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Dedicated to Professor Hossein Hajiabolhassan on his 51st birthday

ON INCIDENCE COLORING OF GRAPH FRACTIONAL POWERS

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Abstract. For any $n \in \mathbb{N}$, the n-subdivision of a graph G is a simple graph $G^{\frac{1}{n}}$ which is constructed by replacing each edge of G with a path of length n. The m-th power of G is a graph, denoted by G^m, with the same vertices of G, where two vertices of G^m are adjacent if and only if their distance in G is at most m. In [M.N. Iradmusa, On colorings of graph fractional powers, Discrete Math. 310 (2010), no. 10–11, 1551–1556] the m-th power of the n-subdivision of G, denoted by $G^{\frac{m}{n}}$ is introduced as a fractional power of G. The incidence chromatic number of G, denoted by $\chi_i(G)$, is the minimum integer k such that G has an incidence k-coloring. In this paper, we investigate the incidence chromatic number of some fractional powers of graphs and prove the correctness of the incidence coloring conjecture for some powers of graphs.

Keywords: incidence coloring, incidence chromatic number, subdivision of graph, power of graph.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

In this paper we only consider simple, finite and nontrivial graphs. As usual, we denote the maximum degree of a graph G by $\Delta(G)$. For each vertex $v \in V(G)$, $N_G(v)$ is the set of all neighbors of v in G and $N_G[v] = N_G(v) \cup \{v\}$. From now on, we use the notation $[n]$ instead of $\{1, \ldots, n\}$.

Let $G = (V, E)$ be a nontrivial graph. Any pair (v, e) is called an incidence of G, if $v \in V$, $e \in E$ and $v \in e$. The set of the incidences of G is denoted by $I(G)$. Precisely, $I(G) = \{(v, e) : v \in V, e \in E, v \in e\}$. The incidence graph of G, denoted by $\mathcal{I}(G)$, is a graph with vertex set $V(\mathcal{I}(G)) = I(G)$ and two vertices (v, e) and (w, f) are adjacent provided one of the following holds:

(i) $v = w$,
(ii) $e = f$,
(iii) the edge $\{v, w\}$ equals e or f.

For an edge $e = \{u, v\} \in E(G)$, we show two incidence vertices (u, e) and (v, e) with (u, v) and (v, u), respectively. Also, $\{(u, v) | \ v \in N_G(u)\}$ is the set of all first incidences of u, denoted by $I^+(u)$ and $\{(v, u) | \ v \in N_G(u)\}$ is the set of all second incidences of u, denoted by $I^-(u)$.

A proper k-coloring of G is a mapping $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ for any two adjacent vertices u and v of G. The minimum integer k that G has a proper k-coloring is the chromatic number of G and denoted by $\chi(G)$.

The concept of incidence coloring is introduced by Brualdi and Massey in 1993 [4].

Definition 1.1 ([4])**.** Let G be a graph. A mapping $c : I(G) \to \{1, \ldots, k\}$ is an incidence k-coloring of G such that any two adjacent incidence vertices have different colors. The incidence chromatic number of G, denoted by $\chi_i(G)$, is the minimum integer k such that G has an incidence k-coloring. In other words, $\chi_i(G) = \chi(\mathcal{I}(G))$.

Definition 1.2 ([11])**.** Let G be a graph and $r \in \mathbb{N}$. An incidence k-coloring of G is an incidence (k, r)-coloring of G if for every vertex $v \in V(G)$, the number of colors used for coloring $I^-(v)$ is at most r. We denote by $\chi_{i,r}(G)$ the smallest number of colors required for an incidence (k, r)-coloring of G.

Observe that $\chi_i(G) \leq \chi_{i,r}(G)$ and $\chi_{i,1}(G) = \chi(G^2)$. In order to prove the second identity, suppose that c is a $(k, 1)$-incidence coloring of a graph G. Then $|\{c((u, v))|(u, v) \in I^-(v)\}| = 1$ for every vertex $v \in V(G)$. Therefore, the mapping c' defined by $c'(v) = c((u, v))$ for every vertex v is well-defined. Moreover, it is not difficult to see that $c'(u) \neq c'(v)$ for every two vertices u and v whose distance in G is 1 or 2. Therefore, c' is a proper vertex-coloring of G^2. Conversely, from every proper k-vertex-coloring of G^2, we obtain an incidence $(k, 1)$-coloring c of G by setting $c'((u, v)) = c(v)$ for every incidence (u, v) of G. Therefore, $\chi_{i,1}(G) = \chi(G^2)$ for every graph G.

One can easily prove that $\chi_i(G) = \max_{l \in [k]}\{\chi_i(G_l\}$, where $G_1, \ldots, G_k$ are the connected components of G. For this reason, in this paper we only consider connected graphs. In addition, if $\{I_1, I_2, \ldots, I_k\}$ is a partition of $I(G)$, then

$$\chi_i(G) = \chi(\mathcal{I}(G)) \leq \chi(\mathcal{I}_1(G)) + \cdots + \chi(\mathcal{I}_k(G)),$$

where $\mathcal{I}_j(G)$ is the subgraph of $\mathcal{I}(G)$ induced by the subset I_j of incidences.

In [4], the authors determined the incidence chromatic number of trees, complete bipartite graphs and complete graphs. In addition, the bounds for incidence chromatic number of various graph classes is found out. Also, they conjectured that the incidence chromatic number of an arbitrary graph G is at most $\Delta(G) + 2$ [4], which is named *the incidence coloring conjecture* (ICC). Although the conjecture was disproved by Guiduli in 1997 who showed that Paley graphs have incidence chromatic number at least $\Delta + O(\log(\Delta))$ [8], the stated upper bound is proved for some classes of graphs such as paths, fans, cycles, wheels, complete tripartite graph and adding edge wheels, which were determined by Chen *et al.* in 1998 [5]. Following this, in 2005, the incidence coloring of graphs with maximum degree $\Delta = 3$ was investigated and determined by Maydanskiy [16]. The bounds for incidence chromatic number of various graph classes is found out. For more information see [2, 6, 8, 11, 15, 19–24, 26–28]. It is worth to note that, despite the fact that the incidence coloring conjecture is proved for some special graphs with maximum degree 4, it is unsolved in general for graphs with maximum degree four, although the upper bound 7 is stated by Gregor, Lužar and Soták in [7].

