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THE METRIC DIMENSION OF CIRCULANT GRAPHS AND THEIR CARTESIAN PRODUCTS

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Abstract. Let $G = (V, E)$ be a connected graph (or hypergraph) and let $d(x, y)$ denote the distance between vertices $x, y \in V(G)$. A subset $W \subseteq V(G)$ is called a resolving set for G if for every pair of distinct vertices $x, y \in V(G)$, there is $w \in W$ such that $d(x, w) \neq d(y, w)$. The minimum cardinality of a resolving set for G is called the metric dimension of G, denoted by $\beta(G)$. The circulant graph $C_n(1, 2, \dots, t)$ has vertex set $\{v_0, v_1, \dots, v_{n-1}\}$ and edges $v_i v_{i+j}$ where $0 \leq i \leq n-1$ and $1 \leq j \leq t$ and the indices are taken modulo n ($2 \leq t \leq \lfloor \frac{n}{2} \rfloor$). In this paper we determine the exact metric dimension of the circulant graphs $C_n(1, 2, \dots, t)$, extending previous results due to Borchert and Gosselin (2013), Grigorious *et al.* (2014), and Vetrík (2016). In particular, we show that $\beta(C_n(1, 2, \dots, t)) = \beta(C_{n+2t}(1, 2, \dots, t))$ for large enough n, which implies that the metric dimension of these circulants is completely determined by the congruence class of n modulo $2t$. We determine the exact value of $\beta(C_n(1, 2, \dots, t))$ for $n \equiv 2 \bmod 2t$ and $n \equiv (t+1) \bmod 2t$ and we give better bounds on the metric dimension of these circulants for $n \equiv 0 \bmod 2t$ and $n \equiv 1 \bmod 2t$. In addition, we bound the metric dimension of Cartesian products of circulant graphs.

Keywords: metric dimension, circulant graph, cartesian product.

Mathematics Subject Classification: 05C12, 05C65, 05C25, 05E18.

1. INTRODUCTION

1.1. DEFINITIONS

A vertex x in a graph G is said to *resolve* a pair u, v of vertices of G if the distance from u to x does not equal the distance from v to x. A set W of vertices of G is a *resolving set* for G if every pair of vertices of G is resolved by some vertex of W. The smallest cardinality of a resolving set for G is called the *metric dimension* of G, and is denoted by $\beta(G)$.

For positive integers t and n, the *circulant graph* $C_n(1, 2, \ldots, t)$ is the simple graph with vertex set $\mathbb{Z}_n = \{v_0, v_1, \ldots, v_{n-1}\}$, the integers modulo n, in which vertex v_i is adjacent to the vertices $v_{i-t}, v_{i-t+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+t-1}, v_{i+t}$ (mod n) in $C_n(1, 2, \ldots, t)$. Observe that the distance between two vertices v_i and v_j in $G = C_n(1, 2, \ldots, t)$ is given by

$$d_G(v_i, v_j) = \begin{cases} \left\lceil \frac{|i-j|}{t} \right\rceil & |i-j| < \lceil \frac{n}{2} \rceil, \\ \left\lceil \frac{n-|i-j|}{t} \right\rceil & |i-j| \geq \lceil \frac{n}{2} \rceil. \end{cases}$$

The *outer cycle* of the circulant graph $G = C_n(1, 2, \ldots, t)$ is a spanning subgraph of G in which the vertex v_i is adjacent to exactly the vertices v_{i+1} and v_{i-1}. See Figure 1 for an example with $n = 13$ and $t = 2$.

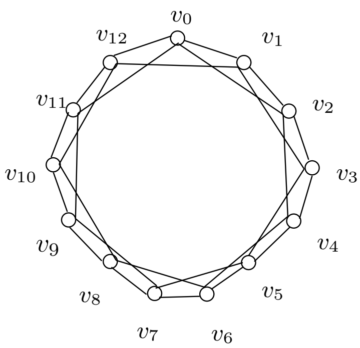

Fig. 1. $C_{13}(1, 2)$

The *Cartesian product* of graphs G_1 and G_2, denoted by $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2) := \{(x, y) : x \in V(G_1), y \in V(G_2)\}$, in which (x, y) is adjacent to (x', y') whenever $x = x'$ and $yy' \in E(G_2)$, or $y = y'$ and $xx' \in E(G_1)$. Observe that if G_1 and G_2 are connected graphs, then $G_1 \square G_2$ is connected. Assuming that isomorphic graphs are equal, the Cartesian product is associative, so $G_1 \square G_2 \square \cdots \square G_d$ is well-defined for graphs $G_1, G_2, \ldots, G_d$. Moreover, for two vertices $\vec{x} = (x_1, x_2, \ldots, x_d)$ and $\vec{y} = (y_1, y_2, \ldots, y_d)$ of the graph $G = G_1 \square G_2 \square \cdots \square G_d$, the distance $d_G(\vec{x}, \vec{y}) = \sum_{i=1}^{d} d_{G_i}(x_i, y_i)$.

1.2. HISTORY AND LAYOUT OF THE PAPER

The concept of the metric dimension of a graph was first introduced by Slater [13, 14], and independently by Harary and Melter [7]. Their introduction of this invariant was motivated by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Khuller *et al.* [10] later studied the metric dimension as an application to the navigation of robots in a graph space, and showed that the problem of determining the metric dimension of a given graph is NP-hard, and they determined the metric dimension of trees. An alternate proof of the formula for the metric dimension of trees was given by Chartrand *et al.* in [5], and they characterized the graphs of order n with metric dimension 1 (paths), $n-1$ (complete graphs) and $n-2$. Their study of the metric dimension was motivated by its applications to a problem in pharmaceutical chemistry. The metric dimension of a graph is related to several other well studied graph invariants such as the *determining number* (the base size of its automorphism group), and a good survey of these invariants and their relation to one another was written by Bailey and Cameron in 2011 [1].

Due to the fact that metric dimension has applications in network discovery and verification, combinatorial optimization, chemistry, and many other areas, researchers focus on computing or bounding the metric dimension of certain classes of graphs. In particular, there is great interest in finding classes of graphs whose metric dimension does not increase with the number of vertices. Such classes of graphs are said to have *bounded metric dimension*. Circulant graphs are an important class of graphs that can be used in the design of local area networks. They have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities. Javaid *et al.* [9] initiated a study of the metric dimension of circulants as some classes of these graphs had been shown to have bounded metric dimension. Imran *et al.* [8] later bounded the metric dimension of $C_n(1,2)$ and $C_n(1,2,3)$, and then Borchert and Gosselin [2] extended their results and determined the exact metric dimension of these two families of circulants for all n.

Proposition 1.1 ([2])**.** (1) *For* $n \geq 6$*,*

$$\beta(C_n(1,2)) = \begin{cases} 4 & \textit{for } n \equiv 1 \bmod 4, \\ 3 & \textit{otherwise.} \end{cases}$$

(2) *For* $n \geq 8$*,*

$$\beta(C_n(1,2,3)) = \begin{cases} 5 & \textit{for } n \equiv 1 \bmod 6, \\ 4 & \textit{otherwise.} \end{cases}$$

More recently, Grigorious *et al.* [6] bounded the metric dimension of the circulant graph $C_n(1,2,\ldots,t)$ for all n and t, as stated in the following result.

Proposition 1.2. *Suppose* $n \equiv r \bmod 2t$ *where* $2 \leq r \leq 2t+1$*. Then*

$$\beta(C_n(1,2,\ldots,t)) \leq \begin{cases} t+1 & 2 \leq r \leq t+1, \\ r-1 & t+2 \leq r \leq 2t+1. \end{cases}$$

These bounds were obtained from resolving sets consisting of consecutive vertices on the outer cycle of $C_n(1,2,\ldots,t)$. Grigorious *et al.* conjectured that these upper bounds on $\beta(C_n(1,2,\ldots,t))$ were also lower bounds, but this was refuted by Vetrík in 2016 [15] when he found the following two infinite families of counterexamples.

Proposition 1.3 ([15])**.**

(1) *If* $n = 2tk + t$ *where* $t \geq 4$ *is even and* $k \geq 2$*, then*

$$\beta(C_n(1, 2, \dots, t)) \leq t.$$

(2) *If* $n = 2tk + t + p$ *where* t *and* p *are even,* $t \geq 4$*,* $2 \leq p \leq t$ *and* $k \geq 1$*, then*

$$\beta(C_n(1, 2, \dots, t)) \leq t + \frac{p}{2}.$$

In addition, Vetrík gave the following lower bounds on $\beta(C_n(1, 2, \dots, t))$.

Proposition 1.4 ([15])**.**

(1) *If* $n \geq t^2 + 1$ *where* $t \geq 2$*, then*

$$\beta(C_n(1, 2, \dots, t)) \geq t.$$

(2) *If* $n = 2tk + r$ *where* $t \geq 2$*, and* $t + 2 \leq r \leq 2t + 1$*, then*

$$\beta(C_n(1, 2, \dots, t)) \geq t + 1.$$

Propositions 1.3 and 1.4 together imply that if $n \equiv t \bmod 2t$, where $n \geq t^2 + 1$ and $t \geq 4$ is even, then $\beta(C_n(1, 2, \dots, t)) = t$, and if $n \equiv (t + 2) \bmod 2t$ where $t \geq 2$, then $\beta(C_n(1, 2, \dots, t)) = t + 1$. In Section 2, we will extend Vetrík's results and find the exact metric dimension of $C_n(1, 2, \dots, t)$ in the cases where n is congruent to 2 or $(t + 1)$ modulo $2t$ (See Theorem 2.7), and we give better bounds on the metric dimension of these circulants for some other congruence classes of n modulo $2t$. We also show that for large enough n, $\beta(C_n(1, 2, \dots, t)) = \beta(C_{n+2t}(1, 2, \dots, t))$, which implies that the metric dimension of these circulants is completely determined by the congruence class of n modulo $2t$ (See Theorem 2.23).

