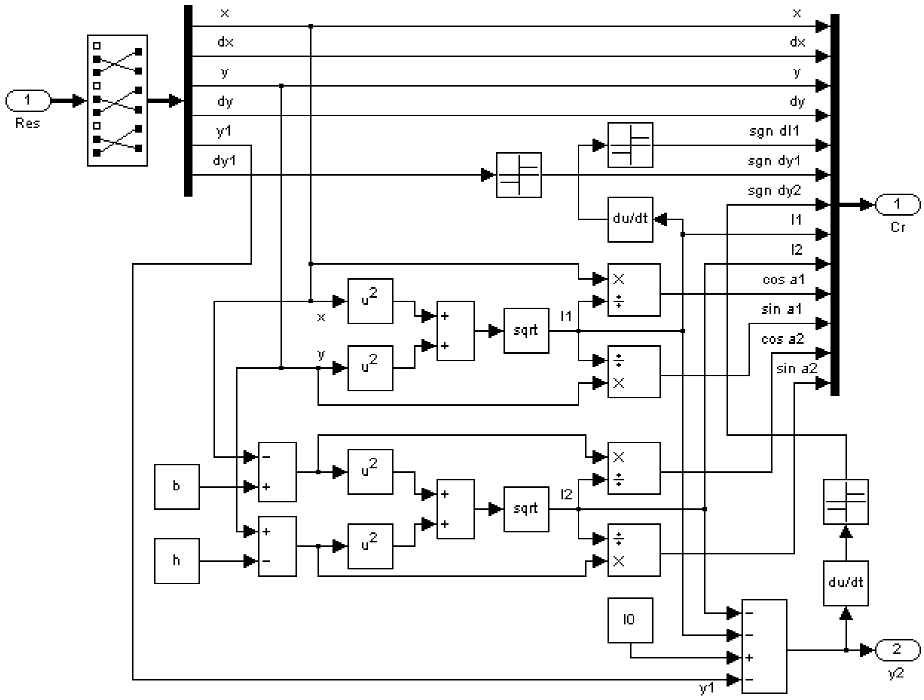

Fig. 5. Block of auxiliary calculations

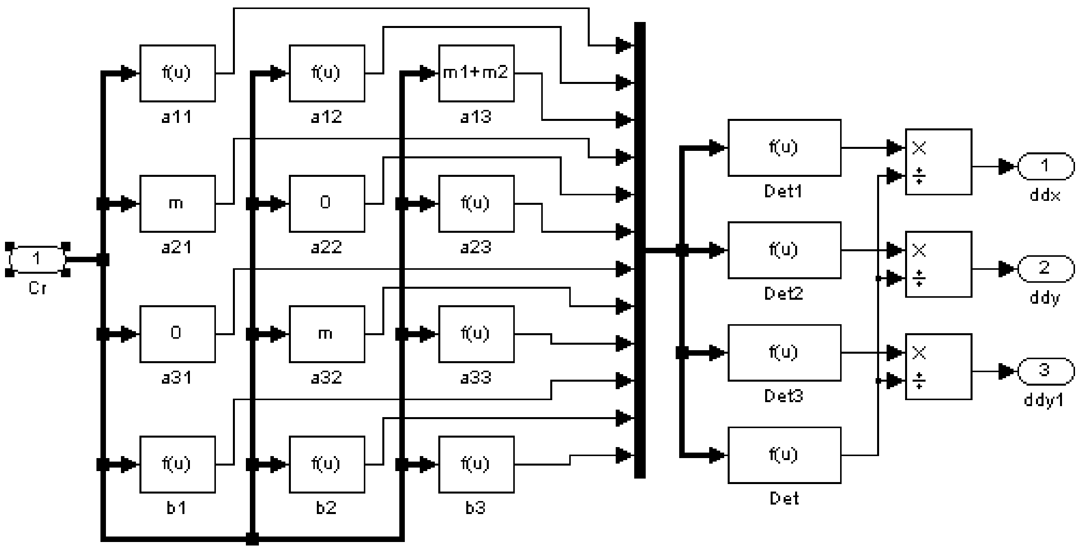

Fig. 6. Block of the calculation of the highest derivations

This important subsystem the detail of which can be found in Fig. 6 performs the calculation of second derivations of the coordinates from equation (26) according to the Cramer’s rule, the equation (27). The system and right-side vector matrix elements are in this case calculated using a user-defined function but due to the simulation speed it is easily possible to imagine other subsystems containing basic blocks in their place. The last subsystem which is not described in detail serves for the calculation of forces present in the rope according to equations (13) and (14).