In this paper, we investigate the correctness of incidence coloring conjecture for some fractional powers of graphs. The concept of fractional power of graphs was first introduced by Iradmusa in 2010 [12]. In the following, to deal with fractional power of graphs we are going to take some required definition into account. Let G be a graph and $m, n \in \mathbb{N}$. The *m-power* of G, denoted by G^m, is defined on the vertex set $V(G)$ by adding edges joining any two distinct vertices x and y with distance at most m. In other words, $E(G^m) = \{\{x, y\} : 1 \leq d_G(x, y) \leq m\}$. Also, the m-distance graph of G, denoted by $G^{[m]}$, is a graph with vertex set $V(G)$ and two vertices v and w are adjacent if $d_G(u, v) = m$. Obviously, $G^{[1]} = G$, $G^m = G^{m-1} \cup G^{[m]}$ and so $G^m = \bigcup_{k=1}^{m} G^{[k]}$, where the union of graphs G_i $(1 \leq i \leq n)$ is the graph $\bigcup_{i=1}^{n} G_i$ with vertex set $\bigcup_{i=1}^{n} V(G_i)$ and edge set $\bigcup_{i=1}^{n} E(G_i)$. In addition, the *n-subdivision* of G, denoted by $G^{\frac{1}{n}}$, is constructed by replacing each edge xy of G with a path of length n with new vertices $(xy)_{\frac{1}{n}}, \ldots, (xy)_{\frac{n-1}{n}}$, where the vertex $(xy)_{\frac{l}{n}}$ has distance l from the vertex x, where $l \in \{0, 1, \ldots, n\}$. Also, $(xy)_{\frac{l}{n}} = (yx)_{\frac{n-l}{n}}$, $(xy)_{\frac{0}{n}} = x$ and $(xy)_{\frac{n}{n}} = y$. Any vertex $(xy)_{\frac{0}{n}}$ of $G^{\frac{1}{n}}$ is called a terminal vertex (or briefly t-vertex) and the other vertices are called internal vertices. Now the fractional power of a graph G is defined as follows.

Definition 1.3 ([12])**.** Let G be a graph and $m, n \in \mathbb{N}$. The graph $G^{\frac{m}{n}}$ is defined to be the *m-power* of the n-subdivision of G. In other words, $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$.

Note that $G^{\frac{1}{1}} = G$ and $G^{\frac{2}{2}}$ is isomorphic to $T(G)$, the total graph of G, which is defined in [1]. We denote the set of terminal vertices of $G^{\frac{m}{n}}$ by $V_t(G^{\frac{m}{n}})$ and the set of internal vertices of $G^{\frac{m}{n}}$ by $V_i(G^{\frac{m}{n}})$. Hence, $V(G^{\frac{m}{n}}) = V_t(G^{\frac{m}{n}}) \cup V_i(G^{\frac{m}{n}})$. We also use the notation $G^{\frac{[m]}{n}}$ for the graph $(G^{\frac{1}{n}})^{[m]}$. For other necessary definitions and notations we refer the reader to the textbook [3].

In this paper, we prove that the incidence coloring conjecture is true for some $\frac{m}{n}$-power of graphs, where $m, n \in \mathbb{N}$ and $\frac{m}{n} \in \mathbb{Q} \cap (0, 1)$. The main results of this paper are as follows.

Theorem 1.4. *Let G be a connected graph with $\Delta(G) \geq 3$, $m, n \in \mathbb{N}$ and $0 < \frac{m}{n} < \frac{1}{2}$. Then*

$$\chi_i(G^{\frac{m}{n}}) = \chi_{i,1}(G^{\frac{m}{n}}) = \Delta(G^{\frac{m}{n}}) + 1 = m\Delta(G) + 1.$$

Theorem 1.5. *Let $n \in \mathbb{N} \setminus \{1\}$ and G be a connected graph. Then*

$$\chi_i(G^{\frac{1}{n}}) = \begin{cases} \Delta(G^{\frac{1}{n}}) + 2 & \textit{if } G = C_l, nl \not\equiv 0 \ (\textit{mod } 3), \\ \Delta(G^{\frac{1}{n}}) + 1 & \textit{otherwise.} \end{cases}$$

Theorem 1.6. *Let $n \in \mathbb{N} \setminus \{1, 2\}$ and G be a connected graph. Then*

$$\chi_i(G^{\frac{2}{n}}) = \begin{cases} \Delta(G^{\frac{2}{n}}) + 2 & \textit{if } G = C_l, nl \not\equiv 0 \ (\textit{mod } 5), \\ \Delta(G^{\frac{2}{n}}) + 1 & \textit{otherwise.} \end{cases}$$

Theorem 1.7. *Let $m \in \mathbb{N}$ and G be a connected graph with $\Delta(G) \geq 3$. Then*

$$\chi_i(G^{\frac{m}{2m}}) \leq \Delta(G^{\frac{m}{2m}}) + 2 = m\Delta(G) + 2.$$

The paper is organized as follows. In Section 2, some preliminary definitions and theorems are mentioned. Section 3 is devoted to the proofs of the main theorems and the last section contains some concluding remarks and open problems.

2. PRELIMINARY DEFINITIONS AND THEOREMS

As we said before, the incidence coloring conjecture is proved for some classes of graphs. In the following, some of such classes, which we are going to use in the proofs of the main theorems, are mentioned. In [5], Chen *et al.* investigated incidence coloring of some graphs including paths and cycles and concluded two following theorems. Let P_n and C_n denote the path and cycle of order n, respectively.

Theorem 2.1 ([5])**.** *Let $n \in \mathbb{N} \setminus \{1\}$. Then* $\chi_i(P_n) = \begin{cases} 2 & \textit{if } n = 2, \\ 3 & \textit{otherwise.} \end{cases}$

Theorem 2.2 ([5])**.** *Let $n \in \mathbb{N} \setminus \{1, 2\}$. Then* $\chi_i(C_n) = \begin{cases} 3 & \textit{if } n \equiv 0 \ (mod\ 3), \\ 4 & \textit{otherwise.} \end{cases}$

In 2008, the incidence coloring of the square of the path was found by Li *et al.* [15]. Also, in 2012, Nakprasit *et al.* proved that, for $n \geq 5$, the incidence chromatic number of C_n^2 is 5, if n is divisible by 5, otherwise, it is equal to 6 [19].

Theorem 2.3 ([15])**.** *Let n, k be positive integers and $n \geq 2$. Then*

$$\chi_i(P_n^k) = \min\{n, 2k + 1\} = \Delta(P_n^k) + 1.$$

Theorem 2.4 ([19])**.** *Let $n \in \mathbb{N} \setminus \{1, 2\}$. Then*

$$\chi_i(C_n^2) = \begin{cases} n & \textit{if } n \leq 5, \\ 5 & \textit{if } n > 5, n \equiv 0 \ (mod\ 5), \\ 6 & \textit{otherwise.} \end{cases}$$

In [17], Montgomery defined a dynamic coloring of a graph G to be a proper coloring in which each vertex neighborhood of size at least two receives at least two distinct colors. The dynamic chromatic number $\chi_d(G)$ is the least number of colors in such a coloring of G. Lai, Montgomery and Poon proved the following theorem, which will be used in the proof of Theorem 1.5.

