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ON THE CONTINGENT OF THE GRAPH OF THE SUM OF TWO MAPPINGS

MAŁGORZATA TUROWSKA

Abstract

It is shown that the graph of the sum of two Lipschitz mappings of the real line into a normed space of infinite dimension, whose graphs have tangents, need not have a tangent. Moreover, it turns out that the contingent of the graph of their linear combination may depend on the coefficients of that combination in quite "nonlinear" way.

Definition 1. [5]. Let $\emptyset \neq M \subset Z$, where Z is a real normed space, and $z \in \mathrm{cl}M$. The set

$$\Big\{v \in Z\colon \exists (z_n)_{n\in\mathbb{N}}, z_n \in M, \lim_{n\to\infty} z_n = z, \exists (\lambda_n)_{n\in\mathbb{N}}, \lambda_n > 0\colon \lim_{n\to\infty} \lambda_n(z_n - z) = v\Big\}$$

is called the tangent cone to M at z and is denoted by $\mathrm{Tan}_M(z)$. The elements of $\mathrm{Tan}_M(z)$ are called vectors tangent to M at z. The set $\mathrm{Tan}_M(z)$ is also called the contingent of M at z ([1], [4]).

We will use a more short term "contingent".

It is well known that $\mathrm{Tan}_M(z)$ is a nonempty closed subset of Z and $0_Z \in \mathrm{Tan}_M(z)$, where 0_Z denotes the zero vector of Z.

By $G(f)$ we denote the graph of a mapping f. We will also write $\mathrm{T}_f(x_0)$ instead of $\mathrm{Tan}_{G(f)}\big((x_0, f(x_0))\big)$.

Theorem 1. [5]. *Let X and Y be real normed spaces. If $f\colon X \to Y$ is differentiable (in the Fréchet sense) at $x_0 \in X$, then $\mathrm{T}_f(x_0)$ is a linear subspace of $X \times Y$ and $\mathrm{T}_f(x_0) = G\big(f'(x_0)\big)$.*

Let us recall a condition (in a sense, converse of Theorem 1) implying the differentiability of a mapping at a point.

Theorem 2. [2], [6]. *Let X and Y be real normed spaces of finite dimension and $U \subset X$ be an open set. Assume that $f\colon U \to Y$ is continuous at $x_0 \in U$ and $\mathrm{T}_f(x_0)$ is a linear subspace of $X \times Y$ which does not contain vertical*

Małgorzata Turowska — Pomeranian University.

vectors (i.e. vectors of the form $(0_X, y) \in X \times Y$, $y \neq 0_Y$). *Then* f *is differentiable at* x_0.

The combination of Theorem 2 and Theorem 1 may be regarded as a geometric criterion for the differentiability of a mapping at a point in the case of finite-dimensional spaces.

Theorem 3. [3]. *Let* X, Y *be real normed spaces, finite-dimensional or not, and* $U \subset X$ *an open set. Let* $g\colon U \to Y$ *be continuous at a point* $x_0 \in U$. *Assume that* $F\colon U \to Y$ *is differentiable at* x_0. *Then*

(*a*) $\mathrm{T}_{F+g}(x_0) = \{(u, F'(x_0)u + v)\colon (u,v) \in \mathrm{T}_g(x_0)\}$;

(*b*) $\mathrm{T}_g(x_0)$ *is homeomorphic to* $\mathrm{T}_{F+g}(x_0)$, *where the natural homeomorphism* $H\colon \mathrm{T}_g(x_0) \to \mathrm{T}_{F+g}(x_0)$ *is given by*

$$H(u,v) = (u, F'(x_0)u + v);$$

(*c*) *if* $\mathrm{T}_g(x_0)$ *is a linear subspace of* $X \times Y$, *then* H *is a linear isomorphism.*

Theorem 4. *Let* Y *be a normed space,* $\dim Y \leq \infty$, *and let* $f\colon \mathbb{R} \to Y$ *be continuous at* t_0. *Then*

$$\mathrm{T}_{\alpha f}(t_0) = \{(u, \alpha v)\colon (u,v) \in \mathrm{T}_f(t_0)\}.$$

for each $\alpha \in \mathbb{R}$.

The proof is immediate making use of Definition 1.

Corollary 5. *Let* Y *be a normed space,* $\dim Y \leq \infty$. *Let* $f\colon \mathbb{R} \to Y$ *be continuous at* 0, $f(0) = 0_Y$, *and such that*

$$\mathrm{T}_f(0) = \{\xi(1,y)\colon \xi \in \mathbb{R}\}$$

with some $y \in Y$. *Then*

$$\mathrm{T}_{\alpha f}(0) = \{\xi(1,\alpha y)\colon \xi \in \mathbb{R}\}$$

for each $\alpha \in \mathbb{R}$.

