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Positivity and stability of fractional discrete-time linear systems. The model without a time shift in the difference*

by

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Abstract: The positivity and stability of fractional discrete-time linear systems described by a new model are addressed. Necessary and sufficient conditions for the positivity and asymptotic stability of the systems are established. New tests for checking of the stability are proposed.

Keywords: positivity, stability, fractional discrete–time linear system

1. Introduction

Having a good mathematical model is very important now in technological domain. Practical applications resulted in much greater importance of the discrete-time models. Models using the fractional-order derivatives or differences are much more accurate (see Wei et al., 2016). These models, initially used for modeling of dynamic systems with so called "memory", currently are applied in modelling of various types of systems. One important class of dynamical systems is characterized by only positive signals. Thus, a dynamical system is called positive if its trajectory, starting from any nonnegative initial state, remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs of Farina and Rinaldi (2000) and Kaczorek (2002). A variety of models, having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. Analysis of positive and fractional electrical circuits has been addressed in Aracena, Demongeot and Goles (2004), Kaczorek (2011a,c; 2013a), Kaczorek and Rogowski (2015), Richard (2009). Decoupling zeros of positive electrical circuits have been introduced in Kaczorek (2013b). Fractional linear

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systems and electrical circuits have been investigated in Kaczorek (2008, 2010, 2011b,c), and singular fractional systems and electrical circuits in Kaczorek (2011d). Robust stability of fractional positive discrete-time linear systems has been analyzed in Buslowicz (2010).

In the fractional-order discrete-time mathematical system description, we can use two models: with and without a discrete-time shift in the fractional-order difference (Kaczorek and Ostalczyk, 2016). In this paper, the novelty consists in the fact that positivity and asymptotic stability conditions of the fractional discrete-time linear systems, described by the two mentioned models are investigated. Necessary and sufficient conditions for the positivity and asymptotic stability of the systems are established. New tests for checking of the stability are proposed.

The paper is organized as follows. In Section 2 the positivity of the fractional systems is analyzed. The asymptotic stability of the systems is addressed in Section 3. The theoretical results are supported by numerical examples. Concluding remarks are given in Section 4.

The following notation will be used: $\mathbb{R}$ - the set of real numbers, $\mathbb{R}^{n\times m}$ - the set of real matrices $n\times m$, $\mathbb{R}_+^{n\times m}$ - the set of real matrices with nonnegative entries, $\mathbb{Z}_+$ - the set of nonnegative integers, M_n- the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

2. Positivity of the fractional models

In this paper the following Grünwald-Letnikov fractional-order difference of $x(k)$, see Ostalczyk (2008)

$$\Delta^\alpha x(k) = \sum_{i=0}^{k} c_i(\alpha)x(k-i)\,,\ k\in\mathbb{Z}=\{0,1,\ldots\}\,, \tag{1}$$

where

$$c_i(\alpha)=\begin{cases} 0 & \text{for} \quad i<0\,,\\ 1 & \text{for} \quad i=0\,,\\ (-1)^i\frac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!} & \text{for} \quad i>0\,,\end{cases}$$

is used. Consider the fractional discrete-time linear system

$$\Delta^\alpha x(k) = Ax(k)+Bu(k),\ \ k\in\mathbb{Z}_+=\{0,1,\ldots\},\ 0<\alpha\le 1, \tag{2a}$$

$$y(k) = Cx(k)+Du(k), \tag{2b}$$

where $x(k)\in\mathbb{R}^n$, $u(k)\in\mathbb{R}^m$, $y(k)\in\mathbb{R}^p$ are the state, input and output vectors, respectively, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times m}$. One should mention that the above model differs from the commonly analyzed model with a discrete-time shift (see Kaczorek, 2011c, or Stanislawski, 2013).