Theorem 2.5 ([14])**.** *If G is a graph with $\Delta(G) \geq 3$, then $\chi_d(G) \leq \Delta(G) + 1$.*

What stands out from two definitions 1.1 and 1.3 is that the graph $\mathcal{I}(G)$ is isomorphic to the induced subgraph of $G^{\frac{3}{3}}$ on $V_i(G^{\frac{3}{3}})$. Hence, we conclude that $\chi_i(G) = \chi(G^{\frac{3}{3}}[V_i(G^{\frac{3}{3}})])$. Although the coloring of fractional powers of graph for powers greater that one is not completely investigated, except coloring of some $\frac{3}{3}$-power of graphs which is investigated in [13, 18, 25], it is highly investigated for powers lower than one in [10, 12]. Iradmusa in [12] proved that $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$, where $n = m + 1$ or $m = 2 < n$.

Theorem 2.6 ([12]). *If G is a graph with $\Delta(G) \geq 3$ and $m \in \mathbb{N}$, then*

$$\chi(G^{\frac{m}{m+1}}) = \omega(G^{\frac{m}{m+1}}) = \begin{cases} \frac{m}{2}\Delta(G) + 1 & \textit{if } m \equiv 0 \ (mod\ 2), \\ \frac{m-1}{2}\Delta(G) + 2 & \textit{if } m \equiv 1 \ (mod\ 2). \end{cases}$$

Also, it was conjectured that $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$ for any graph G with $\Delta(G) \geq 3$ when $\frac{m}{n} \in \mathbb{Q} \cap (0, 1)$. Despite the fact that the conjecture was disproved by Hartke, Liu and Petrickova [10], who proved that the conjecture is not true for the Cartesian product $C_3 \square K_2$ (triangular prism) when $m = 3$ and $n = 5$, they claimed that it is valid for all graphs except when $G = C_3 \square K_2$. In addition, they proved that the conjecture is true when m is even.

Theorem 2.7 ([10]). *If G is a graph with $\Delta(G) \geq 3$ and $1 < m < n$ with m even, then*

$$\chi(G^{\frac{m}{n}}) = \frac{m}{2}\Delta(G) + 1.$$

3. PROOFS OF THE MAIN THEOREMS

It is well-known that any connected graph with maximum degree r is a subgraph of a connected r-regular graph. Also, $\chi_i(H) \leq \chi_i(G)$ for any subgraph H of the graph G. Hence, to prove a theorem about the upper bound on the incidence chromatic number, it is sufficient to proof it for the connected r-regular graphs. We use this fact in the proof of some upper bound results.

Proof of Theorem 1.4. We know that $\chi_i(G) \leq \chi_{i,1}(G) = \chi(G^2)$. Hence,

$$\chi_i(G^{\frac{m}{n}}) \leq \chi_{i,1}(G^{\frac{m}{n}}) = \chi((G^{\frac{m}{n}})^2) = \chi(G^{\frac{2m}{n}})$$

and by Theorem 2.7,

$$\chi(G^{\frac{2m}{n}}) = \omega(G^{\frac{2m}{n}}) = \frac{2m}{2}\Delta(G) + 1 = m\Delta(G) + 1.$$

So

$$\chi_i(G^{\frac{m}{n}}) \leq \chi_{i,1}(G^{\frac{m}{n}}) = m\Delta(G) + 1.$$

On the other hand,

$$\chi_i(G^{\frac{m}{n}}) \geq \Delta(G^{\frac{m}{n}}) + 1 = m\Delta(G) + 1.$$

Therefore,

$$\chi_i(G^{\frac{m}{n}}) = \chi_{i,1}(G^{\frac{m}{n}}) = m\Delta(G) + 1.$$

□

Proof of Theorem 1.5. First suppose that $\Delta(G) \leq 2$. If $G = C_l$, then $G^{\frac{1}{n}} \cong C_{nl}$ and if $G = P_l$, then $G^{\frac{1}{n}} \cong P_{n(l-1)+1}$. Hence, by Theorems 2.1 and 2.2 we have $\chi_i(P_l^{\frac{1}{n}}) = 3$ and $\chi_i(C_l^{\frac{1}{n}}) = 3$ when $|V(C_l^{\frac{1}{n}})| = nl \equiv 0 \pmod 3$. Otherwise, $\chi_i(C_l^{\frac{1}{n}}) = 4$.

Now we suppose that $\Delta(G) \geq 3$. If $n \geq 3$, then by Theorem 1.4 we conclude that $\chi_i(G^{\frac{1}{n}}) = \Delta(G^{\frac{1}{n}}) + 1 = \Delta(G) + 1$. Finally suppose that $n = 2$. By Theorem 2.5, there is a dynamic coloring $c : V(G) \to \{1, \ldots, \Delta(G) + 1\}$ for G. We define an incidence coloring c' for $G^{\frac{1}{2}}$. First, for $uv \in E(G)$ color all incidences in $I^-(u)$ with color $c(u)$. Now we need to color the first incidences in such a way that for any edge $\{u, v\} \in E(G)$,

$$\{c'((u, (uv)_{\frac{1}{2}})), c'((v, (vu)_{\frac{1}{2}}))\} \cap \{c(u), c(v)\} = \varnothing.$$

For each incidence $(u, (uv)_{\frac{1}{2}})$ in $I^+(u)$, there are $\Delta(G) - 1$ available colors. Let $\mathcal{L}((u, (uv)_{\frac{1}{2}}))$ be the set of available colors for the incidence $(u, (uv)_{\frac{1}{2}})$. Since $|c(N_G(u))| \geq 2$ for any t-vertex $u \in V(G)$, $|\bigcup_{x \in I^+(u)} \mathcal{L}(x)| = \Delta(G)$, using Hall's Theorem (see Theorem 16.4 in [3]), we can assign different colors from $\{1, \ldots, \Delta(G) + 1\} \setminus \{c(u)\}$ to the incidences in $I^+(u)$, such that for any edge $\{u, v\} \in E(G)$ with $c'(((uv)_{\frac{1}{2}}, u)) = c(u)$ and $c'(((uv)_{\frac{1}{2}}, v)) = c(v)$, we have

$$\{c'((u, (uv)_{\frac{1}{2}})), c'((v, (vu)_{\frac{1}{2}}))\} \cap \{c(u), c(v)\} = \varnothing.$$

The given coloring is a $(\Delta(G) + 1)$-incidence coloring of $G^{\frac{1}{2}}$. □

Proof of Theorem 1.6. First suppose that $\Delta(G) \leq 2$. If $G = C_l$, then $G^{\frac{2}{n}} \cong C_{nl}^2$ and if $G = P_l$, then $G^{\frac{2}{n}} \cong P_{n(l-1)+1}^2$. Hence, by Theorem 2.3 and 2.4 we have $\chi_i(P_l^{\frac{2}{n}}) = \Delta(P_l^{\frac{2}{n}}) + 1$ and $\chi_i(C_l^{\frac{1}{n}}) = 5$ when $|V(C_l^{\frac{2}{n}})| = nl \equiv 0 \pmod 5$. Otherwise, $\chi_i(C_l^{\frac{2}{n}}) = 6$.