Now, let two mappings $f\colon \mathbb{R} \to Y$, $g\colon \mathbb{R} \to Y$ be given. Assume that the contingents of their graphs are known. What can be said about the contingent of the graph of the sum $f+g$? It turns out that the answer depends on whether Y is finite-dimensional or not.

To compare the results, we assume first that $\dim Y < \infty$. Let $t_0 \in \mathbb{R}$ and $f\colon \mathbb{R} \to Y$, $g\colon \mathbb{R} \to Y$ be given, and assume that the contingents $\mathrm{T}_f(t_0)$, $\mathrm{T}_g(t_0)$ are one-dimensional linear non-vertical subspaces of $\mathbb{R} \times Y$. By Theorem 2, f and g are differentiable at t_0. Then, it is easily checked that

$$\mathrm{T}_f(t_0) = \{\xi(1, f'(t_0))\colon \xi \in \mathbb{R}\};$$

$$\mathrm{T}_g(t_0) = \big\{\xi\big(1, g'(t_0)\big) : \xi \in \mathbb{R}\big\};$$
$$\mathrm{T}_{\alpha f}(t_0) = \big\{\xi\big(1, \alpha f'(t_0)\big) : \xi \in \mathbb{R}\big\} \ \text{ for each } \ \alpha \in \mathbb{R};$$
$$\mathrm{T}_{\alpha f+\beta g}(t_0) = \big\{\xi\big(1, \big(\alpha f'(t_0) + \beta g'(t_0)\big)\big) : \xi \in \mathbb{R}\big\} \ \text{ for each } \ \alpha \in \mathbb{R},\ \beta \in \mathbb{R}.$$

Thus if $\dim Y < \infty$, then in the differentiable case (what is equivalent to the properties of contingents described in Theorems 1, 2) the contingent of the graph of the linear combination of mappings is equal to the linear combination of the contingents with respect to their second components.

If mappings $f\colon \mathbb{R} \to Y$ and $g\colon \mathbb{R} \to Y$ are differentiable then the contingent of the graph of the linear combination of this mappings is equal to the linear combination of the contingents with respect to their second components, it follows from Theorem 3.

This is generally no longer true for infinite-dimensional Y even if the contingents of f and g are linear subspaces. The following example shows that the contingent of the graph of the sum of mappings may be trivial even if the contingent of the graph of each of mappings is a nontrivial vector space.

Now we are coming up to the main result of the paper.

Example 6. Let $Y = l^2$ be the classical Hilbert space, $\{e_n : n \in \mathbb{N}\}$ the standard orthonormal base of Y. Choose any two elements y_1, y_2 of Y and a number $c > 1$. Consider the mappings $f\colon [-1,1] \to Y$, $g\colon [-1,1] \to Y$ defined for $t \in [0,1]$ as follows

$$f(t) = \begin{cases} 0_Y & \text{if } t = 0,\\ \dfrac{ct - c^{1-2n}}{c-1}\cdot e_n + \dfrac{c^{1-2n} - t}{c-1}\cdot y_1 & \text{if } t \in \big(c^{-2n}, c^{1-2n}\big],\ n \in \mathbb{N},\\ \dfrac{c^{2-2n} - t}{c-1}\cdot e_n + \dfrac{ct - c^{2-2n}}{c-1}\cdot y_1 & \text{if } t \in \big(c^{1-2n}, c^{2-2n}\big],\ n \in \mathbb{N}, \end{cases}$$

$$g(t) = \begin{cases} 0_Y & \text{if } t = 0,\\ \dfrac{c^{1-2n} - t}{c-1}\cdot e_{n+1} + \dfrac{ct - c^{1-2n}}{c-1}\cdot y_2 & \text{if } t \in \big(c^{-2n}, c^{1-2n}\big],\ n \in \mathbb{N},\\ \dfrac{ct - c^{2-2n}}{c-1}\cdot e_n + \dfrac{c^{2-2n} - t}{c-1}\cdot y_2 & \text{if } t \in \big(c^{1-2n}, c^{2-2n}\big],\ n \in \mathbb{N}, \end{cases}$$

and $f(t) = -f(-t)$, $g(t) = -g(-t)$ for $t \in [-1, 0)$. Note that f, g are continuous on $[-1,1]$.
We will show that