Cáceres *et al.* [4], and independently Peters-Fransen and Oellermann [11], have studied the metric dimension of Cartesian products of graphs, and they obtained the following result.

Proposition 1.5 ([4, 11])**.** *Let* G *be a graph and let* $n \geq m \geq 3$*. Then*

$$\beta(G) \leq \beta(G \square C_m) \leq \begin{cases} \beta(G) + 1 & \text{if } m \text{ is odd,} \\ \beta(G) + 2 & \text{if } m \text{ is even,} \end{cases}$$

and

$$\beta(C_m \square C_n) = \begin{cases} 3, & \text{if } m \text{ or } n \text{ is odd,} \\ 4, & \text{if } m \text{ and } n \text{ are both even.} \end{cases}$$

In Section 3 we will extend their result to find analogous bounds on the metric dimension of $G \square C_n(1, 2, \dots, t)$ for a given graph G (Theorem 3.2), and this will yield bounds on the metric dimension of Cartesian products of circulant graphs (Corollary 3.3).

2. THE METRIC DIMENSION OF CIRCULANT GRAPHS

In this section we analyze the metric dimension of the circulant graph $C_n(1,2,\ldots,t)$. We use the concept of a *resolving hypergraph*, defined in [2], to visualize the resolving set W of this graph.

Definition 2.1. For a graph G of diameter d and a set of vertices $W \in G$, we define the *resolving hypergraph* of G with respect to $W = \{w_1, w_2, \ldots, w_s\}$ as the hypergraph with vertex set $V(G)$, and for each $i \in \{1,2,\ldots,s\}$ and $j \in \{0,1,2,\ldots,d\}$, there is a hyperedge which contains all vertices at distance j from w_i in G. We denote this hypergraph by $R_W(G)$.

To represent the resolving hypergraph $R_W(G)$, a vertex $v \in W$ is circled to show that it is in the resolving set W, and separating lines are used to separate the set of vertices at distance j from v, for each $v \in W$, for $j \in \{1,2,\ldots,d\}$. The graph is resolved when every two distinct vertices in G are separated by a line (Fig. 2).

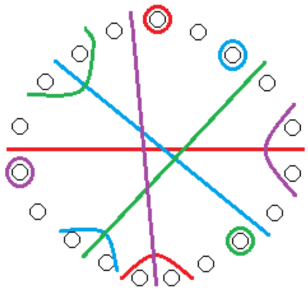

Fig. 2. $C_{19}(1,2,3,4)$ resolved by 4 vertices

Since the circulant $C_n(1,2,\ldots,t)$ is vertex-transitive, we may assume that v_0 is one of the vertices in every resolving set, and throughout the paper we shall denote the set of vertices at distance j from v_0 by L_j, for $j \in \{1,2,\ldots,d\}$. Our figures are all orientated such that v_0 is the topmost circled vertex and the indices of the vertices ascend clockwise. Observe that each separating line in the resolving hypergraph $R_W(G)$ partitions the level L_j into two subsets. If two such separating lines partition L_j into two different pairs of subsets, we say they are *distinct separating lines* in L_j. We denote the set of vertices $\{v_{n-(j-1)t-1}, v_{n-(j-1)t-2}, \ldots, v_{n-jt}\}$ on the left side of L_j by L_j^-, and set of vertices $\{v_{(j-1)t+1}, v_{(j-1)t+2}, \ldots, v_{jt}\}$ on the right side of L_j by L_j^+. See Figure 3 for an example with $n = 21$ and $t = 3$. Given a vertex w in a resolving set W of G, the ends of the separating lines in L_j of the resolving hypergraph $R_{\{v_0,w\}}(G)$ draw between two of the t vertices in L_j^+ (L_j^-) with one *cut*, except for the line separating the vertices at distance d from w, which might make two cuts between vertices in one L_j^+ (L_j^-) if the congruence class of n modulo $2t$ is at most $t-1$. If two cuts in L_j of $R_{\{v_0,w\}}(G)$ draw between two different pairs of vertices in the graph, we say they make *distinct cuts*.

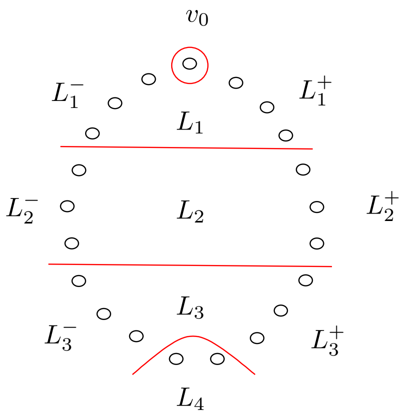

Fig. 3. $R_{\{v_0\}}(C_{21}(1,2,3))$

It will be shown in this section that, for large enough n, the value of $\beta(C_n(1,2,\ldots,n))$ is completely determined by the congruence class of n modulo $2t$ (see Corollary 9). It will be useful to consider n in the form $n = 2tk + r$ for $2 \le r \le 2t+1$, since for each value of r in this range, the diameter of $C_n(1,2,\ldots,t)$ is equal to $k+1$. For each of these values of r, we either determine the exact value of $\beta(C_n(1,2,\ldots,t))$, or we bound this value, and our results are summarized in Section 4.

Remark 2.2. The diameter of any circulant graph $G = C_n(1,2,\ldots,t)$ is $\lceil \frac{n-1}{2t} \rceil$. Observe that if $n = 2tk + r$ for $2 \le r \le 2t+1$, then the diameter of G is equal to $k+1$.

We now prove a couple of technical lemmas which will be used throughout the paper in the proofs of various bounds on $\beta(C_n(1,2,\ldots,t))$.

Lemma 2.3. *Three pairs of vertices* (v_a, v_{a+1}), (v_{a+2+x}, v_{a+3+x}), *and* (v_{a+4+2x}, v_{a+5+2x}) *cannot be resolved by one vertex when* $t \neq x+2$.

Proof. Label (v_a, v_{a+1}), (v_{a+2+x}, v_{a+3+x}), and (v_{a+4+2x}, v_{a+5+2x}) as P_1, P_2, and P_3 respectively. We may assume that for some vertex $v \in V(G)$, we have $d(v, v_a) = d(v, v_{a+1}) + 1$.

Case 1. Assume $n \equiv (x+3) \bmod 2t$. Then the vertices in P_1 and P_2 can only be simultaneously resolved with a vertex antipodal to these pairs. Figure 4 shows an example of three pairs of vertices where two of the pairs are resolved in this way. Observe that $d(v, v_{a+4+2x}) = d(v, v_{a+5+2x})$.

Case 2. Assume $n \not\equiv (x+3) \bmod 2t$. Observe that only vertices greater than t distance away from v_a on the outer cycle will have a unique representation with respect to v. Since P_1 is resolved by v and $t \neq x+2$, we have $d(v, v_{a+2+x}) = d(v, v_{a+3+x}) = d(v, v_a) + 1$. $\square$

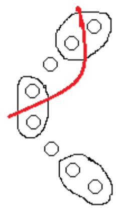

Fig. 4. $n \equiv 4 \bmod 8$; the bottom pair is unresolved

Corollary 2.4. *Two pairs of vertices* (v_a, v_{a+1}) *and* (v_{a+2+x}, v_{a+3+x}) *cannot be resolved by one vertex when* $t \neq x+2$ *and* $n \not\equiv (x+3) \bmod 2t$.

Lemma 2.5. *Let* $G = C_n(1,2,\ldots,t)$. *No* x *consecutive vertices sharing the same edge neighborhood in* $R_{\{v_0\}}(G)$ *can be resolved by* $x-2$ *vertices if* $n \equiv r \bmod 2t$ *where* $x \leq r \leq 2t+2$ *and* $3 \leq x \leq t$.

Proof. Consider a clique of x consecutive vertices on the outer cycle of G for $t \geq 3$. Without loss of generality, we can consider any x consecutive vertices from the clique $L_1^+ = \{v_1, v_2, \ldots, v_t\}$ which all have distance 1 from v_0. When $3 \leq r \leq x-1$, we can make at most two distinct cuts in Q of the resolving hypergraph $R_{\{v_0\}}(G)$ by taking vertices antipodal to the clique. Taking $\lceil \frac{x}{2} \rceil$ vertices that each create two distinct cuts in Q of the resolving hypergraph $R_{\{v_0\}}(G)$ will resolve the clique because $2\lceil \frac{x}{2} \rceil \geq x$. When $x \leq r \leq 2t+2$, we can make at most one distinct cut in Q of the resolving hypergraph $R_{\{v_0\}}(G)$. By the Pigeonhole Principle, taking at most $x-2$ vertices to resolve the x vertices in Q will leave at least one pair of vertices unresolved. □

Corollary 2.6. *Let* $G = C_n(1,2,\ldots,t)$. *No* x *consecutive vertices sharing the same edge neighborhood in* $R_{\{v_0\}}(G)$ *can be resolved by* $\lceil \frac{x}{2} \rceil - 1$ *vertices.*

In [15], Vetrík showed that $\beta(C_n(1,2,\ldots,t)) \geq t+1$ for $n \equiv r \bmod t$ where $t+2 \leq r \leq 2t+1$ (see Proposition 1.4 (2)) by showing that no t vertices of the graph could resolve all pairs of vertices in L_d, the set of vertices at the greatest distance from v_0. We now give an alternate proof of Vetrík's result, and show that $t+1$ is a lower bound on the metric dimension of this circulant for $n \equiv 2, t+1 \bmod 2t$ as well.