Now suppose that $\Delta(G) = \Delta \geq 3$. If $n \geq 5$, then Theorem 1.4 implies that

$$\chi_i(G^{\frac{2}{n}}) = \Delta(G^{\frac{2}{n}}) + 1 = 2\Delta(G) + 1.$$

Therefore, we need to prove the theorem for $n = 3$ and $n = 4$.

Case 1. $n = 3$.
Since $\Delta(G^{\frac{2}{3}}) = 2\Delta(G)$, it follows that $\chi_i(G^{\frac{2}{3}}) \geq 2\Delta(G) + 1$. Now, we prove that $\chi_i(G^{\frac{2}{3}}) \leq 2\Delta(G) + 1$. As mentioned before, we only consider the connected regular graphs.

We partition the incidences of $G^{\frac{2}{3}}$ into two subsets

$$I_1 = \bigcup_{\{u,v\} \in E(G)} \{(u, (vu)_{\frac{1}{3}}), ((uv)_{\frac{1}{3}}, (vu)_{\frac{1}{3}}), ((vu)_{\frac{1}{3}}, (uv)_{\frac{1}{3}}), (v, (uv)_{\frac{1}{3}})\}$$

and $I_2 = I(G^{\frac{2}{3}}) \setminus I_1$. We show that $\chi(\mathcal{I}_1(G^{\frac{2}{3}})) = \Delta(G)$ and $\chi(\mathcal{I}_2(G^{\frac{2}{3}})) = \Delta(G) + 1$, where $\mathcal{I}_j(G^{\frac{2}{3}})$ is the subgraph of $\mathcal{I}(G^{\frac{2}{3}})$ induced by I_j $(j = 1, 2)$. Thus,

$$\chi_i(G^{\frac{2}{3}}) = \chi(\mathcal{I}(G^{\frac{2}{3}})) \leq \chi(\mathcal{I}_1(G^{\frac{2}{3}})) + \chi(\mathcal{I}_2(G^{\frac{2}{3}})) = 2\Delta(G) + 1.$$

Since $G^{\frac{1}{2}}$ is a bipartite graph, $\chi'(G^{\frac{1}{2}}) = \Delta(G^{\frac{1}{2}}) = \Delta(G)$. Suppose that $c_1 : E(G^{\frac{1}{2}}) \to [\Delta]$ is a proper edge coloring of $G^{\frac{1}{2}}$. We define the following coloring for $\mathcal{I}_1(G^{\frac{2}{3}})$:

$$f((x,y)) = \begin{cases} c_1(\{u,(uv)_{\frac{1}{2}}\}) & \text{if } (x,y) = (u,(vu)_{\frac{1}{3}}), \\ c_1(\{u,(uv)_{\frac{1}{2}}\}) & \text{if } (x,y) = ((uv)_{\frac{1}{3}},(vu)_{\frac{1}{3}}). \end{cases}$$

Note that in $G^{\frac{1}{2}}$, $(uv)_{\frac{1}{2}} = (vu)_{\frac{1}{2}}$. Let (x,y) and (r,s) be adjacent in $\mathcal{I}_1(G^{\frac{2}{3}})$. There are three cases for these adjacent vertices:

(i) $(x,y) = (u,(vu)_{\frac{1}{3}})$ and $(r,s) = (u,(wu)_{\frac{1}{3}})$, where $v,w \in N_G(u)$. In this case

$$f((x,y)) = c_1(\{u,(uv)_{\frac{1}{2}}\}) \neq c_1(\{u,(uw)_{\frac{1}{2}}\}) = f((r,s)).$$

(ii) $(x,y) = ((uv)_{\frac{1}{3}},(vu)_{\frac{1}{3}})$ and $(r,s) = ((vu)_{\frac{1}{3}},(uv)_{\frac{1}{3}})$, where $\{u,v\} \in E(G)$. In this case

$$f((x,y)) = c_1(\{u,(uv)_{\frac{1}{2}}\}) \neq c_1(\{v,(vu)_{\frac{1}{2}}\}) = f((r,s)).$$

(iii) $(x,y) = (u,(vu)_{\frac{1}{3}})$ and $(r,s) = ((vu)_{\frac{1}{3}},(uv)_{\frac{1}{3}})$, where $\{u,v\} \in E(G)$. In this case

$$f((x,y)) = c_1(\{u,(uv)_{\frac{1}{2}}\}) \neq c_1(\{v,(vu)_{\frac{1}{2}}\}) = f((r,s)).$$

So f is a proper coloring of $\mathcal{I}_1(G^{\frac{2}{3}})$ and then $\chi(\mathcal{I}_1(G^{\frac{2}{3}})) = \Delta$.

To prove $\chi(\mathcal{I}_2(G^{\frac{2}{3}})) = \Delta + 1$, we use the dynamic coloring of G. We know that $\chi_d(G) \leq \Delta + 1$. Suppose that $c_d : V(G) \to [\Delta+1]$ is a dynamic coloring of G. At first, we assign the color $c_d(u)$ to any incidence of $I^-(u)$ in $G^{\frac{2}{3}}$. Easily one can show that $\mathcal{I}_2(G^{\frac{2}{3}})$ is the union of n disjoint copies of $\mathcal{I}(K_{\Delta+1})$, except of the colored vertices $\{((vu)_{\frac{1}{3}},u),((uv)_{\frac{1}{3}},v)|\ \{u,v\} \in E(G)\}$. Precisely, for any vertex $u \in V(G)$, we have a copy of $K_{\Delta+1}$ in $G^{\frac{2}{3}}$ induced by the vertices of $\{u\} \cup \{(uv)_{\frac{1}{3}}|\ v \in N_G(u)\}$, which is denoted by $K^u_{\Delta+1}$. We know that $\chi_i(K_n) = n$ and any incidence coloring of K_n is also an incidence $(n,1)$-coloring. Since we used only one color $c_d(u) \in [\Delta+1]$ for all incidences of $I^-(u)$ in $K^u_{\Delta+1}$, similar to the proof of Theorem 1.5 we can extend this partial coloring to a proper coloring of $\mathcal{I}_2(G^{\frac{2}{3}})$ by use of Hall's Theorem.

Case 2. $n = 4$.