1) f and g are Lipschitz;
2) f and g are not differentiable at 0;
3) $\mathrm{T}_f(0) = \{\xi(1, y_1) : \xi \in \mathbb{R}\}$;
4) $\mathrm{T}_g(0) = \{\xi(1, y_2) : \xi \in \mathbb{R}\}$;

5) given any two real numbers α, β, we have

$$\mathrm{T}_{\alpha f+\beta g}(0)=\begin{cases}\{(0,0_Y)\} & \text{if } \dfrac{\alpha}{\beta}>0;\\ \left\{\xi\left(1,\dfrac{\alpha^2}{\alpha-\beta}y_1-\dfrac{\beta^2}{\alpha-\beta}y_2\right):\xi\in\mathbb{R}\right\} & \text{if } \dfrac{\alpha}{\beta}<0;\\ \{\xi(1,\alpha y_1+\beta y_2)\colon \xi\in\mathbb{R}\} & \text{if } \alpha=0 \text{ or } \beta=0.\end{cases}$$

Proof.

1) Since $f|_{(0,1]}$ is piecewise differentiable, we get from the definition of f that

$$f'(t)=\begin{cases}\dfrac{ce_n-y_1}{c-1} & \text{if } t\in\left(c^{-2n},c^{1-2n}\right],\ n\in\mathbb{N},\\ \dfrac{cy_1-e_n}{c-1} & \text{if } t\in\left(c^{1-2n},c^{2-2n}\right],\ n\in\mathbb{N}.\end{cases}$$

This yields by the mean value theorem, that f is Lipschitz on each interval $\left(c^{-2n},c^{2-2n}\right]$, $n\in\mathbb{N}$, with the constant

$$L_f=\max\left\{\frac{c+\|y_1\|}{c-1},\frac{c\|y_1\|+1}{c-1}\right\}.$$

It follows that f is Lipschitz with the constant L_f on each $\left[c^{-2n},1\right]$, $n\in\mathbb{N}$, hence by continuity, f is Lipschitz on $[0,1]$. Finally f is Lipschitz on $[-1,1]$ with the same constant L_f, what was to be shown. The same argument applies to g, and we obtain that g is Lipschitz with constant $L_g=\max\left\{\dfrac{c+\|y_2\|}{c-1},\dfrac{c\|y_2\|+1}{c-1}\right\}$.

2) Observe that f is not differentiable at $t=0$ because the limit of the ratio

$$\frac{f(c^{1-2n})}{c^{1-2n}}=e_n$$

does not exist for $n\to\infty$. The same holds for g.

3) Next we show that

$$\mathrm{T}_f(0)=\{\xi(1,y_1)\colon \xi\in\mathbb{R}\}. \tag{1}$$

Let $v=(\xi,\mu)\in\mathrm{T}_f(0)$, $v\neq 0$. There exist a sequence $(t_n)_{n\in\mathbb{N}}$, $t_n\in[-1,1]$, $t_n\to 0$, and a sequence $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n>0$, such that

$$\lambda_n t_n\to\xi \quad\text{and}\quad \lambda_n f(t_n)\to\mu \quad\text{as}\quad n\to\infty. \tag{2}$$

First assume that $t_n>0$, $n\in\mathbb{N}$. Generally, the sequence $(t_n)_{n\in\mathbb{N}}$ is composed of two subsequences: the terms of the first one belong to the intervals of the form $\left(c^{-2m},c^{1-2m}\right]$ while the terms of the second subsequence are in the intervals of the form $\left(c^{1-2m},c^{2-2m}\right]$. Since the arguments in both cases are alike, we will restrict ourselves to the first case, i.e. suppose that all t_n hit the intervals of the form $\left(c^{-2m}, c^{1-2m}\right]$. It is easy to see that there exists a sequence $(k_n)_{n\in\mathbb{N}}$, $k_n \in \mathbb{N}$, $k_n \to \infty$, such that $c^{-2k_n} < t_n \le c^{1-2k_n}$ for $n \in \mathbb{N}$. Now (2) can be written as follows

$$\lambda_n t_n \underset{n\to\infty}{\longrightarrow} \xi \tag{3}$$

and

$$\lambda_n f(t_n) = \lambda_n \frac{ct_n - c^{1-2k_n}}{c-1} \cdot e_{k_n} + \lambda_n \frac{c^{1-2k_n} - t_n}{c-1} \cdot y_1 \underset{n\to\infty}{\longrightarrow} \mu. \tag{4}$$

Multiplying (4) by e_{k_n} (scalar multiplication), we obtain

$$\lim_{n\to\infty} \lambda_n \frac{ct_n - c^{1-2k_n}}{c-1} = 0, \tag{5}$$

and from (5) and (3) we get

$$\lim_{n\to\infty} \frac{\lambda_n c^{1-2k_n}}{c-1} = \frac{c\xi}{c-1}. \tag{6}$$

Consequently, (6) and (3) imply

$$\lim_{n\to\infty} \lambda_n \frac{c^{1-2k_n} - t_n}{c-1} = \lim_{n\to\infty} \frac{\lambda_n c^{1-2k_n}}{c-1} - \lim_{n\to\infty} \frac{\lambda_n t_n}{c-1} = \frac{c\xi}{c-1} - \frac{\xi}{c-1} = \xi.$$

Therefore from (4) we get $\mu = \xi y_1$, hence $v = \xi(1, y_1)$.