Theorem 2.7. *Let* $G = C_n(1,2,\ldots,t)$ *where* $n \equiv r \bmod 2t$ *and* $t+1 \leq r \leq 2t+2$, *then* $\beta(G) \geq t+1$.

Proof. Let W be a resolving set for G and take $v_0 \in W$. Observe that for any vertex $v \in W$ where $v \neq v_0$, at most two distinct cuts are made in the set L_1 in the resolving hypergraph $R_{\{v,v_0\}}(G)$ (Fig. 5). This is because each of the distinct cuts created by v are at least t vertices apart from each other. Taking any $t-1$ additional vertices to be in W with v_0 creates at most $2(t-1) = 2t-2$ distinct cuts in L_1. Since there are $2t$ vertices in L_1, then by the Pigeonhole Principle, there will be at least one pair of vertices in L_1 left unresolved. □

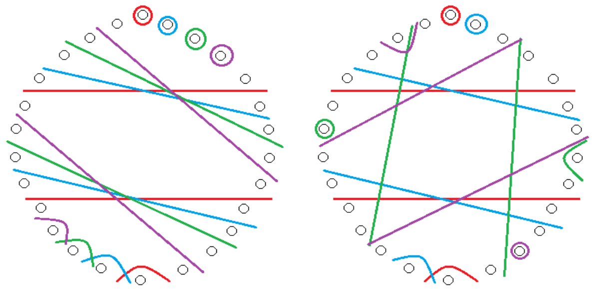

Fig. 5. $R_W(C_{26}(1,2,3,4))$; L_1 has at least one unresolved pair

The next corollary follows from Proposition 1.2 and Theorem 2.7.

Corollary 2.8. *Let* $G = C_n(1,2,\ldots,t)$ *where* $n \geq 2t+2$. *If* $n \equiv r \bmod 2t$ *where* $r = 2$, $t+1$ *or* $t+2$, *then* $\beta(G) = t+1$.

Theorem 2.9. *Let* $G = C_n(1,2,\ldots,t)$ *where* t *is odd. If* $n \equiv (t+3) \bmod 2t$ *then* $\beta(G) \leq t+1$

Proof. Let $n = 2tk + (t+3)$. Then $diam(G) = k+1$. Let $W_1, W_2, W_3, W_4 \subseteq V(G)$ where

$$\begin{aligned} W_1 &= \{v_0\}, \\ W_2 &= \{v_2, v_4, \ldots, v_{t-1}\}, \\ W_3 &= \{v_{n-2}, v_{n-4}, \ldots, v_{n-(t-1)}\}, \\ W_4 &= \{v_{kt+1}\}. \end{aligned}$$

Note that $|W_1| = |W_4| = 1$ and $|W_2| = |W_3| = \frac{t-1}{2}$. Observe that $W_1 \cup W_2 \cup W_3$ leaves only three pairs of unresolved vertices, listed below. (Figure 6 shows the case for $k = 3$. For $k \geq 3$, each of the layers $L_1, L_2, \ldots, L_k$ in $R_{v_0}(G)$ have the same cut pattern as this example.)

$$\begin{aligned} &\{v_{n-1}, v_1\}, \\ &\{v_{n-kt-1}, v_{n-kt-2}\}, \\ &\{v_{kt+1}, v_{kt+2}\}. \end{aligned}$$

We have

$$\begin{aligned} d(v_{kt+1}, v_{n-1}) &= d(v_{kt+1}, v_1) + 1, \\ d(v_{kt+1}, v_{n-kt-1}) &= d(v_{kt+1}, v_{n-kt-2}) + 1. \end{aligned}$$

So these pairs of vertices can all be resolved by taking v_{kt+1} as a resolving vertex thus $W_1 \cup W_2 \cup W_3 \cup W_4$ resolves G. $\square$

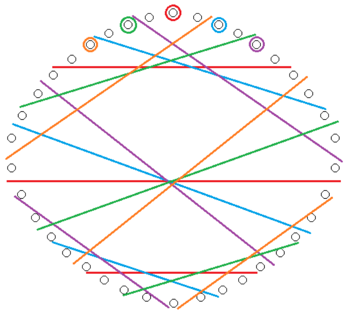

Fig. 6. $R_{W_1 \cup W_2 \cup W_3}(C_{38}(1, 2, \ldots, 5))$

The next corollary follows from Theorems 2.7 and 2.9.

Corollary 2.10. *Let $G = C_n(1, 2, \ldots, t)$ where t is odd. If $n \equiv (t + 3) \bmod 2t$ then $\beta(G) = t + 1$.*

From the empirical evidence in the tables in the Appendix, it appears that $t + 1$ is also a lower bound on $\beta(C_n(1, 2, \ldots, t))$ for $n \equiv r \bmod 2t$ where $3 \leq r \leq t-1$ when n is large enough. We were able to prove this in the case where $t = 4$ and $r = 3$ for $n \geq 27$, and where $t = 5$ and $r = 3$ for $n \geq 23$, as the next two theorems state.

Theorem 2.11. *Let $G = C_n(1, 2, 3, 4)$. If $n \equiv 3 \bmod 8$ and $n \geq 27$ then $\beta(G) \geq 5$*

Proof. Let W be a resolving set for G and suppose to the contrary that $|W| \leq 4$ where $v_0 \in W$. We consider the vertices on the $v_1 - v_{4(d-1)+1}$ path of the outer cycle to be the right side of the graph and the vertices on the $v_{n-1} - v_{n-4(d-1)-1}$ path of the outer cycle to be the left side of the graph. In order to resolve G, some vertex $v \in W$ must create a separating line that partitions the two vertices in L_d into separate sets. The only vertices that can do this are $\{v_{n-1}, v_{n-5}, \ldots, v_{n-4(d-1)-1}\}$ and $\{v_1, v_5, \ldots, v_{4(d-1)+1}\}$ but by symmetry we can consider only the former set as possible options for vertices in W.

Case 1. Say $v_{n-1} \in W$. Then

$$\begin{gathered}\{v_{4(d-1)-1}, v_{4(d-1)-2}, v_{4(d-1)-3}\},\\ \{v_{n-4(d-1)+2}, v_{n-4(d-1)+1}, v_{n-4(d-1)}\}\end{gathered}$$

are two cliques of three unresolved vertices in L_d. We need to take vertices to be in W such that at least one vertex in W makes three distinct cuts between the vertices in the above cliques while the other vertex in W makes at least two distinct cuts between the vertices in the above cliques. So, the only way for these cliques to be resolved is if v_{n-3} and v_2 are in W but then $r(v_{n-2}|W) = r(v_1|W)$.

Case 2. Say $v_{n-5} \in W$. Then

$$\begin{gathered}\{v_{n-4(d-1)+2}, v_{n-4(d-1)+1}, v_{n-4(d-1)}\},\\ \{v_{4(d-1)}, v_{4(d-1)-1}, v_{4(d-1)-2}\},\\ \{v_{4(d-2)-1}, v_{4(d-2)-2}, v_{4(d-2)-3}\}\end{gathered}$$

are three cliques of three unresolved vertices. We need to take our remaining vertices in W to be antipodal to some of the above cliques in order to obtain the required number of distinct cuts for resolving them. Specifically, the only way for these cliques to be resolved is if either one additional vertex from the left side of L_2 and one additional vertex from the right side of L_1 are in W or two additional vertices from the left side of L_1 are in W. But there also exists the clique of unresolved vertices $\{v_{n-1}, v_{n-2}, v_{n-3}, v_{n-4}\}$ which can only be resolved if at least one vertex in W is from the right side of L_{d-1}.