4. RESULTS AND CONCLUSIONS

The results of the solution for the selected system parameters i.e. the distance of the fixed pulleys $b = 10$ m, elevation of the end pulley $h = 1.3$ m, the ballast weight and the free pulley weight $m = m_1 = m_2 = 10$ kg, negligible inertia moments, braking forces $F_b = 4$ N, $F_{b1} = 3$ N, $F_{b1} = 5$ N and initial conditions $x_0 = 1$ m, $y_0 = x_0h/b = 0.13$ m and zero initial velocities are displayed in Fig. 7, 8, 9 and 10.

The above-stated results confirm the functionality of the elaborated simulation model; the correctness of the mathematical model can be determined based on the compliance of the steady states with the system static balance solution.

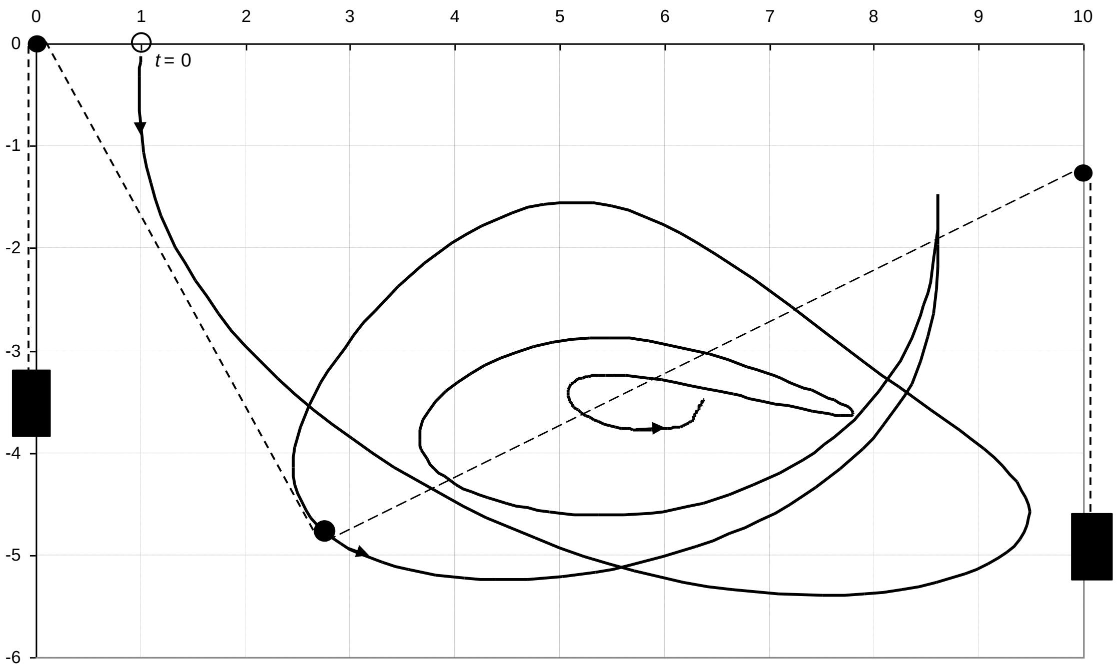

Fig. 7. Free pulley movement trajectory

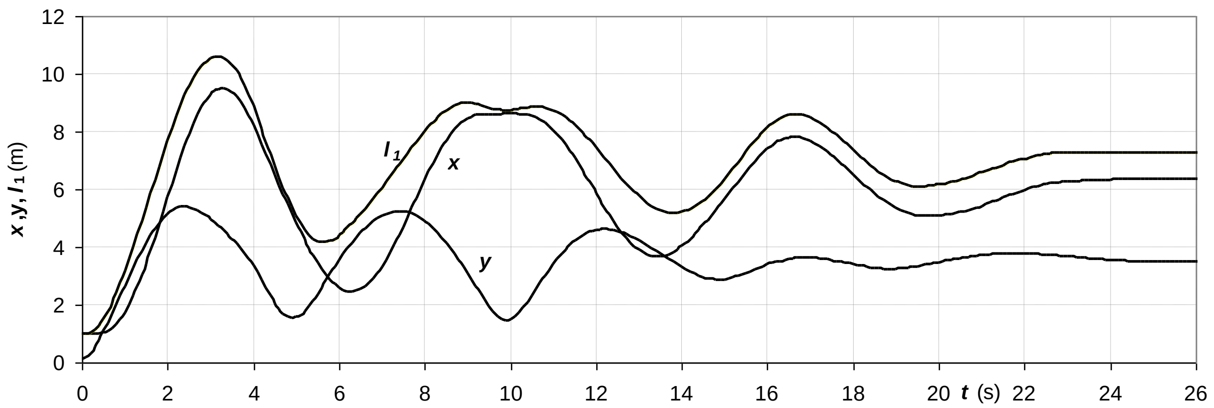