The case $t_n \in \left(c^{1-2m}, c^{2-2m}\right]$, as was mentioned above, is treated analogously.

Now let us suppose that (2) holds for $t_n < 0$, $n \in \mathbb{N}$. Since f is odd, we have $\lambda_n f(-t_n) \to -\mu$, whence by the previous argument, we have $-\mu = (-\xi)y_1$ which again yields $v = \xi(1, y_1)$.

Conversely, take any nonzero vector $v = \xi(1, y_1)$ and consider two sequences $(t_n)_{n\in\mathbb{N}}$, $(\lambda_n)_{n\in\mathbb{N}}$ defined by

$$t_n = c^{-2n} \operatorname{sgn} \xi,\ \lambda_n = c^{2n}|\xi|, \quad n \in \mathbb{N}.$$

Since f is odd and $f\left(c^{-2n}\right) = c^{-2n} y_1$, we have for each $n \in \mathbb{N}$

$$\lambda_n \big(t_n, f(t_n)\big) = c^{2n}|\xi| \big(c^{-2n} \operatorname{sgn} \xi, f\left(c^{-2n} \operatorname{sgn} \xi\right)\big) =$$

$$= \big(\xi, c^{2n}|\xi| \operatorname{sgn} \xi f\left(c^{-2n}\right)\big) = \xi(1, y_1) = v.$$

Thus (1) is proved.

4) The mapping g is similar to f. Applying the same arguments as for f, we obtain

$$\mathrm{T}_g(0) = \{\xi(1, y_2) \colon \xi \in \mathbb{R}\}.$$

5) First, let us prove that

$$\mathrm{T}_{f+g}(0) = \{(0, 0_Y)\}. \tag{7}$$

We will proceed by contradiction. Assume that (ξ, v) is a nonzero element of $\mathrm{T}_{f+g}(0)$ and $\xi > 0$. By Definition 1, there exists a sequence $(t_k)_{k\in\mathbb{N}}$, $t_k \in (0,1]$, $t_k \to 0$, and there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$, $\lambda_k > 0$, such that

$$\lambda_k t_k \to \xi \quad \text{and} \quad \lambda_k(f+g)(t_k) \to v \quad \text{as} \quad k \to \infty.$$

There are two cases to consider.

(i) First suppose that there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $t_k \in \left(c^{-2n_k}, c^{1-2n_k}\right]$ for each $k \in \mathbb{N}$. We then get

$$\begin{aligned}\lambda_k(f+g)(t_k) =&\lambda_k \cdot \frac{ct_k - c^{1-2n_k}}{c-1} \cdot e_{n_k} + \lambda_k \cdot \frac{c^{1-2n_k} - t_k}{c-1} \cdot y_1 + \\ &+ \lambda_k \cdot \frac{c^{1-2n_k} - t_k}{c-1} \cdot e_{n_k+1} + \lambda_k \cdot \frac{ct_k - c^{1-2n_k}}{c-1} \cdot y_2 \underset{k\to\infty}{\longrightarrow} v.\end{aligned} \tag{8}$$

Scalar multiplication of (8) by e_{n_k} yields

$$\lambda_k \left(ct_k - c^{1-2n_k}\right) \underset{k\to\infty}{\longrightarrow} 0.$$

It follows that

$$\lambda_k c^{1-2n_k} \underset{k\to\infty}{\longrightarrow} c\xi.$$

Multiplying (8) by e_{n_k+1}, we obtain $\lambda_k \left(c^{1-2n_k} - t_k\right) \underset{k\to\infty}{\longrightarrow} 0$. It follows that

$$\lambda_k c^{1-2n_k} \underset{k\to\infty}{\longrightarrow} \xi,$$

whence $c\xi = \xi$; a contradiction because of $\xi \neq 0$ and $c > 1$.