Case 3. Say we take any one vertex from $\{v_{n-9}, v_{n-13}, \ldots, v_{n-4(d-1)-1}\}$ to be in W. Then

$$\{v_{n-1}, v_{n-2}, v_{n-3}, v_{n-4}\},$$
$$\{v_{n-5}, v_{n-6}, v_{n-7}, v_{n-8}\}$$

are two consecutive cliques of four that are unresolved. We need six distinct cuts between the vertices in these cliques in order to resolve them. The only way to make this many distinct cuts in these cliques is by taking two vertices to be in W that each create three distinct cuts in the cliques. Observe that since $n \equiv 3 \bmod 8$ and $t = 4$, only one additional vertex can make at most three distinct cuts in the cliques and the final vertex that we choose can only make at most two distinct cuts in the above cliques. Figure 7 shows an example of how the final vertex we take can only make at most two distinct cuts in the above cliques. □

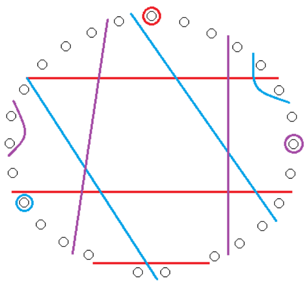

Fig. 7. $C_{27}(1,2,3,4)$

Corollary 2.12. *Let $G = C_n(1,2,3,4)$ where $n = 8k+3$. Then*

$$\beta(G) = \begin{cases} 4 & \text{if } k \in \{1,2\}, \\ 5 & \text{if } k \geq 3. \end{cases}$$

Proof. The value of $\beta(G)$ when $k \geq 3$ follows from Proposition 1.2 and Theorem 2.11. For $k \in \{1,2\}$, observe that $\{v_0, v_2, v_3, v_{10}\}$ resolves G when $n = 11$ and $\{v_0, v_2, v_7, v_{14}\}$ resolves G when $n = 19$. □

Theorem 2.13. *Let $G = C_n(1,2,\ldots,5)$. If $n \equiv 3 \bmod 10$ and $n \geq 23$ then $\beta(G) \geq 6$.*

Proof. Let W be a resolving set for G and suppose to the contrary that $|W| \leq 5$ where $v_0 \in W$. We consider the vertices on the $v_1 - v_{5(d-1)+1}$ path of the outer cycle to be the right side of the graph and the vertices on the $v_{n-1} - v_{n-5(d-1)-1}$ path of the outer cycle to be the left side of the graph. In order to resolve G, some vertex $v \in W$ must create a separating line that partitions the two vertices in L_d into separate sets. The only vertices that can do this are $\{v_{n-1}, v_{n-6}, \ldots, v_{n-5(d-1)-1}\}$ and $\{v_1, v_6, \ldots, v_{5(d-1)+1}\}$ but by symmetry we can consider only the former set as possible options for vertices in W.

Case 1. Say $v_{n-1} \in W$. Then

$$\begin{aligned}&\{v_{5(d-1)-1}, v_{5(d-1)-2}, v_{5(d-1)-3}, v_{5(d-1)-4}\},\\&\{v_{n-5(d-1)+3}, v_{n-5(d-1)+2}, v_{n-5(d-1)+1}, v_{n-5(d-1)}\}\end{aligned}$$

are two cliques of four unresolved vertices in L_d. In order to achieve the required six distinct cuts for resolving the above cliques, we need at least two vertices that are both antipodal to one of the above cliques. Alternatively we can use one vertex that is antipodal to one of the above cliques and use another vertex that is antipodal to the other clique. Specifically, our options for resolving these cliques is by taking either v_{n-3} and v_{n-4} to be in W or v_3 and v_{n-4} to be in W. In the former case, we only have one vertex left to take to be in W after v_0, v_{n-1}, v_{n-3}, v_{n-4} which is insufficient for simultaneously resolving the two sets of unresolved vertices $\{v_1, v_{n-2}, v_{n-5}\}$ and $\{v_7, v_8\}$. In the latter case, we only have one vertex left to take to be in W after v_0, v_{n-1}, v_3, v_{n-4} which is insufficient for resolving the set of unresolved vertices $\{v_1, v_{n-2}, v_{n-3}\}$ that is a clique of three.

Case 2. Say $v_{n-6} \in W$. Then

$$\begin{aligned}&\{v_{n-5(d-1)+3}, v_{n-5(d-1)+2}, v_{n-5(d-1)+1}, v_{n-5(d-1)}\},\\&\{v_{5(d-1)}, v_{5(d-1)-1}, v_{5(d-1)-2}, v_{5(d-1)-3}\},\\&\{v_{5(d-1)-1}, v_{5(d-1)-2}, v_{5(d-1)-3}, v_{5(d-1)-4}\}\end{aligned}$$

are three cliques of four unresolved vertices. In order to simultaneously resolve these cliques with our three remaining vertices, we need a vertex that makes two distinct cuts in the clique $\{v_{5(d-1)}, v_{5(d-1)-1}, v_{5(d-1)-2}, v_{5(d-1)-3}\}$ which is done with a vertex antipodal that clique. So, the only way to resolve these cliques is if $v_{n-3} \in W$ but then we only have two vertices left to take to be in W after v_0, v_{n-6}, v_{n-3} which is insufficient for resolving the set of unresolved vertices $\{v_{n-1}, v_{n-2}, v_{n-4}, v_{n-5}\}$ since subsequent vertices in W can only make at most one distinct cut in the set yet we need to make a total of three distinct cuts.

Case 3. Say $v_{n-11} \in W$. Then

$$\begin{aligned}&\{v_{n-1}, v_{n-2}, v_{n-3}, v_{n-4}, v_{n-5}\},\\&\{v_{n-5}, v_{n-6}, v_{n-7}, v_{n-8}, v_{n-9}\}\end{aligned}$$

are two consecutive cliques of five vertices that are unresolved. Since we can only use at most three additional vertices to resolve these cliques, we need the three additional vertices in W to be antipodal to the cliques in a way that creates eight distinct cuts in total. So, the only way to obtain the required number of distinct cuts for resolving the above cliques is if $v_{5(d-2)-2}$, $v_{5(d-1)-4}$, $v_{5(d-1)-1} \in W$ but then $\{v_1, v_2\}$ remains a pair of unresolved vertices.

Case 4. Say we take any one vertex from $\{v_{n-16}, v_{n-21}, \dots, v_{n-5(d-1)-1}\}$ to be in W. Then

$$\begin{aligned}&\{v_{n-1}, v_{n-2}, v_{n-3}, v_{n-4}, v_{n-5}\},\\ &\{v_{n-6}, v_{n-7}, v_{n-8}, v_{n-9}, v_{n-10}\},\\ &\{v_{n-11}, v_{n-12}, v_{n-13}, v_{n-14}, v_{n-15}\}\end{aligned}$$

are three consecutive cliques of five vertices that are unresolved. So we need twelve distinct cuts to resolve the above cliques. Observe that since $n \equiv 3 \bmod 10$ and $t = 5$, only the next two choices of vertices we take to be in W will be able to make at most four distinct cuts in the above cliques and the final choice of vertex that we take will then make at most three distinct cuts in the above cliques. Figure 8 shows an example of how the final vertex we take can only make at most three distinct cuts in the above cliques. □

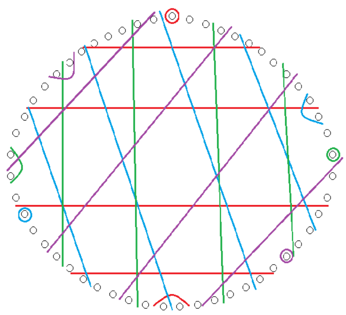

Fig. 8. $R_{W_1 \cup W_2}(C_{53}(1, 2, \dots, 5))$

Corollary 2.14. *Let $G = C_n(1, 2, \dots, 5)$ where $n = 10k + 3$. Then*

$$\beta(G) = \begin{cases} 5 & \text{if } k = 1, \\ 6 & \text{if } k \geq 2. \end{cases}$$

Proof. The value of $\beta(G)$ when $k \geq 2$ follows from [6] and Theorem 2.13. For $k = 1$, observe that $\{v_0, v_1, v_2, v_4, v_5\}$ resolves G when $n = 13$. □

In [15], Vetrík showed that $\beta(C_n(1, 2, \dots, t)) \geq t$ for $n \geq t^2 + 1$ (see Proposition 1.4 (1)). We now use an alternate proof to show that this bound holds for all n. The result is stated in the following theorem for the congruence classes modulo $2t$ for which we do not already have a better lower bound of $t + 1$ on the metric dimension.

Theorem 2.15. *Let $G = C_n(1, 2, \ldots, t)$ where $n \equiv r \bmod 2t$ and $3 \leq r \leq t$, then $\beta(G) \geq t$.*

Proof. Suppose to the contrary that $\beta(G) \leq t-1$. Let W be a resolving set for G with $v_0 \in W$. Consider the sets of vertices $L_1^+ = \{v_1, v_2, \ldots, v_t\}$ and $L_1^- = \{v_{n-1}, v_{n-2}, \ldots, v_{n-t}\}$. Note that all of the vertices in L_1^+ (L_1^-) have distance 1 from each other and from v_0.

Case 1. Suppose $(n-1) \mod 2t > t-2$. Then each of the $t-2$ vertices $v \in W$ besides v_0 will make at most one distinct cut in L_1^+. By Lemma 2.5, at least one pair of vertices in L_1^+ will be unresolved.

Case 2. Suppose $(n-1) \mod 2t \leq t-2$. We may resolve the vertices in L_1^+ by taking the $t-2$ vertices at furthest distance from $t-2$ consecutive vertices in L_1^+. However, by doing this, each of the $t-2$ vertices $v \in W$ besides v_0 will make at most one distinct cut in L_1^+ so at least one pair of vertices in L_1^+ will be left unresolved by the argument from the first case (Fig. 9). $\square$

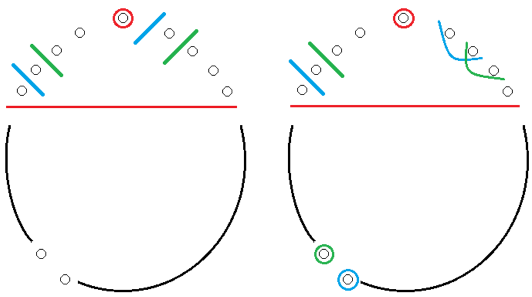

Fig. 9. Case 1 and Case 2 respectively

The empirical evidence in the Appendix shows that for small values of n and t, the metric dimension of $\beta(C_n(1, 2, \ldots, t))$ seems to be t for some of the smaller congruence classes of n modulo $2t$. The next theorem shows that this is always true when $n = 2t+r$ and $3 \leq r \leq \lfloor \frac{t}{2} \rfloor + 1$.