Since $\Delta(G^{\frac{2}{4}}) = 2\Delta$, $\chi_i(G^{\frac{2}{4}}) \geq 2\Delta+1$. It is enough to prove that $\chi_i(G^{\frac{2}{4}}) \leq 2\Delta+1$. Let c be a dynamic coloring of G with colors $\{1,\ldots,\Delta+1\}$. At first, for each $u \in V(G)$, color all incidences in $I^-(u)$ with color $c(u)$. We are going to have an incidence coloring for $G^{\frac{2}{4}}$, named c', such that for any edge $\{u,v\} \in E(G)$,

$$\{c'((u,(uv)_{\frac{1}{4}})), c'((v,(vu)_{\frac{1}{4}}))\} \cap \{c(u),c(v)\} = \varnothing.$$

Let $\mathcal{L}(x)$ be the set of available colors for the incidence x. Since $|\mathcal{L}((u,(uv)_{\frac{1}{4}}))| = \Delta - 1$ for any incidence $(u,(uv)_{\frac{1}{4}})$ and $|c(N_G(u))| \geq 2$ for any t-vertex u, $|\bigcup_{v\in N_G(u)} \mathcal{L}((u,(uv)_{\frac{1}{4}}))| = \Delta$. Therefore, by Hall's Theorem, we can assign different colors from $\{1,\ldots,\Delta+1\} \setminus \{c(u)\}$ to the incidences in $\{(u,(uv)_{\frac{1}{4}})|\ v \in N_G(u)\}$ such that $\{c'((u,(uv)_{\frac{1}{4}}))\} \cap \{c(u),c(v)\} = \varnothing$. Hence, by repeating this coloring on each vertex, the desired result can be achieved. Also, for any t-vertex $u \in V(G)$, color the incidence $((uv)_{\frac{1}{4}}, (uw)_{\frac{1}{4}})$ with color $c'((u, (uw)_{\frac{1}{4}}))$, where $v, w \in N_G(u)$.

Now we extend this partial coloring to an incidence coloring of $G^{\frac{2}{4}}$. We assign different colors from $\{\Delta+2, \ldots, 2\Delta+1\}$ to the incidences of $\{(u, (uv)_{\frac{2}{4}}) | \, v \in N_G(u)\}$. Note that color two incidences $((uv)_{\frac{2}{4}}, u)$ and $((uv)_{\frac{2}{4}}, v)$ with colors $c(u)$ and $c(v)$, respectively. It is enough to color the incidences of the cycle $C = ((uv)_{\frac{1}{4}}, (uv)_{\frac{3}{4}}, (uv)_{\frac{2}{4}}, (uv)_{\frac{1}{4}})$ for any edge $\{u, v\} \in E(G)$. Color two incidences $((uv)_{\frac{1}{4}}, (uv)_{\frac{2}{4}})$ and $((uv)_{\frac{3}{4}}, (uv)_{\frac{2}{4}})$ as same as $(u, (uv)_{\frac{2}{4}})$ and assign color

$$\alpha \in \{\Delta+2, \ldots, 2\Delta+1\} \setminus \{c'(u, (uv)_{\frac{2}{4}}), c'(v, (uv)_{\frac{2}{4}})\}$$

to the incidences $((uv)_{\frac{1}{4}}, (uv)_{\frac{3}{4}})$ and $((uv)_{\frac{2}{4}}, (uv)_{\frac{3}{4}})$. Also, color two incidences $((uv)_{\frac{2}{4}}, (uv)_{\frac{1}{4}})$ and $((uv)_{\frac{3}{4}}, (uv)_{\frac{1}{4}})$ with $c'((u, (uv)_{\frac{1}{4}}))$ and

$$\beta \in \{\Delta+2, \ldots, 2\Delta+1\} \setminus \{c'((u, (uv)_{\frac{2}{4}})), \alpha\},$$

respectively. One can easily show that this coloring is an incidence $(2\Delta+1)$-coloring for $G^{\frac{2}{4}}$. □

Remark 3.1. A stronger result can be drawn by reviewing the proof of Theorems 1.5 and 1.6. In fact we use at most two colors for the coloring of second incidences of each vertex. Therefore, we can replace χ_i with $\chi_{i,2}$.

Let G and H be two graphs. The direct product $G \times H$ of graphs G and H is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$, and vertices (g, h) and (g', h') are adjacent in $G \times H$ if and only if g is adjacent to g', and h is adjacent to h' [9]. Also, the induced subgraph of G by a subset S of $V(G)$ is denoted by $G[S]$ and the line graph of G is denoted by $L(G)$ (see [3, p. 23]).

Lemma 3.2. *Let $m \in \mathbb{N}$ and G be a nontrivial graph. Then*

$$G^{\frac{[m]}{2m}} = G^{\frac{1}{2}} \cup \frac{m-1}{2}(L(G^{\frac{1}{2}}) \times K_2)$$

when m is an odd integer and

$$G^{\frac{[m]}{2m}} = G^{\frac{1}{2}} \cup L(G^{\frac{1}{2}}) \cup \frac{m-2}{2}(L(G^{\frac{1}{2}}) \times K_2)$$

when m is an even integer.

Proof. Let $\Gamma = G^{\frac{[m]}{2m}}$. We partition the vertex set of Γ to the subsets

$$V_k = \bigcup_{uv \in E(G)} \{(uv)_{\frac{k}{2m}}, (uv)_{\frac{m-k}{2m}}, (vu)_{\frac{k}{2m}}, (vu)_{\frac{m-k}{2m}}\},$$

where $0 \le k \le \lfloor \frac{m}{2} \rfloor$. One can easily see that $E(\Gamma) = \bigcup_{k=0}^{\lfloor \frac{m}{2} \rfloor} E(\Gamma[V_k])$ because two ends of any edge belong to the same part of the partition. So $\Gamma = \bigcup_{k=0}^{\lfloor \frac{m}{2} \rfloor} \Gamma_k$, where $\Gamma_k = \Gamma[V_k]$.

Now we show that $\Gamma_0 = G^{\frac{1}{2}}$, $\Gamma_{\frac{m}{2}} = L(G^{\frac{1}{2}})$ when m is even and $\Gamma_k = L(G^{\frac{1}{2}}) \times K_2$ when $0 < k < \frac{m}{2}$. We have

$$V_0 = \bigcup_{\{u,v\}\in E(G)} \{(uv)_{\frac{0}{2m}}, (uv)_{\frac{m}{2m}}, (vu)_{\frac{0}{2m}}\}$$

and

$$E(\Gamma_0) = \bigcup_{\{u,v\}\in E(G)} \{\{(uv)_{\frac{0}{2m}}, (uv)_{\frac{m}{2m}}\}, \{(uv)_{\frac{m}{2m}}, (vu)_{\frac{0}{2m}}\}\}$$

which follows that $\Gamma_0 = G^{\frac{1}{2}}$. If m is an even number then

$$V_{\frac{m}{2}} = \bigcup_{\{u,v\}\in E(G)} \{(uv)_{\frac{\frac{m}{2}}{2m}}, (vu)_{\frac{\frac{m}{2}}{2m}}\}.$$

Suppose that $E_{\frac{m}{2},1} = \bigcup_{\{u,v\}\in E(G)} \{(uv)_{\frac{\frac{m}{2}}{2m}} (vu)_{\frac{\frac{m}{2}}{2m}}\}$ and

$$E_{\frac{m}{2},2} = \bigcup_{u\in V(G)} \bigcup_{v,w\in N_G(u)} \{\{(uv)_{\frac{\frac{m}{2}}{2m}}, (uw)_{\frac{\frac{m}{2}}{2m}}\}\}.$$