(ii) Assume now that there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $t_k \in \left(c^{1-2n_k}, c^{2-2n_k}\right]$ for each $k \in \mathbb{N}$. We then get

$$\lambda_k(f+g)(t_k) = \lambda_k t_k e_{n_k} + \lambda_k \frac{ct_k - c^{2-2n_k}}{c-1} \cdot y_1 + \lambda_k \frac{c^{2-2n_k} - t_k}{c-1} \cdot y_2 \underset{k\to\infty}{\longrightarrow} v \tag{9}$$

Multiplying (9) by e_{n_k} we obtain $\lambda_k t_k \to 0$, whence $\xi = 0$; a contradiction.

We conclude that

$$\mathrm{T}_{f+g}(0) = \{(0, 0_Y)\}.$$

Thus if $\dim Y = \infty$, the contingent of the graph of the sum of mappings with nontrivial linear contingents may be trivial.

In the case when α or β equals zero, we have the same situation as in Corollary 5, so we omit the proof.

Now assume that $\frac{\alpha}{\beta} > 0$. If $\alpha = \beta$ then from Corollary 5 and (7) we obtain

$$\mathrm{T}_{\alpha(f+g)}(0) = \{(0, 0_Y)\}.$$

Consider the case $\alpha \neq \beta$.

Suppose that (ξ, w) is a nonzero element of the set $\mathrm{T}_{\alpha f+\beta g}(0)$. Without loss of generality we may assume that $\xi > 0$. Then by Definition 1, there exists a sequence $(t_k)_{k\in\mathbb{N}}$, $t_k \in (0,1]$, $t_k \to 0$, and there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$, $\lambda_k > 0$, such that

$$\lambda_k t_k \to \xi \quad \text{and} \quad \lambda_k(\alpha f + \beta g)(t_k) \to w \quad \text{as} \quad k \to \infty.$$

The following two cases are possible:

(j) There exists a sequence $(n_k)_{k\in\mathbb{N}}$, $n_k \in \mathbb{N}$, such that

$$t_k \in \left[c^{-2n_k}, c^{1-2n_k}\right).$$

Then

$$\lambda_k \alpha f(t_k) + \lambda_k \beta g(t_k) =$$

$$= \lambda_k \alpha \frac{ct_k - c^{1-2n_k}}{c-1} \cdot e_{n_k} + \lambda_k \alpha \frac{c^{1-2n_k} - t_k}{c-1} \cdot y_1 + \tag{10}$$

$$+ \lambda_k \beta \frac{c^{1-2n_k} - t_k}{c-1} \cdot e_{n_k+1} + \lambda_k \beta \frac{ct_k - c^{1-2n_k}}{c-1} \cdot y_2 \underset{k\to\infty}{\longrightarrow} w.$$

Since $0 < \lambda_k c^{-2n_k} \leq \lambda_k t_k \leq \lambda_k c^{1-2n_k} = c\lambda_k c^{-2n_k}$ for $k \in \mathbb{N}$ and $\lambda_k t_k \to \xi$, we obtain that the sequence $\left(\lambda_k c^{1-2n_k}\right)_{k\in\mathbb{N}}$ is bounded by $c\xi$. Multiplying (10) by e_{n_k}, we obtain

$$\lambda_k \alpha \tfrac{ct_k - c^{1-2n_k}}{c-1} + \lambda_k \alpha \tfrac{c^{1-2n_k} - t_k}{c-1} \cdot \left\langle y_1, e_{n_k} \right\rangle + \lambda_k \beta \tfrac{ct_k - c^{1-2n_k}}{c-1} \cdot \left\langle y_2, e_{n_k} \right\rangle \underset{k\to\infty}{\longrightarrow} 0$$

(where $\langle \cdot, \cdot \rangle$ stands for the scalar product). This implies

$$\lambda_k \alpha \left(ct_k - c^{1-2n_k}\right) \underset{k\to\infty}{\longrightarrow} 0.$$

It follows

$$\alpha c \lambda_k \left(t_k - c^{-2n_k}\right) \underset{k\to\infty}{\longrightarrow} 0,$$

and we obtain

$$\lambda_k c^{-2n_k} \underset{k\to\infty}{\longrightarrow} \xi.$$

Now, multiplying (10) by e_{n_k+1}, we get

$$\lambda_k \alpha \tfrac{c^{1-2n_k} - t_k}{c-1} \cdot \left\langle y_1, e_{n_k+1} \right\rangle + \lambda_k \beta \tfrac{c^{1-2n_k} - t_k}{c-1} + \lambda_k \beta \tfrac{ct_k - c^{1-2n_k}}{c-1} \cdot \left\langle y_2, e_{n_k+1} \right\rangle \underset{k\to\infty}{\longrightarrow} 0,$$

whence

$$\beta\left(\lambda_k c^{1-2n_k}-\lambda_k t_k\right)\underset{k\to\infty}{\longrightarrow}0.$$

Since $\lambda_k t_k \to \xi$ and $\lambda_k c^{-2n_k}\to\xi$, we have $\beta\xi=0$; a contradiction because $\xi>0$ and $\beta\neq 0$. In this case we have $\xi=0$ and $w=0_Y$.