$$\begin{aligned}\Delta^\alpha x(k+1) &= Ax(k)+Bu(k)\,,\ k\in\mathbb{Z}_+=\{0,1,\ldots\}\,,0<\alpha\le 1\,, && \text{(3a)}\\ y(k) &= Cx(k)+Du(k)\,. && \text{(3b)}\end{aligned}$$

THEOREM 1 *The solution of the equation (3a) for given initial condition x_{-j}, $j=1,2,\ldots$ and input $u(k)$, $k=0,1,\ldots$ has the form*

$$x(k)=-\sum_{j=1}^{\infty}\sum_{i=0}^{k}c_{k+j-1}(\alpha)\Phi(i)x_{-j}+\sum_{i=0}^{k}\Phi(i)Bu(k-i)\,,\ k=1,2,\ldots\,, \tag{4}$$

where

$$\Phi(k)=\begin{cases}[I_n-A]^{-1} & \textit{for}\quad k=0\,,\\ [I_n-A]^{-1}\sum\limits_{j=1}^{k}c_j(\alpha)\Phi(k-j) & \textit{for}\quad k=1,2,\ldots\end{cases} \tag{5}$$

PROOF. The proof is given in Ostalczyk (2008). ■

From (1) and (2a) it follows that $x(0)=Ax(0)+Bu(0)$ and

$$x(0)=[I_n-A]^{-1}Bu(0)=\Phi(0)Bu(0) \tag{6}$$

since $c_0(\alpha)=1$. Therefore, $x(0)$ and $u(0)$ are related by (6). In a particular case, for $x_{-j}=0$, $j=2,3,\ldots$ from (4), after shifting i by one, we obtain the solution of (3a) for given $x(0)$ and $u(k)$ in the form

$$x(k)=\Phi(k)x(0)+\sum_{i=1}^{k}\Phi(k-i)\,[I_n-A]^{-1}Bu(i)\,, \tag{7}$$

where

$$\Phi(k)=[I_n-A]^{-1}\sum_{j=1}^{k}c_j(\alpha)\Phi(k-j)\,,\ \Phi(0)=I_n\,. \tag{8}$$

LEMMA 1 *The matrix*

$$\Phi(0)=[I_n-A]^{-1}\in\mathbb{R}_+^{n\times n} \tag{9}$$

if and only if the matrix $A\in\mathbb{R}_+^{n\times n}$ of the positive system

$$x(k+1)=Ax(k) \tag{10}$$

is asymptotically stable.

PROOF. It is well-known, see Kaczorek (2002), that the discrete-time linear system (10) is positive if and only if the matrix $A \in \mathbb{R}_+^{n\times n}$. The positive system (10) is asymptotically stable if and only if the continuous-time linear system

$$\dot{x} = (A - I_n)x \tag{11}$$

is asymptotically stable. If the continuous-time system (11) is asymptotically stable then the condition (9) is satisfied, see Kaczorek (2002). ■

EXAMPLE 1 *It is easy to check that the discrete-time system (10) with the matrix*

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} \in \mathbb{R}_+^{2\times 2} \tag{12}$$

is asymptotically stable and positive. The characteristic polynomial of (12) has the form

$$\det[I_2 z - A] = \begin{vmatrix} z-0.4 & -0.1 \\ -0.2 & z-0.3 \end{vmatrix} = z^2 - 0.7z + 0.1 \tag{13}$$

and its zeros are $z_1 = 0.2$, $z_2 = 0.5$. *The corresponding continuous-time system (11)*

$$A - I_2 = \begin{bmatrix} -0.6 & 0.1 \\ 0.2 & 0.7 \end{bmatrix} \in M_2 \tag{14}$$

is also asymptotically stable since the characteristic polynomial

$$\det[I_2 s - A + I_2] = \begin{bmatrix} s+0.6 & -0.1 \\ -0.2 & s+0.7 \end{bmatrix} = s^2 + 1.3s + 0.4 \tag{15}$$

has positive coefficients and its zeros are $s_1 = -0.8$, $s_2 = -0.5$. *In this case we have*

$$\Phi(0) = \begin{bmatrix} 0.6 & -0.1 \\ -0.2 & 0.7 \end{bmatrix}^{-1} = \frac{1}{0.4}\begin{bmatrix} 0.7 & 0.1 \\ 0.2 & 0.6 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}. \tag{16}$$