Thus, $E(\Gamma_{\frac{m}{2}}) = E_{\frac{m}{2},1} \cup E_{\frac{m}{2},2}$ and easily we can show that $\Gamma_{\frac{m}{2}}$ is isomorphic to $L(G^{\frac{1}{2}})$. Now suppose that $1 < k < \frac{m}{2}$. In this case, Γ_k has four vertices on each edge of G. Suppose that

$$E_{k,1} = \bigcup_{\{u,v\}\in E(G)} \{\{(uv)_{\frac{k}{2m}}, (vu)_{\frac{m-k}{2m}}\}, \{(uv)_{\frac{m-k}{2m}}, (vu)_{\frac{k}{2m}}\}\}$$

and

$$E_{k,2} = \bigcup_{u\in V(G)} \bigcup_{v,w\in N_G(u)} \{\{(uv)_{\frac{k}{2m}}, (uw)_{\frac{m-k}{2m}}\}, \{(uv)_{\frac{m-k}{2m}}, (uw)_{\frac{k}{2m}}\}\}.$$

So $E(\Gamma_k) = E_{k,1} \cup E_{k,2}$. Note that

$$V(L(G^{\frac{1}{2}})) = \bigcup_{\{u,v\}\in E(G)} \{\{u, (uv)_{\frac{1}{2}}\}, \{v, (uv)_{\frac{1}{2}}\}\}.$$

By definition of direct product, in $L(G^{\frac{1}{2}}) \times K_2$, for each vertex $\{u, (uv)_{\frac{1}{2}}\} \in V(L(G^{\frac{1}{2}}))$, there are two vertices in the copy of K_2 related to the vertex $\{u, (uv)_{\frac{1}{2}}\}$. We denote these vertices by $\{u, (uv)_{\frac{1}{2}}\}^0$ and $\{u, (uv)_{\frac{1}{2}}\}^1$. Now the following function is an isomorphism from Γ_k to $L(G^{\frac{1}{2}}) \times K_2$:

$$f((uv)_{\frac{i}{2m}}) = \begin{cases} \{u, (uv)_{\frac{1}{2}}\}^0 & \text{if } i = k, \\ \{u, (uv)_{\frac{1}{2}}\}^1 & \text{if } i = m - k. \end{cases} \qquad \square$$

In the proof of Lemma 3.2, we decomposed $E(\Gamma_k)$ into two subsets $E_{k,1}$ and $E_{k,2}$, where $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$. Suppose that $q = |E(G)|$. Similar to the proof of Lemma 3.2, one can show that the induced subgraph of Γ_k by $E_{k,1}$ is isomorphic to qK_2 (when $k = \frac{m}{2}$) or $q(K_2 \times K_2) = 2qK_2$ (when $k \neq \frac{m}{2}$) and the induced subgraph of Γ_k by $E_{k,2}$ is isomorphic to $\bigcup_{v \in V(G)} K_{d(v)}$ (when $k = \frac{m}{2}$) or $\bigcup_{v \in V(G)} (K_{d(v)} \times K_2)$ (when $k \neq \frac{m}{2}$). Therefore,

$$\bigcup_{k=1}^{\lfloor \frac{m}{2} \rfloor} \Gamma_k[E_{k,1}] = (m-1)qK_2,$$

which is denoted by $G_1^{\frac{[m]}{2m}}$ and

$$\bigcup_{k=1}^{\lfloor \frac{m}{2} \rfloor} \Gamma_k[E_{k,2}] = \begin{cases} \frac{m-1}{2}(\bigcup_{v \in V(G)} (K_{d(v)} \times K_2)) & \text{if } m \equiv 1 \ (mod\ 2), \\ (\bigcup_{v \in V(G)} K_{d(v)}) \bigcup \frac{m-2}{2}(\bigcup_{v \in V(G)} (K_{d(v)} \times K_2)) & \text{if } m \equiv 0 \ (mod\ 2), \end{cases}$$

which is denoted by $G_2^{\frac{[m]}{2m}}$. So, we have the following lemma.

Lemma 3.3. *Let $m \in \mathbb{N}$ and G be a nontrivial graph of size q. With assumptions of the previous paragraph,*

$$G^{\frac{[m]}{2m}} = G^{\frac{1}{2}} \cup G_1^{\frac{[m]}{2m}} \cup G_2^{\frac{[m]}{2m}}.$$

In [29], Yang proved that

$$\chi_i(G \times H) \leq \min\{\chi_i(G)\Delta(H), \Delta(G)\chi_i(H)\}$$

for every graphs G and H. Therefore, if $n \geq 2$ then

$$\chi_i(K_n \times K_2) = \min\{\chi_i(K_n)\Delta(K_2), \Delta(K_n)\chi_i(K_2)\} = n.$$

Lemma 3.4. *Let G be a graph with $\Delta(G) \geq 3$ and $2 \leq m \in \mathbb{N}$. Then*

$$\chi_i(G_2^{\frac{[m]}{2m}}) = \Delta(G).$$

Proof. Suppose that m is an even number. We have

$$\begin{aligned} \chi_i(G_2^{\frac{[m]}{2m}}) &= \max\{\chi_i(K_{d(v)} \times K_2), \chi_i(K_{d(v)})|\ v \in V(G)\} \\ &= \max\{d(v)|\ v \in V(G)\} \\ &= \Delta(G). \end{aligned}$$

The proof is similar when m is an odd number. □

Proof of Theorem 1.7. Let $\Delta(G) = \Delta$. The assertion for $m = 1$ and $m = 2$ follows immediately from Theorems 1.5 and 1.6. Now suppose that $m \geq 3$. We have

$$G^{\frac{m}{2m}} = G^{\frac{m-1}{2m}} \cup G^{\frac{[m]}{2m}} = (G^{\frac{m-1}{2m}} \cup G_1^{\frac{[m]}{2m}}) \cup G^{\frac{1}{2}} \cup G_2^{\frac{[m]}{2m}},$$

where $V(G^{\frac{1}{2}}) \cap V(G_2^{\frac{[m]}{2m}}) = \varnothing$. We define an incidence coloring for $G^{\frac{m}{2m}}$ in four steps:

Step 1. Proper incidence coloring of $G^* = G^{\frac{m-1}{2m}} \cup G_1^{\frac{[m]}{2m}}$ with $l = (m-1)\Delta + 2$ colors of $A = \{a_1, \ldots, a_l\}$: Applying Theorem 2.6, we have $\chi(G^{\frac{2m-1}{2m}}) = (m-1)\Delta + 2$. Let $f_1 : V(G^{\frac{2m-1}{2m}}) \to A$ be a proper coloring. Now we show that the following incidence coloring of G^* is proper.

$$g_1 : I(G^*) \to A, \quad g_1((x, y)) = f_1(y).$$

Suppose that (x_1, y_1) and (x_2, y_2) are two adjacent incidences in $\mathcal{I}(G^*)$. Therefore, $x_1 = x_2$ or $y_1 = x_2$ or $(x_2, y_2) = (y_1, x_1)$. In the first case,

$$d_{G^{\frac{1}{2m}}}(y_1, y_2) \leq d_{G^{\frac{1}{2m}}}(y_1, x_1) + d_{G^{\frac{1}{2m}}}(x_1, y_2).$$