Observe that this conclusion is independent of the choice of nonzero numbers α, β.

(jj) For each $k\in\mathbb{N}$ there exists $n_k\in\mathbb{N}$, $n_k\to\infty$, such that $t_k\in\left[c^{1-2n_k},c^{2-2n_k}\right)$. Then

$$\begin{aligned}
&\lambda_k(\alpha f+\beta g)(t_k)=\\
&=\alpha\lambda_k\frac{c^{2-2n_k}-t_k}{c-1}\cdot e_{n_k}+\alpha\lambda_k\frac{ct_k-c^{2-2n_k}}{c-1}\cdot y_1+\\
&+\beta\lambda_k\frac{ct_k-c^{2-2n_k}}{c-1}\cdot e_{n_k}+\beta\lambda_k\frac{c^{2-2n_k}-t_k}{c-1}\cdot y_2\underset{k\to\infty}{\longrightarrow}w.
\end{aligned}\tag{11}$$

Since $0<\lambda_k c^{1-2n_k}\le\lambda_k t_k\le\lambda_k c^{2-2n_k}=c\lambda_k c^{1-2n_k}$ for each $k\in\mathbb{N}$ and $\lambda_k t_k\to\xi$, the sequence $\left(\lambda_k c^{2-2n_k}\right)_{k\in\mathbb{N}}$ is bounded by $c\xi$. Multiplying (11) by e_{n_k}, we obtain

$$\begin{aligned}
&\alpha\lambda_k\frac{c^{2-2n_k}-t_k}{c-1}+\alpha\lambda_k\frac{ct_k-c^{2-2n_k}}{c-1}\cdot\left\langle y_1,e_{n_k}\right\rangle+\\
&+\beta\lambda_k\frac{ct_k-c^{2-2n_k}}{c-1}+\beta\lambda_k\frac{c^{2-2n_k}-t_k}{c-1}\cdot\left\langle y_2,e_{n_k}\right\rangle\underset{k\to\infty}{\longrightarrow}0.
\end{aligned}$$

Therefore

$$\alpha\lambda_k c^{2-2n_k}-\alpha\lambda_k t_k+c\beta\lambda_k t_k-\beta\lambda_k c^{2-2n_k}\underset{k\to\infty}{\longrightarrow}0.$$

Since $\lambda_k t_k\to\xi$ as $k\to\infty$, we get

$$\lambda_k c^{2-2n_k}\underset{k\to\infty}{\longrightarrow}\frac{(\alpha-c\beta)\xi}{\alpha-\beta}.\tag{12}$$

Since the sequence $\left(\lambda_k c^{2-2n_k}\right)_{k\in\mathbb{N}}$ is bounded by ξ and $c\xi$, we obtain, in view of (12),

$$1\le\frac{\alpha-c\beta}{\alpha-\beta}\le c.$$

The following two cases are possible:

(•) $\alpha>0$ and $\beta>0$.

If $\alpha>\beta$ then we have $\alpha-\beta\le\alpha-c\beta\le c\alpha-c\beta$. It follows that $c=1$; a contradiction because of $c>1$.

If $\alpha<\beta$ then $\alpha-\beta\ge\alpha-c\beta\ge c\alpha-c\beta$. We get $c=1$; a contradiction.

(••) $\beta < 0$ and $\alpha < 0$.
Assume $\alpha > \beta$. Then $\alpha - \beta \leq \alpha - c\beta \leq c\alpha - c\beta$. Thus $c = 1$; a contradiction.
If $\alpha < \beta$ then $\alpha - \beta \geq \alpha - c\beta \geq c\alpha - c\beta$. It follows that $c = 1$; a contradiction.

In this situation (i.e. whenever α and β are of the same sign) we get $\xi = 0$ and $w = 0_Y$ (because $\alpha f + \beta g$ is Lipschitz).

We have thus shown

$$\mathrm{T}_{\alpha f+\beta g}(0) = \{(0, 0_Y)\} \quad \text{if} \quad \frac{\alpha}{\beta} > 0.$$

Now, assume that $\dfrac{\alpha}{\beta} < 0$.