Theorem 2.16. *Let $G = C_n(1, 2, \ldots, t)$ where $n = 2t + r$ and $3 \leq r \leq \lfloor \frac{t}{2} \rfloor + 1$, then $\beta(G) = t$.*

Proof. Let $W_1, W_2 \subseteq V(G)$ where

$$W_1 = \{v_0, v_1, \ldots, v_{r-2}\},$$
$$W_2 = \{v_r, v_{r+1}, \ldots, v_t\}.$$

Note that $|W_1| = r-1$ and $|W_2| = t-(r-1)$. We have that the only vertices at distance $d = 2$ from $v_i \in W_2$ are $v_{i+t+1}, v_{i+t+2}, \ldots, v_{i+t+(r-1)}$. Since the vertices in W_2 are consecutive, the vertices $v_{n-1}, v_{n-2}, \ldots, v_{n-t+1}$ all have unique representations with respect to W_2. Similarly, for $v_i \in W_1$, the only vertices at distance 2 from v_i are

$v_{i+t+1}, v_{i+t+2}, \dots, v_{i+t+(r-1)}$, and since the vertices in W_1 are consecutive, the vertices $v_{t+1}, v_{t+2}, \dots, v_{t+2r-3}$ all have unique representations with respect to W_1. Also, it is clear that the only vertex which has distance 1 to all of the vertices in $W_1 \cup W_2$ is v_{r-1}. Thus all of the vertices in the resolving hypergraph $R_{W_1 \cup W_2}(G)$ are resolved (Fig. 10). It follows from Theorem 2.15 that $\beta(G) = t$ under these conditions. □

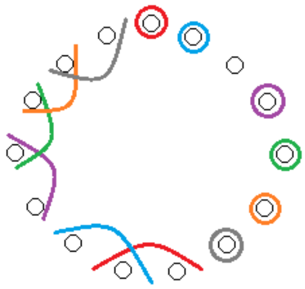

Fig. 10. $C_{15}(1, 2, \dots, 6)$

From Proposition 1.4 and Theorem 2.7, it follows that $\beta(C_n(1, 2, \dots, t)) \geq t + 1$ for $n \equiv 1 \bmod 2t$. We improve this bound slightly in the next theorem.

Theorem 2.17. *Let $G = C_n(1, 2, \dots, t)$ where $t \geq 2$ and $n \equiv 1 \bmod 2t$, then $\beta(G) \geq t + 2$.*

Proof. Let W be a resolving set for G and suppose to the contrary that $|W| \leq t + 1$, where $v_0 \in W$. We consider the vertices on the $v_1 - v_{dt}$ path of the outer cycle as the right side of the graph. Similarly, the vertices on the $v_{n-1} - v_{n-dt}$ path of the outer cycle will be considered the left side of the graph. In order to resolve G, some vertex $v \in W \cap L_j$ must create a separating line in the resolving hypergraph $R_{\{v,v_0\}}(G)$ such that v_{n-dt} and v_{dt} in L_d are partitioned into separate sets. The only vertices that can do this have a nonnegative integer multiple of t distance from v_0 on either the left or right side of the outer cycle. By symmetry, we consider only the vertices $v_{n-t}, v_{n-2t}, \dots, v_{n-dt}$ as possible choices for v. Note that taking any of these vertices to be in W leaves $2d - j$ consecutive cliques of unresolved vertices, where each clique has cardinality t. By Lemma 2.5, subsequent vertices in W beyond v_0 and v must not leave an unresolved clique of cardinality t. In addition, the subsequent vertices beyond v_0 and v must each create distinct cuts in both $\{v_{dt+1}, v_{dt+2}, \dots, v_{dt+t}\}$ and $\{v_{n-dt-1}, v_{n-dt-2}, \dots, v_{n-dt-t}\}$. Thus, no two vertices on the same side of the outer cycle may share the same congruence modulo t for otherwise both vertices will create the same cut in either $\{v_{td+1}, v_{td+2}, \dots, v_{td+t}\}$ or $\{v_{n-dt-1}, v_{n-dt-2}, \dots, v_{n-dt-t}\}$.

Note that the order in which we choose the vertices for W does not matter. So, the subsequent choices of vertices must belong to one of the sets

$$\begin{aligned}&\{v_{dt-1}, v_{dt-2}, \ldots, v_{dt-(t-2)}\},\\&\{v_{(d-1)t-1}, v_{(d-1)t-2}, \ldots, v_{(d-1)t-(t-2)}\},\\&\qquad\vdots\\&\{v_{jt-1}, v_{jt-2}, \ldots, v_{jt-(t-2)}\}.\end{aligned}$$

Since the j sets listed above contain $t-2$ vertices of unique congruence modulo t, we only have $t-2$ additional choices of vertices to be in W that do not leave a clique of t (Fig. 11). This means that there are only $t-2$ choices of vertices to be in W that make distinct cuts in every clique of t in the resolving hypergraph $R_{\{v,v_0\}}(G)$ yet we need the $t-1$ vertices to each make these distinct cuts. Seeing that we have at most $t-1$ additional vertices to take in W beyond v_0 and v, but we only have $t-2$ possible choices of vertices to resolve the $2d-j$ cliques of t, by Lemma 2.5, $\beta(G) \geq t+2$. $\square$

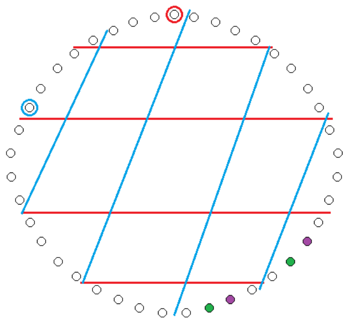

Fig. 11. $R_{\{v,v_0\}}(C_{41}(1,2,3,4))$; The colored vertices are the only choices that do not leave a clique of t

From Grigorious' result in Proposition 1.2, we know that $\beta(C_n(1,2,\ldots,t)) \leq r-1$ when $n = 2tk + r$ for $r \in \{2t, 2t+1\}$. In the next two results, we improve this upper bound to $2t-2$.

Theorem 2.18. *Let $G = C_n(1,2,\ldots,t)$ where $n \equiv 1 \bmod 2t$ and $t \geq 4$, then $\beta(G) \leq 2t-2$.*

Proof. Let $n = 2tk + (2t+1)$. Then $diam(G) = k+1$. Let $W_1, W_2, W_3 \subseteq V(G)$ where

$$\begin{aligned}W_1 &= \{v_0, v_{n-t+1}, v_{tk+3}, v_{n-t(k+1)+1}\},\\W_2 &= \{v_3, v_4, \ldots, v_t\},\\W_3 &= \{v_{n-t(k+1)+3}, v_{n-t(k+1)+4}, \ldots, v_{n-t(k+1)+(t-2)}\}.\end{aligned}$$

Note that $|W_1| = 4$, $|W_2| = t-2$, and $|W_3| = t-4$. For any t, the vertex v_i for $i \in \{3,4,\ldots,t\}$ will always have distance j from only the first tj closest vertices from

v_i on the outer cycle for $j \in \{1, 2, \ldots, d\}$ (Figure 12 shows the case where $t = 5$ and $k = 2$. For $k \geq 2$, each of the layers $L_2, \ldots, L_k$ in $R_{v_0}(G)$ have the same cut pattern as this example.)

When $t = 4$, the sets of vertices left unresolved by W_1 are

$$\{v_{n-2}, v_{n-1}, v_1\},$$
$$\{v_{kt+2}, v_{kt+4}, v_{kt+5}\},$$
$$\{v_{t-1}, v_{t-2}\}, \{v_{2t-1}, v_{2t-2}\}, \ldots, \{v_{kt-1}, v_{kt-2}\}.$$

We have:

$$\begin{aligned} d(v_3, v_{n-2}) &= d(v_3, v_{n-1}) + 1, \\ d(v_3, v_{kt+4}) &= d(v_3, v_{kt+2}) + 1, \\ d(v_3, v_{xt-1}) &= d(v_3, v_{xt-2}) + 1 \text{ for } x \in \{1, 2, \ldots, k\}, \\ d(v_4, v_{n-1}) &= d(v_3, v_1) + 1, \\ d(v_4, v_{kt+5}) &= d(v_3, v_{kt+4}) + 1. \end{aligned}$$

So taking $W_1 \cup W_2$ resolves these sets. Thus $\beta(G) \leq t+2$ when $t = 4$. Since $t+2 = 2t-2$ when $t = 4$, we can say $\beta(G) \leq 2t - 2$ when $t = 4$. If $t > 4$, the set of vertices left unresolved by W_1 are resolved by W_2 in a similar way as for $t = 4$, except for the set

$$R = \{v_{n-t(k+1)+3}, v_{n-t(k+1)+4}, \ldots, v_{n-t(k+1)+(t-1)}\}.$$

Since $|R| = t - 3$, taking any $t - 4$ vertices from R will resolve G. Specifically, taking $W_1 \cup W_2 \cup W_3$ resolves G, so $\beta(G) \leq 2t - 2$. □

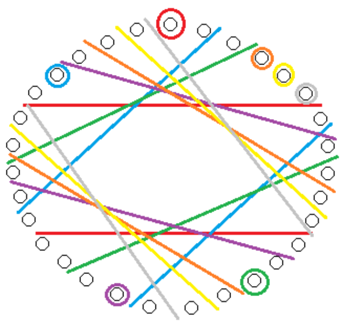

Fig. 12. $R_{W_1 \cup W_2}(C_{31}(1, 2, \ldots, 5))$

Theorem 2.19. *Let* $G = C_n(1, 2, \ldots, t)$ *where* $n \equiv 0 \bmod 2t$ *and* $t \geq 5$, *then* $\beta(G) \leq 2t - 2$.