DEFINITION 1 *The fractional system (3) is called (internally) positive if* $x(k) \in \mathbb{R}_+^n$ *and* $y(k) \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$ *for any initial conditions* $x_{-j} \in \mathbb{R}_+^n$, $j = 1, 2, \ldots$ *and all inputs* $u(k) \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$.

THEOREM 2 *The fractional system (3) is positive if and only if*

$$A \in \mathbb{R}_+^{n\times n} \text{ is asymptotically stable and } B \in \mathbb{R}_+^{n\times m},\ C \in \mathbb{R}_+^{p\times n},\ D \in \mathbb{R}_+^{p\times m}. \tag{17}$$

PROOF. *Sufficiency.* By Lemma 1 $\Phi(0) \in \mathbb{R}_+^{n\times n}$ if $A \in \mathbb{R}_+^{n\times n}$ is asymptotically stable. From (2) it follows that for $0 < \alpha \leq 1$ we have $c_i^\alpha < 0$ for $i = 1, 2, \ldots$. Therefore, for $x_{-j} \in \mathbb{R}_+^n$, $j = 1, 2, \ldots$ we have

$$-\sum_{j=1}^{\infty}\sum_{i=0}^{k} c_{k+j-1}(\alpha)\Phi(i)x_{-j} \in \mathbb{R}_+^n \text{ for } k = 1, 2, \ldots \tag{18}$$

and $x(k) \in \mathbb{R}_+^n$, $k = 1, 2, \ldots$, since by (5) $\Phi(0) \in \mathbb{R}_+^n$, $B \in \mathbb{R}^{n\times m}$ and $u(i) \in \mathbb{R}_+^m$ for $i \in \mathbb{Z}_+$.
Necessity. From (6) for $u(0) = 0$ we have $x(0) = 0$. Assuming zero initial conditions $x_{-j} = 0$, $j = 1, 2, \ldots$, $u(1) \in \mathbb{R}_+^m$ from (4) we obtain $x(1) = \Phi(0)Bu(1) \in \mathbb{R}_+^n$ only if $\Phi(0) \in \mathbb{R}_+^{n\times n}$ since $Bu(1)$ can be arbitrary. Therefore, $A \in \mathbb{R}_+^{n\times n}$ should be, by Lemma 1, asymptotically stable. If $u(1) = 0$ then from (3b) we have $y(1) = Cx(1)$ and $C \in \mathbb{R}_+^{p\times n}$ since $x(1) \in \mathbb{R}_+^n$ can be arbitrary. Similarly, if $x(1) = 0$ then $y(1) = Du(1) \in \mathbb{R}_+^p$ and $D \in \mathbb{R}_+^{p\times m}$ since $u(1) \in \mathbb{R}_+^m$ can be arbitrary. ■

REMARK 1 *1. It is well known (see Kaczorek, 2011c) that if the fractional discrete-time linear system is described by the equation (3a) instead of (2a) then it is positive if and only if*

$$A \in \mathbb{R}_+^{n\times n}, \quad B \in \mathbb{R}_+^{n\times m}. \tag{19}$$

From the comparison of models we conclude that the necessary and sufficient conditions for the positivity of the model (2) are stronger, i.e. the matrix $A \in \mathbb{R}_+^{n\times n}$ should be asymptotically stable.