Let $(\xi, w) \in \mathrm{T}_{\alpha f+\beta g}(0)$. Without loss of generality we may assume $\xi \geq 0$. By Definition 1, there exists a sequence $(t_k)_{k\in\mathbb{N}}$, $t_k \in (0, 1]$, $t_k \to 0$, and there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$, $\lambda_k > 0$, such that

$$\lambda_k t_k \to \xi \quad \text{and} \quad \lambda_k(\alpha f + \beta g)(t_k) \to w \quad \text{as} \quad k \to \infty. \tag{13}$$

The following two cases are possible:

($\star$) There exists a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, such that

$$t_k \in \left[c^{-2n_k}, c^{1-2n_k}\right).$$

This case is similar to the case (j), so we omit the details. We get $\xi = 0$ and $w = 0_Y$.

($\star\star$) For each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$t_k \in \left[c^{1-2n_k}, c^{2-2n_k}\right).$$

Then

$$\begin{aligned} &\lambda_k(\alpha f + \beta g)(t_k) = \\ &= \alpha\lambda_k \frac{c^{2-2n_k} - t_k}{c-1} \cdot e_{n_k} + \alpha\lambda_k \frac{ct_k - c^{2-2n_k}}{c-1} \cdot y_1 + \\ &+ \beta\lambda_k \frac{ct_k - c^{2-2n_k}}{c-1} \cdot e_{n_k} + \beta\lambda_k \frac{c^{2-2n_k} - t_k}{c-1} \cdot y_2 \underset{k\to\infty}{\longrightarrow} w. \end{aligned} \tag{14}$$

Since $0 < \lambda_k c^{1-2n_k} \leq \lambda_k t_k \leq \lambda_k c^{2-2n_k} = c\lambda_k c^{1-2n_k}$ for each $k \in \mathbb{N}$, and $\lambda_k t_k \to \xi$, the sequence $(\lambda_k c^{2-2n_k})_{k\in\mathbb{N}}$ is bounded by $c\xi$.

Scalar multiplication of (14) by e_{n_k} yields

$$\begin{aligned} &\alpha\lambda_k \frac{c^{2-2n_k} - t_k}{c-1} + \alpha\lambda_k \frac{ct_k - c^{2-2n_k}}{c-1} \cdot \langle y_1, e_{n_k}\rangle + \\ &+ \beta\lambda_k \frac{ct_k - c^{2-2n_k}}{c-1} + \beta\lambda_k \frac{c^{2-2n_k} - t_k}{c-1} \cdot \langle y_2, e_{n_k}\rangle \underset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

Thus

$$\alpha\lambda_k c^{2-2n_k} - \alpha\lambda_k t_k + c\beta\lambda_k t_k - \beta\lambda_k c^{2-2n_k} \underset{k\to\infty}{\longrightarrow} 0.$$

Since $\lambda_k t_k \to \xi$ as $k \to \infty$, we get

$$\lambda_k c^{2-2n_k} \to \frac{(\alpha - c\beta)\xi}{\alpha - \beta} \quad \text{as} \quad k \to \infty. \tag{15}$$

Since the sequence $\left(\lambda_k c^{2-2n_k}\right)_{k\in\mathbb{N}}$ is bounded by ξ and $c\xi$, we obtain in view of (15), that

$$1 \le \frac{\alpha - c\beta}{\alpha - \beta} \le c.$$

We have

$$\begin{aligned}
&\lambda_k(\alpha f + \beta g)(t_k) = \\
&= \frac{e_{n_k}}{c-1}\left(\alpha\lambda_k c^{2-2n_k} - \alpha\lambda_k t_k + c\beta\lambda_k t_k - \beta\lambda_k c^{2-2n_k}\right) + \\
&+ \frac{\alpha y_1}{c-1}\left(c\lambda_k t_k - \lambda_k c^{2-2n_k}\right) + \frac{\beta y_2}{c-1}\left(\lambda_k c^{2-2n_k} - \lambda_k t_k\right) = \\
&= \frac{e_{n_k}}{c-1}\left((\alpha-\beta)\,\lambda_k c^{2-2n_k} + (c\beta - \alpha)\,\lambda_k t_k\right) + \\
&+ \frac{\alpha y_1}{c-1}\left(c\lambda_k t_k - \lambda_k c^{2-2n_k}\right) + \frac{\beta y_2}{c-1}\left(\lambda_k c^{2-2n_k} - \lambda_k t_k\right).
\end{aligned}$$

Passing to the limit as $k \to \infty$ we get from (15) and (13)

$$\lambda_k(\alpha f + \beta g)(t_k) \underset{k\to\infty}{\longrightarrow} \xi\left(\frac{\alpha^2}{\alpha-\beta}y_1 - \frac{\beta^2}{\alpha-\beta}y_2\right).$$