Proof. Let $n = 2tk + 2t$. Then the diameter is $d = k + 1$. Let $W_1, W_2, W_3 \subseteq V(G)$ where

$$\begin{aligned} W_1 &= \{v_0, v_{n-t+1}, v_{tk+3}, v_{n-t(k+1)+1}, v_{n-t(k+1)+2}\}, \\ W_2 &= \{v_3, v_4, \dots, v_t\}, \\ W_3 &= \{v_{n-t(k+1)+4}, v_{n-t(k+1)+5}, \dots, v_{n-t(k+1)+(t-2)}\}. \end{aligned}$$

Note that $|W_1| = 5$, $|W_2| = t - 2$, and $|W_3| = t - 5$. For any t, the vertex v_i for $i \in \{3, 4, \dots, t\}$ will always have distance j from only the first tj closest vertices from v_i on the outer cycle for $j \in \{1, 2, \dots, k+1\}$. (Figure 13 shows the case for $t = 7$ and $k = 2$. For $k \geq 2$, each of the layers $L_2, \dots, L_k$ in $R_{v_0}(G)$ have the same cut pattern as this example.) When $t = 5$, the sets of vertices left unresolved by W_1 are

$$\begin{aligned} &\{v_{n-2}, v_{n-1}, v_1\}, \\ &\{v_{kt+2}, v_{kt+4}, v_{kt+5}\}, \\ &\{v_{t-1}, v_{t-2}, v_{t-3}\}, \{v_{2t-1}, v_{2t-2}, v_{2t-3}\}, \dots, \{v_{kt-1}, v_{kt-2}, v_{kt-3}\}. \end{aligned}$$

We have:

$$\begin{aligned} d(v_3, v_{kt+4}) &= d(v_3, v_{kt+2}) + 1, \\ d(v_3, v_{xt-2}) &= d(v_3, v_{xt-3}) + 1 \text{ for } x \in \{1, 2, \dots, d-1\}, \\ d(v_4, v_{n-2}) &= d(v_4, v_{n-1}) + 1, \\ d(v_4, v_{xt-1}) &= d(v_3, v_{xt-2}) + 1, \\ d(v_5, v_{n-1}) &= d(v_5, v_1) + 1. \end{aligned}$$

So taking $W_1 \cup W_2$ resolves these sets. Thus for $t = 5$, $\beta(G) \leq t+3$. But $t+3 = 2t-2$ in this case, so we can say $\beta(G) \leq 2t - 2$ when $t = 5$. When $t > 5$, the set of vertices left unresolved by W_1 are resolved by W_2 in a similar way as for $t = 5$, except for the set

$$R = \{v_{n-t(k+1)+4}, v_{n-t(k+1)+5}, \dots, v_{n-t(k+1)+(t-1)}\}$$

Since $|R| = t - 4$, taking any $t - 5$ vertices from R will resolve G. Specifically, taking $W_1 \cup W_2 \cup W_3$ resolves G so $\beta(G) \leq 2t - 2$. $\square$

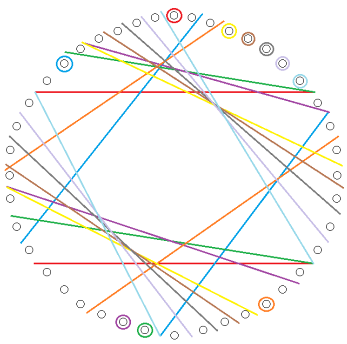

Fig. 13. $R_{W_1 \cup W_2}(C_{42}(1, 2, \dots, 7))$

Theorem 2.20. *If* $G = C_n(1,2,3,4)$ *where* $n \equiv 0 \bmod 8$ *then* $\beta(G) \le 6$.

Proof. Let $n = 8k+8$. Then $diam(G) = k+1$. Let $W_1, W_2 \subseteq V(G)$ where

$$W_1 = \{v_0, v_2, v_3, v_{n-3}\}$$
$$W_2 = \{v_{n-kt-1}, v_{n-kt-2}\}$$

The vertices left unresolved by W_1 are

$$\{v_{n-1}, v_1\},$$
$$\{v_{n-kt-1}, v_{n-kt-2}, v_{n-kt-3}\},$$
$$\{v_{kt+1}, v_{kt+2}\}.$$

A vertex v will always have distance j from only the first tj closest vertices from v on the outer cycle for $j \in \{1,2,\dots,k+1\}$. (Figure 14 shows the case where $t=4$ and $k=4$. For each $k \ge 2$, each of the layers $L_2,\dots,L_k$ in $R_{v_0}(G)$ have the same cut pattern as this example.) We have:

$$d(v_{n-kt-1}, v_1) = d(v_{n-kt-1}, v_{n-1}) + 1,$$
$$d(v_{n-kt-2}, v_{kt+1}) = d(v_{n-kt-2}, v_{kt+2}) + 1.$$

So taking $W_1 \cup W_2$ resolves these sets. Thus $\beta(G) \le 2t-2 = 6$. □

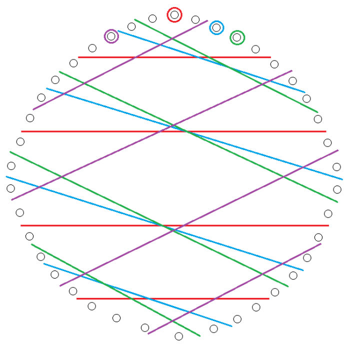

Fig. 14. $R_{W_1}(C_{40}(1,2,3,4))$

The following corollary comes from Theorems 2.18, 2.19, and 2.20.

Corollary 2.21. *If* $n \equiv 0 \bmod 2t$ *or* $n \equiv 1 \bmod 2t$ *where* $t \ge 4$ *then we have* $\beta(C_n(1,2,\dots,t)) \le 2t-2$.

Note that when $n \equiv 0 \bmod 2t$ and t is even, the bound $\beta(C_n(1,2,\dots,t)) \le \frac{3t}{2}$ presented in [15] is better (see Proposition 1.3). The next result follows from Theorems 2.17 and 2.20.

Corollary 2.22. *If* $n \equiv 1 \bmod 8$ *then* $\beta(C_n(1,2,3,4)) = 6$.

From previous results and from the empirical data in the Appendix, it seems that $\beta(C_n(1,2,\ldots,t))$ depends on the congruence class of n modulo $2t$. In the next three theorems, we show that this is indeed true for large enough n.

Theorem 2.23. *Let $G = C_n(1,2,\ldots,t)$ and $G' = C_{n+2t}(1,2,\ldots,t)$ where $n = 2tk+r$. If $k \geq 3$, then $\beta(G') \leq \beta(G)$.*

Proof. We obtain $C_{n+2t}(1,2,\ldots,t)$ from $C_n(1,2,\ldots,t)$ by adding one set of $2t$ vertices in L_i of $C_n(1,2,\ldots,t)$ for some $i \notin \{0,1,d\}$. We can assume that v_0 is in both W and W'. Notice that vertices $v \in L_j \cap W$ and the distinct cuts that are made in these L_j remain where they are in the L_j upon adding the $2t$ vertices to L_i. Observe that for every vertex $v \in L_j \cap W$, the distinct cuts that are made in the resolving hypergraph $R_{\{v,v_0\}}(G')$ are the same in L_i as they are in $R_{\{v,v_0\}}(G)$ for every $i \notin L_j \cap W$. So the vertices in $L_j \cap W$ will still leave the same distinct cuts in L_i if they did so before adding the $2t$ vertices (Fig. 15). Thus the vertices in $V(G')$ are resolved if they were resolved in $V(G)$ before adding the $2t$ vertices. □

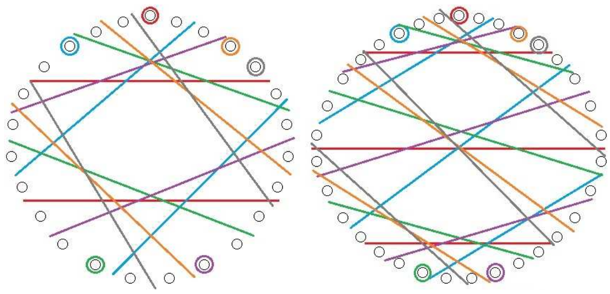

Fig. 15. $C_{25}(1,2,3,4)$ and $C_{33}(1,2,3,4)$ respectively

Theorem 2.24. *Let $G = C_n(1,2,\ldots,t)$ and $G' = C_{n+2t}(1,2,\ldots,t)$ where $n = 2tk + r$. If*

a) $2 \leq r \leq t+2$ *and* $k > t-1$ *or*
b) $t+3 \leq r \leq 2t-2$ *and* $k > r-3$ *or*
c) $2t-1 \leq r \leq 2t+1$ *and* $k > 2t-4$

then $\beta(G) \leq \beta(G')$.