EXAMPLE 2 *Consider the models described by the equations (2) and (3) with the matrices*

$$A = \begin{bmatrix} 1 & 2 \\ 0.5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{20}$$

The model (3a) with (20) is positive since the condition (19) is satisfied. The matrix A, defined by (20), has positive entries, but it is not asymptotically stable, since the characteristic polynomial

$$\det[I_2 s - A + I_2] = \begin{vmatrix} s & -2 \\ -0.5 & s-1 \end{vmatrix} = s^2 - s - 1 \tag{21}$$

has negative coefficients. In this case we have

$$\Phi(0) = [I_2 - A]^{-1} = \begin{bmatrix} 0 & -2 \\ -0.5 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ -0.5 & 0 \end{bmatrix} \tag{22}$$

and the condition (9) is not satisfied. Therefore, the model (3a) with (21) is not positive.

3. Stability of the fractional positive systems

Consider the system (3a) for $u(k) = 0$, $k \in \mathbb{Z}_+$, i.e.

$$\Delta^\alpha x(k+1) = Ax(k)\,, \tag{23}$$

which, using (1), can be rewritten in the form

$$x(k) = \sum_{i=1}^{k} \overline{A} x(k-i)\,, \tag{24}$$

where

$$\overline{A} = -\left[I_n - A\right]^{-1} c_i(\alpha)\,. \tag{25}$$

The solution of (24)-(25) for given initial condition $x(0) \in \mathbb{R}_+^n$ has the form

$$x(k) = \Phi(k)x(0)\,, \tag{26}$$

where

$$\Phi(k) = -\left[I_n - A\right]^{-1} \sum_{i=1}^{k} c_i(\alpha)\Phi(k-i),\ \Phi(0) = I_n\,. \tag{27}$$

DEFINITION 2 *The positive system (23) is called asymptotically stable if*

$$\lim_{k\to\infty} x(k) = 0 \textit{ for all } x(0) \in \mathbb{R}_+^n\,. \tag{28}$$

LEMMA 2 *The positive system (23) is asymptotically stable if*

$$\lim_{k\to\infty} \left(\left[I_n - A\right]^{-1}\right)^k = 0\,. \tag{29}$$

PROOF. Note that (27) can be rewritten in the form

$$\begin{aligned}\Phi(k) =& A_k^{-1}\left\{I_n + \frac{\alpha-1}{2!}\left[I_n - A\right] + \cdots + \frac{(\alpha-1)\ldots(\alpha-k+1)}{k!}\left[I_n - A\right]^{k-1}\right\}\\ &\text{for } k = 1, 2, \ldots\,,\end{aligned} \tag{30}$$

and where $A_k^{-1} = \left(-\left[I_n - A\right]^{-1}\right)^k$. The positive system (3.1) is asymptotically stable if, by Definition 2, $\lim_{k\to\infty} \Phi(k) = 0$ and this condition is satisfied if (29) holds. ■

THEOREM 3 *The fractional positive system (23) is asymptotically stable if*

$$\left|\frac{\alpha}{1-\lambda_{max}}\right| < 1\,, \tag{31}$$

where λ_{max} is the maximal eigenvalue of the matrix A.

PROOF. It is well known (see Gantmacher, 1959) that if λ_{max}, $k = 1, \ldots, n$ are the eigenvalues of the matrix A, then the eigenvalues of the matrix $[I_n - A]^{-1} \alpha$ are $(1 - \lambda_k)^{-1} \alpha$ for $k = 1, \ldots, n$. The positive system (23) is asymptotically stable if, by Definition 2, $\lim_{k\to\infty} \Phi(k) = 0$ and from (29) it follows that this condition is satisfied if (31) holds. ∎

If the eigenvalues of the matrix are real, then from (31) we have the condition

$$\alpha - 1 < \lambda_{max} < 1 - \alpha\,. \tag{32}$$

EXAMPLE 3 *(Continuation of Example 1) Consider the positive system with the matrix (12). The condition (32) for $\alpha = 0.5$ is not satisfied. It is satisfied for $\alpha = 0.4$, since $\lambda_{max} = 0.5 < 0.6$. Therefore, the positive fractional system with the matrix (12) and $\alpha = 0.4$ is asymptotically stable.*