Thus

$$w = \xi\left(\frac{\alpha^2}{\alpha-\beta}y_1 - \frac{\beta^2}{\alpha-\beta}y_2\right).$$

Suppose that $\xi < 0$. Since $\alpha f + \beta g$ is odd, we obtain, applying the previous argument, that $w = \xi\left(\frac{\alpha^2}{\alpha-\beta}y_1 - \frac{\beta^2}{\alpha-\beta}y_2\right)$. Thus we have shown that

$$\mathrm{T}_{\alpha f+\beta g}(0) \subset \left\{\xi\left(1, \frac{\alpha^2}{\alpha-\beta}y_1 - \frac{\beta^2}{\alpha-\beta}y_2\right) : \xi \in \mathbb{R}\right\}. \tag{16}$$

To show the reverse inclusion, take any vector

$$\left(\xi, \xi\left(\frac{\alpha^2}{\alpha-\beta}y_1 - \frac{\beta^2}{\alpha-\beta}y_2\right)\right)$$

with $\xi > 0$.

Put $t_n = \dfrac{c^{2-2n}(\alpha-\beta)}{\alpha - c\beta}$ and $\lambda_n = \dfrac{(\alpha - c\beta)c^{2n-2}\xi}{\alpha-\beta}$ for each $n \in \mathbb{N}$. Then

$$c^{1-2n} < \frac{c^{2-2n}(\alpha-\beta)}{\alpha - c\beta} < c^{2-2n},$$

because $\dfrac{\alpha}{\beta} < 0$. We obviously have $\lambda_n t_n = \xi$ for each $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\lambda_n(\alpha f + \beta g)(t_n) = \\
&= \frac{(\alpha - c\beta)c^{2n-2}\xi}{(\alpha-\beta)(c-1)}\left((\alpha-\beta)c^{2-2n} + (c\beta-\alpha)\frac{c^{2-2n}(\alpha-\beta)}{\alpha-c\beta}\right)e_n + \\
&+ \frac{\alpha(\alpha-c\beta)c^{2n-2}\xi}{(c-1)(\alpha-\beta)}\left(c\frac{(\alpha-\beta)c^{2-2n}}{(\alpha-c\beta)} - c^{2-2n}\right)y_1 + \\
&+ \frac{\beta(\alpha-c\beta)c^{2n-2}\xi}{(c-1)(\alpha-\beta)}\left(c^{2-2n} - \frac{(\alpha-\beta)c^{2-2n}}{(\alpha-c\beta)}\right)y_2 = \frac{\alpha^2\xi}{\alpha-\beta}\cdot y_1 - \frac{\beta^2\xi}{\alpha-\beta}\cdot y_2
\end{aligned}$$

for each $n \in \mathbb{N}$. If $\xi < 0$, then we take $t'_n = -t_n$ and $\lambda'_n = -\lambda_n$ because $\alpha f + \beta g$ is odd.

We have proved that

$$\left\{\xi\left(1, \frac{\alpha^2}{\alpha-\beta}y_1 - \frac{\beta^2}{\alpha-\beta}y_2\right) : \xi \in \mathbb{R}\right\} \subset \mathrm{T}_{\alpha f+\beta g}(0). \tag{17}$$

From (16) and (17) we get

$$\mathrm{T}_{\alpha f+\beta g}(0) = \left\{\xi\left(1, \frac{\alpha^2}{\alpha-\beta}y_1 - \frac{\beta^2}{\alpha-\beta}y_2\right) : \xi \in \mathbb{R}\right\} \quad \text{if} \quad \frac{\alpha}{\beta} < 0.$$

□

The question arises as to whether the contingent of the graph of the sum of two mappings with trivial contingents should be trivial.
The answer is NO.

For instance, take $F = f + g$ where f and g are from the Example 6 and put $G = -F$. Then F, G have trivial contingents at 0 whereas their sum $W = F + G = 0$ obviously has a horizontal contingent

$$\mathrm{T}_W(0) = \{\xi(1, 0_Y) : \xi \in \mathbb{R}\},$$

thus nontrivial.

On the other hand, the contingent of the graph of the sum $f+g$ is trivial, if f is differentiable at 0 and $\mathrm{T}_g(0)$ is trivial. This follows from Theorem 3 (a) or (b).

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Małgorzata Turowska
Pomeranian University
Institute of Mathematics,
ul. Arciszewskiego 22d, 62 200 Słupsk, Poland
E-mail address: malgorzata.turowska@apsl.edu.pl