Also, we know that at most one of the edges $\{x_1, y_1\}$ and $\{x_1, y_2\}$ is in $G_1^{\frac{[m]}{2m}}$. So

$$d_{G^{\frac{1}{2m}}}(y_1, y_2) \leq d_{G^{\frac{1}{2m}}}(y_1, x_1) + d_{G^{\frac{1}{2m}}}(x_1, y_2) \leq m + m - 1 = 2m - 1.$$

Therefore, y_1 and y_2 are adjacent in $G^{\frac{2m-1}{2m}}$ and

$$g_1((x_1, y_1)) = f_1(y_1) \neq f_1(y_2) = g_1((x_2, y_2)).$$

In the second case,

$$d_{G^{\frac{1}{2m}}}(y_1, y_2) = d_{G^{\frac{1}{2m}}}(x_2, y_2) \leq m.$$

Therefore, x_2 and y_2 are adjacent in $G^{\frac{2m-1}{2m}}$ and

$$g_1((x_1, y_1)) = f_1(y_1) = f_1(x_2) \neq f_1(y_2) = g_1((x_2, y_2)).$$

In the third case,

$$d_{G^{\frac{1}{2m}}}(y_1, y_2) = d_{G^{\frac{1}{2m}}}(y_1, x_1) \leq m.$$

Therefore, y_1 and y_2 are adjacent in $G^{\frac{2m-1}{2m}}$ and

$$g_1((x_1, y_1)) = f_1(y_1) \neq f_1(y_2) = g_1((x_2, y_2)).$$

Hence, g_1 is an incidence coloring of G^*.

Step 2. Proper incidence coloring of $G_2^{\frac{[m]}{2m}}$ with Δ colors of $B = \{b_1, \ldots, b_\Delta\}$: This follows from Lemma 3.4.

Step 3. Proper incidence coloring of $G^{\frac{1}{2}}$ (with $V(G^{\frac{1}{2}}) = V_0$ which is defined in the proof of Lemma 3.2) with $\Delta + 1$ colors of $B' = \{b_0, b_1, \dots, b_\Delta\}$: This follows from Theorem 1.5.

Step 4. Changing the color b_0 in Step 3 to the available colors from the set A: Let $B_0 \subset V_0$ is the set of incidences of $G^{\frac{1}{2}}$ with color b_0. Now if $(u, (uv)_{\frac{m}{2m}}) \in B_0$ we change its color to $f_1((uv)_{\frac{m}{2m}})$ and if $((uv)_{\frac{m}{2m}}, u) \in B_0$ we change its color to $f_1(u)$. Notice that $(uv)_{\frac{m}{2m}} = (vu)_{\frac{m}{2m}}$ in $G^{\frac{m}{2m}}$. We prove that this new coloring of $I(G^*) \cup B_0$ is also proper.

(1) Consider the incidence $(u, (uv)_{\frac{m}{2m}}) \in B_0$ with color $f_1((uv)_{\frac{m}{2m}})$. Because

$$E(G_1^{\frac{[m]}{2m}}) \cap E(G^{\frac{1}{2}}) = \varnothing,$$

we only consider the incidences of $G^{\frac{m-1}{2m}}$. The set of incidences in $G^{\frac{m-1}{2m}}$ adjacent to $(u, (uv)_{\frac{m}{2m}})$ is

$$\{(u, (uw)_{\frac{i}{2m}}), ((uw)_{\frac{i}{2m}}, u), ((uv)_{\frac{m}{2m}}, (uv)_{\frac{j}{2m}}) \mid \\ w \in N_G(u), 1 \le i \le m-1, 1 \le j \le 2m-1, j \neq m\}$$

and the set of their colors is

$$F_1 = \{f_1(u)\} \cup \{f_1((uw)_{\frac{i}{2m}}), f_1((uv)_{\frac{j}{2m}}) \mid \\ w \in N_G(u), 1 \le i \le m-1, 1 \le j \le 2m-1, j \neq m\}.$$

Since

$$\{u\} \cup \{(uw)_{\frac{i}{2m}}, (uv)_{\frac{j}{2m}} \mid w \in N_G(u), 1 \le i \le m-1, m+1 \le j \le 2m-1\} \\ \subset N_{G^{\frac{2m-1}{2m}}}((uv)_{\frac{m}{2m}}),$$

$f_1((uv)_{\frac{m}{2m}}) \notin F_1$. Thus, there is no color conflict in this case.

(2) Consider the incidence $((uv)_{\frac{m}{2m}}, u) \in B_0$ with color $f_1(u)$. Because

$$E(G_1^{\frac{[m]}{2m}}) \cap E(G^{\frac{1}{2}}) = \varnothing,$$

we only consider the incidences of $G^{\frac{m-1}{2m}}$. The set of incidences in $G^{\frac{m-1}{2m}}$ adjacent to $((uv)_{\frac{m}{2m}}, u)$ is

$$\{(u, (uw)_{\frac{i}{2m}}), ((uv)_{\frac{m}{2m}}, (uv)_{\frac{j}{2m}}), ((uv)_{\frac{j}{2m}}, (uv)_{\frac{m}{2m}}) \mid \\ w \in N_G(u), 1 \le i \le m-1, 1 \le j \le 2m-1, j \neq m\}$$

and the set of their colors is

$$F_2 = \{f_1((uv)_{\frac{m}{2m}})\} \\ \cup \{f_1((uw)_{\frac{i}{2m}}), f_1((uv)_{\frac{j}{2m}}) \mid w \in N_G(u), 1 \le i \le m-1, 1 \le j \le 2m-1, j \neq m\}.$$

Since

$$\{(uv)_{\frac{m}{2m}}\} \cup \{(uw)_{\frac{i}{2m}}, (uv)_{\frac{j}{2m}} \mid w \in N_G(u), 1 \leq i \leq m-1, m+1 \leq j \leq 2m-1\}$$
$$\subset N_{G^{\frac{2m-1}{2m}}}(u),$$

$f_1(u) \notin F_2$. So the coloring of $I(G^*) \cup B_0$ is an incidence coloring.

Finally, since

$$V(G^{\frac{1}{2}}) \cap V(G_2^{\frac{[m]}{2m}}) = \varnothing,$$

by use of the colorings defined in Steps 2, 3 and 4, we can define an incidence $(m\Delta + 2)$-coloring of $G^{\frac{m}{2m}}$. This completes the proof. □

4. PROBLEMS

In this paper, we prove that $\chi_i(G^r) = \Delta(G^r) + 1$ for any graph with maximum degree at least 3 and any fractional positive number r from the set

$$\Big\{\frac{1}{n} \,\big|\, 1 < n \in \mathbb{N}\Big\} \cup \Big\{\frac{2}{n} \,\big|\, 2 < n \in \mathbb{N}\Big\} \cup \Big\{\frac{m}{n} \,\big|\, 0 < \frac{m}{n} < \frac{1}{2}\Big\}.$$

Also we show that $\Delta(G^{\frac{m}{2m}}) + 2$ is an upper bound for the incidence chromatic number of $G^{\frac{m}{2m}}$ when $\Delta(G) \geq 3$. Besides the presented results there are several open problems. Below we list some of them.