By Definition 2 and (26) the positive system (23) is asymptotically stable if and only if

$$\lim_{k\to\infty} \Phi(k) = \lim_{k\to\infty} [I_n - A]^{-1} \sum_{j=1}^{k} c_j(\alpha)\Phi(k-j) = 0\,. \tag{33}$$

Therefore, from (33) we have

$$1 - \frac{1}{1 - \lambda_i(A)} \sum_{j=1}^{\infty} c_j(\alpha) z^{-j} = 0 \tag{34}$$

and

$$1 - \lambda_i(A) - \sum_{j=1}^{\infty} c_j(\alpha) z^{-j} = 0\,, \tag{35}$$

where $\lambda_i(A)$, $i = 1, \ldots, n$ are the eigenvalues of the matrix A. It is well known (see Kaczorek, 2011c) that

$$(z-1)^{\alpha} = z^{\alpha} - \sum_{j=1}^{\infty} c_j(\alpha) z^{\alpha-j} = 0\,. \tag{36}$$

Upon multiplying (36) by $z^{-\alpha}$ we obtain

$$\sum_{j=1}^{\infty} c_j(\alpha) z^{-j} = 1 - (z-1)^{\alpha} z^{-\alpha}\,. \tag{37}$$

Substitution of (37) into (35) yields

$$\lambda_i(A) = (z-1)^{\alpha} z^{-\alpha}\,. \tag{38}$$

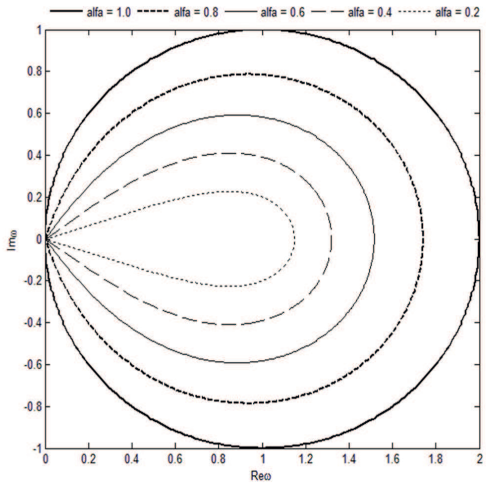

Figure 1. Stability regions for different fractional orders

From (38) for $\eta = \lambda_i(A)$ and $z = e^{j\omega}$, $\omega \in [0, 2\pi]$ we have

$$\eta(\omega) = \left(e^{j\omega} - 1\right)^{\alpha} \left(e^{j\omega}\right)^{-\alpha} \text{ for } \omega \in [0, 2\pi]\,. \tag{39}$$

Note that the closed curve (39) divides the complex η–plane into two regions. Let $S(\alpha)$ be the bounded region inside the closed curve.

Therefore, the following theorem has been proved.

Theorem 4 *The positive fractional system (23) is asymptotically stable if and only if the condition*

$$eta \in S(\alpha) \tag{40}$$

is satisfied.

Using (39) the regions for different values of α can be computed.

Example 4 *(Continuation of Example 3) It was shown in Example 3 that the positive system with the matrix (12) and $\alpha = 0.4$ is asymptotically stable. From Fig. 1 it follows also that the condition (40) is satisfied and the system is asymptotically stable.*

Using (26)- (27) we can compute the solution of the system. From Fig. 2 it follows that the system is asymptotically stable. Analogous stability regions evaluated for model (3a) can be found in Stanislawski (2013).