Proof. Let W be a metric basis for G and W' be a metric basis for G' and suppose that v_0 is in both W and W'. We obtain $C_n(1,2,\ldots,t)$ from $C_{n+2t}(1,2,\ldots,t)$ by removing one set of $2t$ vertices in L_i of $C_{n+2t}(1,2,\ldots,t)$ where L_i does not contain vertices in the metric basis. We can guarantee that a L_i exists because the $k+1$ choices of sets of $2t$ vertices from L_j to remove from $V(G')$ is greater than our best known upper bound given a congruence class of n modulo $2t$. Notice that vertices $v' \in L_j \cap W'$ and the distinct cuts that are made in these L_j remain where they are upon removing the $2t$ vertices from L_i. Observe that for every vertex $v' \in L_j \cap W'$, the distinct cuts that are made in the resolving hypergraph $R_{\{v',v_0\}}(G)$ are the same in L_i as they are in $R_{\{v',v_0\}}(G')$ for every $i \notin L_j \cap W'$. So the vertices in $L_j \cap W'$ will still leave the same distinct cuts in L_i if they did so before removing the $2t$ vertices. Thus the vertices in $V(G)$ are resolved if they were resolved in $V(G')$ before removing the $2t$ vertices. □

The next corollary follows from Theorems 2.23 and 2.24.

Corollary 2.25. *Let $G = C_n(1, 2, \ldots, t)$ and $G' = C_{n+2t}(1, 2, \ldots, t)$ where $n = 2tk + r$ and $t \geq 4$. If*

a) $2 \leq r \leq t + 2$ *and* $k > t - 1$ *or*
b) $t + 3 \leq r \leq 2t - 2$ *and* $k > r - 3$ *or*
c) $2t - 1 \leq r \leq 2t + 1$ *and* $k > 2t - 4$

then $\beta(G) = \beta(G')$.

The following theorem shows that if the vertices in the metric basis belong only to L_0, L_1, or L_d, then $\beta(C_n(1, 2, \ldots, t)) = \beta(C_{n+2t}(1, 2, \ldots, t))$ when k is at least three. Although we could not prove that there always exists a metric basis where vertices belong to just L_0, L_1, or L_d, it may be possible to show that such a metric basis exists for certain t or certain congruence classes of n modulo $2t$, in which case this result could be of some use.

Theorem 2.26. *Let $G = C_n(1, 2, \ldots, t)$ and $G' = C_{n+2t}(1, 2, \ldots, t)$ where $n = 2tk+r$ and $k \geq 3$. If W is a metric basis for G and W' is a metric basis for G' where all vertices in W and W' belong to L_0, L_1, or L_d, then $\beta(G) = \beta(G')$.*

Proof. We obtain $C_n(1, 2, \ldots, t)$ from $C_{n+2t}(1, 2, \ldots, t)$ by removing one set of $2t$ vertices in L_i of $C_{n+2t}(1, 2, \ldots, t)$ for some $i \notin \{0, 1, d\}$. Similarly, we obtain $C_{n+2t}(1, 2, \ldots, t)$ from $C_n(1, 2, \ldots, t)$ by adding one set of $2t$ vertices in L_i of $C_n(1, 2, \ldots, t)$ for some $i \notin \{0, 1, d\}$. We can assume that v_0 is in both W and W'. Notice that since the vertices in the metric basis for G and G' remain where they are in L_0, L_1, or L_d, the separating lines made in L_0, L_1, and L_d remain where they are when we add (or remove) the $2t$ vertices in L_i for some $i \notin \{0, 1, d\}$. Observe that for every vertex $v \in L_1 \cap W$ (or $v' \in L_1 \cap W'$), the separating lines made in the resolving hypergraph $R_{\{v,v_0\}}(G)$ or $R_{\{v',v_0\}}(G')$ are the same within every L_i for $i \notin \{0, 1, d\}$. So the vertices in L_1 will still leave the same separating lines in L_i if they did so before adding (or removing) the $2t$ vertices. Similar arguments holds for each $v \in L_d \cap W$ (or $v' \in L_d \cap W'$). Thus the vertices in $V(G')$ are resolved if they were resolved in $V(G)$ before adding the $2t$ vertices and the vertices in $V(G)$ are resolved if they were resolved in $V(G')$ before removing the $2t$ vertices. □

3. THE METRIC DIMENSION OF CARTESIAN PRODUCTS OF CIRCULANT GRAPHS

We are motivated to study the metric dimension of Cartesian products of the circulant graphs $C_n(1, 2, \ldots, t)$ since their metric dimension is equal to the metric dimension of Cayley hypergraphs on finite Abelian groups. Let Γ be a group, let $\Omega \subseteq \Gamma \setminus \{1\}$, and let t be an integer such that $2 \leq t \leq \max\{|\omega| : \omega \in \Omega\}$. The *$t$-Cayley hypergraph of Γ over Ω*, denoted $H = t\text{-}Cay[\Gamma : \Omega]$, is the hypergraph with vertex set Γ in which a subset $S \subseteq \Gamma$ is in $E(H)$ if and only if there is $x \in \Gamma$ and $\omega \in \Omega$ such that $S = \{x\omega^i : 0 \leq i \leq t-1\}$. Note that a 2-Cayley hypergraph is a Cayley graph. This definition is due to Buratti [3], and is a subclass of the more general Cayley hypergraphs, or *group hypergraphs* which were defined by Shee in [12]. Specifically, we consider the t-Cayley hypergraph $H = t\text{-}Cay(\Gamma, \Omega)$ where Γ is a finite Abelian group, so we may assume the Γ is a direct product of cyclic groups of prime-power order, say $\Gamma = \mathbb{Z}_{n_1} \bigoplus \mathbb{Z}_{n_2} \bigoplus \ldots \bigoplus Z_{n_s}$ where n_i is a prime-power for $1 \leq i \leq s$. The canonical set of generators for this group is

$$\Omega = \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$$

and so we require $2 \leq t \leq \max\{n_i : 1 \leq i \leq s\}$. It was shown in [2] that this Cayley hypergraph has the same metric dimension as the Cartesian product

$$C_{n_1}(1, 2, \ldots, t-1) \square C_{n_2}(1, 2, \ldots, t-1) \square \cdots \square C_{n_s}(1, 2, \ldots, t-1),$$

and in this section we establish better bounds on their metric dimension. We start with a technical lemma.

Lemma 3.1. *Let $G = C_n(1, 2, \ldots, t)$ and let $v_0, v_p, v_q \in V(G)$. If $d(v_p, v_0) < d(v_q, v_0)$, then there exists a vertex $v_d \in V(G)$ at diameter distance from v_0 such that $d(v_p, v_d) > d(v_q, v_d)$.*

Proof. Let $d(v_p, v_0) < d(v_q, v_0)$. Then v_p and v_q must be in different L_j such that v_p is in a L_j closer to v_0. So take $v_p \in L_p$ and $v_q \in L_q$ for $1 \leq p < q \leq d$. Let $V_R = \{v_1, v_2, \ldots, v_{td-1}\}$ be the set of vertices on the right side of the outer cycle from v_0 and $V_L = \{v_{n-1}, v_{n-2}, \ldots, v_{n-td-1}\}$ be the set of vertices on the left side of the outer cycle from v_0.

Case 1. Assume $v_p, v_q \in V_R$ or $v_p, v_q \in V_L$. We consider only the former since the argument for the latter is similar. Then $d(v_p, v_{t(d-1)+1}) = d - p$ and $d(v_q, v_{t(d-1)+1}) = d - q$. Thus $d(v_p, v_{t(d-1)+1}) > d(v_q, v_{t(d-1)+1})$ since $p < q$.

Case 2. Assume $v_p \in V_R$ and $v_q \in V_L$. Then $d(v_p, v_{n-t(d-1)-1}) = d - p$ and $d(v_q, v_{n-t(d-1)-1}) = d - q$. Thus $d(v_p, v_{n-t(d-1)-1}) > d(v_q, v_{n-t(d-1)-1})$ since $p < q$. □

Theorem 3.2. *Let H be any graph and $G = C_n(1, 2, \ldots, t)$ where $n = 2tk + r$ for $1 \leq r \leq 2t$, then $\beta(H \square G) \leq \beta(H) + \max\{r, t+1\}$.*

Proof. Note that $H \square G$ consists of n copies of H labeled $H_1, H_2, \ldots, H_n$ where corresponding vertices in each copy form a copy of G. Let $W = \{w_1, w_2, \ldots, w_m\}$ be a resolving set for H. Now let $W_i = \{w_{1i}, w_{2i}, \ldots, w_{mi}\}$ be the set of vertices of H_i corresponding to W in H for $i \in \{1, 2, \ldots, n\}$. We show that either $W_1 \cup \{w_{1(tk+1)}, w_{1(tk+2)}, \ldots, w_{1(tk+r)}\}$ or $W_1 \cup \{w_{1(tk+1)}, w_{1(tk+2)}, \ldots, w_{1(tk+t+1)}\}$ is a resolving set for $G' = H \square G$. Let u and v be any two vertices of G'. For any $l \in \{1, 2, \ldots, n\}$, let $u_l, v_l \in G$ be the vertices that correspond to u and v in H_l, the l-th copy of H.