Problem 4.1. Let G be a graph with maximum degree $\Delta \geq 3$, $\frac{1}{2} < \frac{m}{n} < 1$ and $m \geq 3$. What is the upper bound for $\chi_i(G^{\frac{m}{n}})$? Is the incidence coloring conjecture (ICC) true for these graphs?

Let G be a graph with maximum degree $\Delta \geq 3$. By Theorem 1.4, we have $\chi_i(G^{\frac{3}{n}}) = 3\Delta(G) + 1$ when $n \geq 7$ and $3\Delta(G) + 1 \leq \chi_i(G^{\frac{3}{6}}) \leq 3\Delta(G) + 2$. So the following problem arise naturally:

Problem 4.2. Let G be a graph with maximum degree $\Delta \geq 3$. What are the exact values or upper bounds for $\chi_i(G^{\frac{3}{4}})$, $\chi_i(G^{\frac{3}{5}})$ and $\chi_i(G^{\frac{3}{6}})$? Is the incidence coloring conjecture (ICC) true for these graphs?

We know that $\frac{2}{2}$-power of graph G is isomorphic to $T(G)$, the total graph of G. In addition, G is an induced subgraph of $G^{\frac{2}{2}}$.

Problem 4.3. Find an upper bound for $\chi_i(G^{\frac{2}{2}})$ in terms of $\Delta(G)$ and $\chi_i(G)$.

We prove that the conjecture ICC is true for $\frac{m}{2m}$-power of graphs with maximum degree at least 3. According to the results, we conjecture the following:

Conjecture 4.4. For any graph G with $\Delta(G) \geq 3$, $\chi_i(G^{\frac{m}{2m}}) = m\Delta(G) + 1$.

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REFERENCES

[1] M. Behzad, *Graphs and Their Chromatic Numbers*, Ph.D. Thesis, Michigan State University, 1965.

[2] M. Bonamy, H. Hocquard, S. Kerdjoudj, A. Raspaud, *Incidence coloring of graphs with high maximum average degree*, Discrete Appl. Math. **227** (2017), 29–43.

[3] J.A. Bondy, U.S.R. Murty, *Graph Theory*, Springer, New York, 2008.

[4] R.A. Brualdi, J.Q. Massey, *Incidence and strong edge colorings of graphs*, Discrete Math. **122** (1993), 51–58.

[5] D. Chen, X. Liu, S. Wang, *The incidence coloring number of graph and the incidence coloring conjecture*, Math. Econom. (People's Republic of China) **15** (1998), 47–51.

[6] P. Gregor, B. Lužar, R. Soták, *On incidence coloring conjecture in Cartesian products of graphs*, Discrete Appl. Math. **213** (2016), 93–100.

[7] P. Gregor, B. Lužar, R. Soták, *Note on incidence chromatic number of subquartic graphs*, J. Comb. Optim. **34** (2017), 174–181.

[8] B. Guiduli, *On incidence coloring and star arboricity of graphs*, Discrete Math. **163** (1997), 275–278.

[9] R. Hammack, W. Imrich, S. Klavzar, *Handbook of Product Graphs*, 2nd ed., Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2011.

[10] S. Hartke, H. Liu, S. Petrickova, *On coloring of fractional powers of graphs*, arxiv:1212.3898v1 (2013).

[11] M. Hosseini Dolama, E. Sopena, X. Zhu, *Incidence coloring of k-degenerated graphs*, Discrete Math. **283** (2004), 121–128.

[12] M.N. Iradmusa, *On colorings of graph fractional powers*, Discrete Math. **310** (2010), 1551–1556.

[13] M.N. Iradmusa, *A short proof of 7-colorability of $\frac{3}{3}$-power of sub-cubic graphs*, Iranian J. Sci. Tech., Transactions A: Sci. **44** (2020), no. 1, 225–226.

[14] H.J. Lai, B. Montgomery, H. Poon, *Upper bounds of dynamic chromatic number*, Ars Combin. **68** (2003), 193–201.

[15] D. Li, M. Liu, *Incidence coloring of the squares of some graphs*, Discrete Math. **308** (2008), 6569–6574.

[16] M. Maydanskiy, *The incidence coloring conjecture for graphs of maximum degree 3*, Discrete Math. **292** (2005), 131–141.

[17] B. Montgomery, *Dynamic Coloring*, Ph.D. Dissertation, West Virginia University, 2001.

[18] M. Mozafari-Nia, M.N. Iradmusa, *A note on coloring of $\frac{3}{3}$-power of subquartic graphs*, Australas. J. Combin. **79** (2021), 454–460.

[19] K. Nakprasit, K. Nakprasit, *Incidence colorings of the powers of cycles*, Int. J. Pure Appl. Math. **71** (2012), 143–148.

[20] K.J. Pai, J.M. Chang, J.S. Yang, R.U. Wu, *Incidence coloring on hypercubes*, Theor. Comput. Sci. **557** (2014), 59–65.

[21] K.J. Pai, J.M. Chang, J.S. Yang, R.U. Wu, *On the incidence coloring number of folded hypercubes*, 2014 International Computer Science and Engineering Conference (ICSEC), 7–11, 2014.

[22] W.C. Shiu, P.C.B. Lam, D.L. Chen, *On incidence coloring for some cubic graphs*, Discrete Math. **252** (2002), 259–266.

[23] W.C. Shiu, P.K. Sun, *Graphs which are not* $(\Delta+1)$*-incidence colorable with erratum to the incidence chromatic number of outerplanar graphs*, Technical Report of Department of Mathematics, Hong Kong Baptist University, **419** (2006), 397–405.

[24] P.K. Sun, W.C. Shiu, *Some results on incidence coloring, star arboricity and domination number*, Australas. J. Combin. **54** (2012), 107–114.

[25] F. Wang, X. Liu, *Coloring 3-power of 3-subdivision of subcubic graph*, Discrete Math. Algorithms and Appl. **10** (2018), 1850041.

[26] S.D. Wang, D.L. Chen, S.C. Pang, *The incidence coloring number of Halin graphs and outerplanar graphs*, Discrete Math. **256** (2002), 397–405.

[27] S. Wang, J. Xu, F. Ma, C. Xu, *The* $(\Delta+2,2)$*-incidence coloring of outerplanar graphs*, Progress in Natural Sci. **18** (2008), 575–578.

[28] J. Wu, *Some results on the incidence coloring number of a graph*, Discrete Math. **309** (2009), 3866–3870.

[29] D. Yang, *Fractional incidence coloring and star arboricity of graphs*, Ars Combin. **105** (2012), 213–224.

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