Fig. 8. Time curve of free pulley coordinates and rope section length l_1

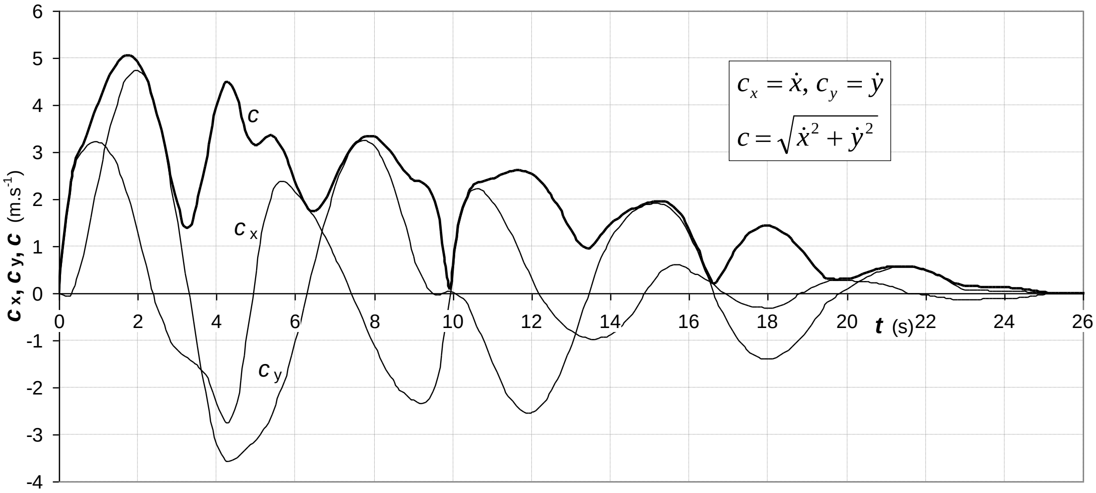

Fig. 9. Time curve of free pulley velocity components c_x, c_y and its resulting velocity c

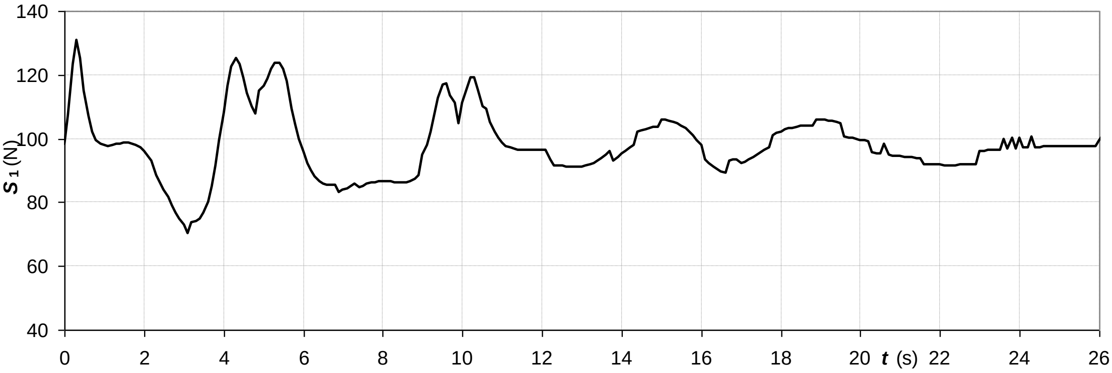

Fig. 10. Time course of the S_1 force in the rope section length l_1

The system described in the article falls within a class of theoretic assignments suitable rather for the lessons on dynamics or the elaboration of simulation models, and it probably will not be used in technical practice. However, the described methodology of the mathematical modeling and simulation of system dynamics is fit for use in various engineering fields, among others in research, and machine and device design and construction.

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