Case 1. Suppose u and v belong to the same H_i. Let u_1 and v_1 be the vertices of H_1 that correspond to u and v in H. Since u and v are in the same H_i, $d_{G'}(u,u_1) = d_{G'}(v,v_1)$. Since W resolves H, there is some $w_q \in W$ such that $d_{H_1}(u_1, w_{1q}) \neq d_{H_1}(v_1, w_{1q})$. But $d_{G'}(u, w_{1q}) = d_{G'}(u,u_1) + d_{G'}(u_1, w_{1q})$ and $d_{G'}(v, w_{1q}) = d_{G'}(v, u_1) + d_{G'}(v_1, w_{1q})$ so $d_{G'}(u, w_{1q}) \neq d_{G'}(v, w_{1q})$.

Case 2. Suppose u and v belong to different H_i. Let $u \in V(H_i)$ and $v \in V(H_j)$ for $1 \leq i < j \leq n$. Let $u_1, v_1 \in H_1$ and $u_k, v_k \in H_k$ be the vertices that correspond to u and v.

Case 2.1. Assume $d_{G'}(u,u_1) = d_{G'}(v,v_1)$ and $u_1 = v_1$. If we take a vertex from each copy of H_x for $x \in \{tk+1, tk+2, \ldots, \max\{tk+r, tk+t+1\}\}$, then u and v will be resolved since there are at least $t+1$ consecutive vertices in the set.

Case 2.2. Assume $1 \leq i < j \leq tk+r$ or $tk+r \leq i < j \leq n$ where $d_{G'}(u,u_1) \neq d_{G'}(v,v_1)$. We consider only the case where $1 \leq i < j \leq tk+r$ since the argument for $tk+r \leq i < j \leq n$ is similar. Then $d_{G'}(u,u_1) < d_{G'}(v,v_1)$ and $d_{G'}(u,u_l) > d_{G'}(v,v_l)$ for some $l \in \{tk+1, tk+2, \ldots, tk+r\}$ by Lemma 2. In the case that $d_{G'}(u, w_{11}) = d_{G'}(v, w_{11})$, we have $d_{G'}(u_1, w_{11}) + d_{G'}(u,u_1) = d_{G'}(v_1, w_{11}) + d_{G'}(v, v_1)$. Hence $d_{G'}(u_1, w_{11}) > d_{G'}(v_1, w_{11})$ and $d_{G'}(u_l, w_{1l}) > d_{G'}(v_l, w_{1l})$. Thus $d_{G'}(u, w_{1l}) = d_{G'}(u,u_l) + d_{G'}(u_l, w_{1l}) > d_{G'}(v, v_l) + d_{G'}(v_l, w_{1l}) = d_{G'}(v, w_{1l})$. So w_{1l} resolves u and v if w_{11} does not.

Case 2.3. Assume $1 \leq i \leq tk+r < j \leq n$ where $d_{G'}(u,u_1) \neq d_{G'}(v,v_1)$. In this case, there exists a $l \in \{tk+1, tk+2, \ldots, tk+r\}$ such that there exists a shortest path from u_1 to $u_l \in G_l$ which passes through u and has length $d-1$ and there exists a shortest path from v_1 to v_l which passes through v and has length d. Then the sum of the coordinates of u with respect to w_{11}, w_{1l} will always have the same parity as $d-1$ and the sum of the coordinates v with respect to w_{11}, w_{1l} will always have the same parity as d. Thus the set $\{w_{11}, w_{1l}\}$ resolves u and v. $\square$

It is not difficult to prove that $\max\{\beta(H), \beta(G)\} \leq \beta(H \Box G)$, and it was shown in [4] as part of another result. Thus the following corollary comes from repeated application of Theorem 3.2.

Corollary 3.3. *Let r_i correspond to $C_{n_i}(1,2,\ldots,t_i)$ for $n_i = 2t_ik_i + r_i$ with and $i \in \{1,2,\ldots,m\}$. Let $r_1 \geq r_2 \geq \ldots \geq r_m$ and set*

$$H = C_{n_1}(1,2,\ldots,t_1) \Box C_{n_2}(1,2,\ldots,t_2) \Box \ldots \Box C_{n_m}(1,2,\ldots,t_m),$$

then

$$\max\{\beta(C_{n_i}(1,2,\ldots,t_i))\} \leq \beta(H) \leq \beta(C_{n_1}(1,2,\ldots,t_1)) + \sum_{i=2}^{m} \max\{r_i, t+1\}.$$

4. SUMMARY

Here we state the bounds on the metric dimension of circulant graphs introduced in this paper. Let $G = C_n(1,2,\ldots,t)$ where $n = 2tk+r$, $t \geq 4$ and $k \geq 2$.

1. If $n \equiv r \bmod 2t$ where $t+1 \le r \le 2t+2$ then $\beta(G) \ge t+1$ (Theorem 2.7).
2. If $n \equiv r \bmod 2t$ where $3 \le r \le t$ then $\beta(G) \ge t$ (Theorem 2.15).
3. If $n \equiv r \bmod 2t$ where $r = 2, t+1, t+2$ then $\beta(G) = t+1$ (Theorem 2.7).
4. If $n \equiv 0 \bmod 2t$ where t is odd then $t+1 \le \beta(G) \le 2t-2$ (Theorems 2.7, 2.19, and 2.20).
5. If $n \equiv 1 \bmod 2t$ then $t+2 \le \beta(G) \le 2t-2$ (Theorems 2.17 and 2.18).
6. If $n \equiv 1 \bmod 8$ where $t = 4$ then $\beta(G) = 6$ (Theorems 2.17 and 2.18).
7. If $n \equiv (t+3) \bmod 2t$ where t is odd then $\beta(G) = t+1$ (Theorems 2.7 and 2.9).

Each of the tables in the Appendix of this paper list the metric dimension β of the circulant $C_n(1, 2, \ldots, t)$ where $n \equiv r \bmod 2t$, for a given range of n. The authors would like to thank Robert Bailey for computing these values using a program in GAP. We make the following conjecture based on the data found in the Appendix.

Conjecture 4.1. Let $G = C_n(1, 2, \ldots, t)$ where $n = 2tk + r$, $t \ge 4$ and $k \ge 2$.

1. If $n \equiv t \bmod 2t$ where t is odd then $\beta(G) = t+1$.
2. If $n \equiv 0 \bmod 2t$ then $\beta(G) \ge t+2$.
3. If $n \equiv (t+3) \bmod 2t$ where t is even then $\beta(G) = t+2$.
4. If $n \equiv r \bmod 2t$ where t is even, $3 \le r \le t-1$, and $k = 1$ then $\beta(G) = t$.

A. APPENDIX

Table 1. The metric dimension of $C_n(1, 2, 3, 4)$, $10 \le n \le 26$

r	**2**	**3**	**4**	**5**	**6**	**7**	**0**	**1**
β	5	4	4	5	5	6	6	6

Table 2. The metric dimension of $C_n(1, 2, 3, 4)$, $27 \le n \le 120$

r	**2**	**3**	**4**	**5**	**6**	**7**	**0**	**1**
β	5	5	4	5	5	6	6	6

Table 3. The metric dimension of $C_n(1, 2, \ldots, 5)$, $12 \le n \le 21$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**0**	**1**
β	6	5	6	6	6	6	6	7	7	8

Table 4. The metric dimension of $C_n(1, 2, \ldots, 5)$, $22 \le n \le 90$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**0**	**1**
β	6	6	6	6	6	6	6	6	7	8

Table 5. The metric dimension of $C_n(1, 2, \ldots, 6)$, $14 \leq n \leq 25$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**0**	**1**
β	7	6	6	6	6	7	7	8	8	8	8	9

Table 6. The metric dimension of $C_n(1, 2, \ldots, 6)$, $26 \leq n \leq 61$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**0**	**1**
β	7	6	6	6	6	7	7	8	7	8	8	9

Table 7. The metric dimension of $C_n(1, 2, \ldots, 6)$, $62 \leq n \leq 84$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**0**	**1**
β	7	7	6	6	6	7	7	8	7	8	8	9

Table 8. The metric dimension of $C_n(1, 2, \ldots, 7)$, $16 \leq n \leq 29$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**0**	**1**
β	8	7	7	7	7	8	8	8	8	9	9	10	10	10

Table 9. The metric dimension of $C_n(1, 2, \ldots, 7)$, $30 \leq n \leq 43$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**0**	**1**
β	8	7	7	8	7	8	8	8	8	9	8	9	10	10

Table 10. The metric dimension of $C_n(1, 2, \ldots, 7)$, $44 \leq n \leq 68$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**0**	**1**
β	8	8	8	8	8	8	8	8	8	9	8	9	10	10

Table 11. The metric dimension of $C_n(1, 2, \ldots, 8)$, $18 \leq n \leq 33$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**0**	**1**
β	9	8	8	8	8	8	8	9	9	10	10	10	10	11	11	12

Table 12. The metric dimension of $C_n(1, 2, \ldots, 8)$, $34 \leq n \leq 48$

r	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**0**
β	9	8	8	8	8	9	8	9	9	10	10	10	10	10	11

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