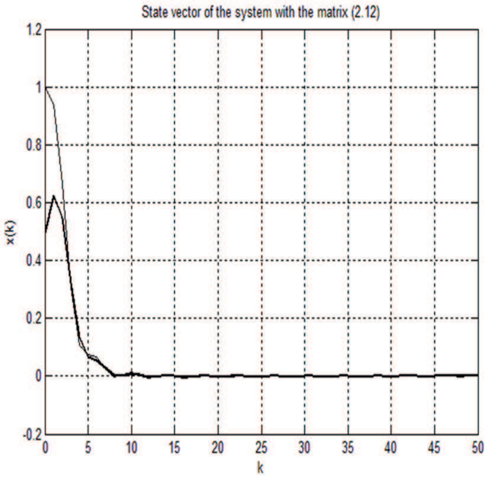

Figure 2. State vector of the positive system with the matrix (12)

4. Concluding remarks

The positivity and stability of the fractional discrete-time linear systems, described by the new model have been addressed. Necessary and sufficient conditions for the positivity of the systems have been established (Theorem 2). Sufficient conditions (Theorem 3) and necessary and sufficient conditions for stability (Theorem 4) have been also proposed. The respective considerations have been illustrated by numerical examples. The respective considerations can be easily extended to descriptor fractional positive discrete-time linear systems.

References

Aracena J., Demongeot J. and Goles E. (2004) Positive and negative circuits in discrete neural networks. *IEEE Trans. Neural Networks*, **15** (1), 77–83.

Busłowicz M. (2010) Robust stability of positive discrete-time linear systems of fractional order. *Bull. Pol. Acad. Sci. Techn.*, **58** (4), 567–572.

Farina L. and Rinaldi S. (2000) *Positive Linear Systems; Theory and Applications*. J. Wiley, New York

Gantmacher F.R. (1959) *The Theory of Matrices*. Chelsea Pub. Comp., London.

Kaczorek T. (2013) Constructibility and observability of standard and positive electrical circuits. *Electrical Review*, **89**(7), 132–136.

Kaczorek T. (2013) Decoupling zeros of positive electrical circuits. *Archives of Electrical Engineering*, **62** (4), 553–568.

Kaczorek T. (2008) Fractional positive continuous-time linear systems and their reachability. *Int. J. Appl. Math. Comput. Sci.*, **18** (2), 223-228.

Kaczorek T. (2002) *Positive 1D and 2D Systems*. Springer-Verlag, London.

Kaczorek T. (2010) Positive linear systems with different fractional orders. *Bull. Pol. Acad. Sci. Techn.*, **58** (3), 453–458.

Kaczorek T. (2011a) Positive electrical circuits and their reachability. *Archives of Electrical Engineering*, **60** (3), 283–301.

Kaczorek T. (2011b) Positive linear systems consisting of n subsystems with different fractional orders. *IEEE Trans. Circuits and Systems*, **58** (6), 1203–1210.

Kaczorek T. (2011c) *Selected Problems of Fractional Systems Theory*. Springer-Verlag, Berlin.

Kaczorek T. (2011d) Singular fractional linear systems and electrical circuits. *Int. J. Appl. Math. Comput. Sci.*, **21** (2), 379–384.

Kaczorek T. and Ostalczyk P. (2016) Responses comparison of the Two Discrete-Time Linear Fractional State-Space Models. *Fractional Calculus and Applied Analysis*, **19** (4), 789–805.

Kaczorek T. and Rogowski K. (2015) Fractional Linear Systems and Electrical Circuits. *Studies in Systems, Decision and Control*, **13**, Springer.

Ostalczyk P. (2008) *Epitome of the Fractional Calculus: Theory and its Applications in Automatics*. Publishing Department of Technical University of Lodz, Łódź (in Polish).

Richard A. (2009) Positive circuits and maximal number of fixed points in discrete dynamical systems. *Discrete Applied Mathematics*, **157**, 3281–3288.

Stanisławski R. (2013) Advances in modeling fractional difference systems - new accuracy, stability and computational results. *Studia i monografie*, issue 343, Politechnika Opolska, Opole.

Wei R., Tse P.W., Du B., Wang Y. (2016) An innovative fixed-pole numerical approximation for fractional order systems. *ISA Transactions*, **62**, May 2016, 94–102.