BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 62, No. 2, 2014
DOI: 10.2478/bpasts-2014-0020

Dedicated to Professor J. Klamka
on the occasion of his 70th birthday

Existence and approximate controllability of Sobolev type fractional stochastic evolution equations

N.I. MAHMUDOV*

Department of Mathematics, Eastern Mediterranean University, Gazimagusa, TRNC, Mersin10, Turkey

Abstract. We study the existence of mild solutions and the approximate controllability concept for Sobolev type fractional semilinear stochastic evolution equations in Hilbert spaces. We prove existence of a mild solution and give sufficient conditions for the approximate controllability. In particular, we prove that the fractional linear stochastic system is approximately controllable in $[0, b]$ if and only if the corresponding deterministic fractional linear system is approximately controllable in every $[s, b]$, $0 \leq s < b$. An example is provided to illustrate the application of the obtained results.

Key words: approximate controllability, Sobolev type fractional stochastic evolution equations.

1. Introduction

Many social, physical, biological and engineering problems can be described by fractional partial differential equations. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations. In the last two decades, fractional differential equations (see, for example, Samko et al [1] and references therein) has attracted many scientists, and notable contributions have been made to both theory and applications of fractional differential equations. Recently, the existence of mild solutions, stability and (approximate) controllability of (fractional) semilinear evolution system in Banach spaces have been reported by many researchers, see [2–32]. The approximate controllability of systems represented by nonlinear evolution equations has been investigated by several authors, in which the authors effectively used the fixed point approach.

The Sobolev type (fractional) equation appears in a variety of physical problems such as flow of fluid through fissured rocks, thermodynamics, propagation of long waves of small amplitude and shear in second order fluids and so on. Brill [33] and Showalter [34] established the existence of solutions of semilinear Sobolev type evolution equations in Banach space. There is an extensive literature in which Sobolev type of equations are investigated, in the abstract framework. Moreover, the fractional differential equations of Sobolev type appear in the theory of control of dynamical systems, when the controlled system or/and the controller is described by a fractional differential equation of Sobolev type. Complete controllability of evolution systems of Sobolev type in Banach spaces have been studied by Balachandran and Dauer [16], Ahmed [24], Feckan et al. [25]. Besides, noise or stochastic perturbation is unavoidable and omnipresent in nature as well as in man-made systems. Therefore, it is great significance to import the stochastic effects into the investigation of fractional differential systems. of great significance to import the stochastic effects into the investigation of fractional differential systems. Up to now, there is no work reported on existence and approximate controllability of fractional evolution equations of Sobolev type. Motivated by this fact, in this paper, we make an attempt to fill this gap by studying the existence and approximate controllability of Sobolev-type fractional stochastic differential systems in Hilbert spaces.

The rest of the paper is organized as follows. In Sec. 2 we recall some basic definitions and results from the stochastic analysis and the theory of fractional calculus. In Sec. 3, we discuss existence of the mild solution of fractional Sobolev type stochastic evolution equations in Hilbert spaces. In Sec. 4 the concept of approximate controllability is discussed. We prove that the fractional linear stochastic system is approximately controllable on $[0, b]$ if and only if the corresponding deterministic fractional linear system is approximately controllable on every $[s, b]$, $0 \leq s < b$. Moreover, we give sufficient condition for the approximate controllability of the fractional Sobolev type semilinear stochastic differential equation in Hilbert spaces. Finally, in Sec. 5, an example is provided to illustrate the applications of the obtained results.

2. Preliminaries

In this section, we shall recall notations, some basic definitions and lemmas from the stochastic analysis and the fractional calculus theory which will be used in the main results [1, 35].

- Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t \uparrow\subset \mathfrak{F}, t \geq 0\}, P)$ denote a complete probability space equipped with a family of nondecreasing subsigma algebras. Let $\mathbb{E}\{\cdot\}$ denote the integration with respect to the measure P. All random processes considered in the paper are assumed to be strongly $\mathfrak{F}_t$-progressively measurable processes unless stated otherwise. Let E be a separable Hilbert space, $\{w(t), t \geq 0\}$ be a Wiener process with values in E with covariance operator Q, where Q

*e-mail: nazim.mahmudov@emu.edu.tr

is a positive nuclear operator in E. We assume that there exists a complete orthonormal system $\{e_k\}$ in E, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, ...$ and a sequence $\{\beta_k\}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \langle e_k, e\rangle \beta_k(t), \ e \in E, \ t \geq 0.$$

Further we assume that $\mathfrak{F}_t$ is generated by $\{w(s) : 0 \leq s \leq t\}$.

- Let $L_2^0 := L_2\left(Q^{1/2}E, H\right)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}E$ to H. The space L_2^0 is a separable Hilbert space, equipped with the norm $\|\Psi\|_{L_2^0}^2 = tr\left[\Psi Q \Psi^*\right]$.
- We use $L_2(\mathfrak{F}, H)$, to denote the Banach space of strongly $\mathfrak{F}$-measurable, H-valued random variables satisfying $\mathbb{E}\|x\|^2 < \infty$. Since for each $t \geq 0$ the subsigma algebras $\mathfrak{F}_t$ are complete, $L_2(\mathfrak{F}_t, H)$ are closed subspaces of $L_2(\mathfrak{F}, H)$ and hence they are also Banach spaces. Similarly, $L_2^{\mathfrak{F}}([0, b], H)$ will denote the Banach space of $\mathfrak{F}_t$-progressively measurable random processes defined in $[0, b]$, taking values from X satisfying $\mathbb{E}\int_0^b \|x(t)\|_H^2 dt < \infty$.
- $C([0, b], L_2(\mathfrak{F}, H))$ is the Banach space of continuous maps from $[0, b]$ into $L_2(\mathfrak{F}, H)$ satisfying the condition $\sup_{t\in[0,b]} \mathbb{E}\|x(t)\|^2 < \infty$. $\mathfrak{H}_2$ is the closed subspace of $C([0, b], L_2(\mathfrak{F}, H))$ consisting of measurable and $\mathfrak{F}_t$-adapted processes $x(t)$. Then $\mathfrak{H}_2$ is a Banach space with the norm topology given by $\|x\|_{\mathfrak{H}_2} = \left(\sup_{t\in[0,b]} \mathbb{E}\|x(t)\|^2\right)^{1/2}$.
- $B_r := \left\{x \in \mathfrak{H}_2 : \|x\|_{\mathfrak{H}_2}^2 \leq r\right\}$.

The operators $A : D(A) \subset H \to H$ and $C : D(C) \subset H \to H$ satisfy the following hypotheses:

(S$_1$) A and C are linear operators, and A is closed.
(S$_2$) $D(C) \subset D(A)$ and C is bijective,
(S$_3$) $C^{-1} : H \to D(C)$ is compact.

The hypotheses (S$_1$)-(S$_3$) and the closed graph theorem imply the boundedness of the linear operator $AC^{-1} : H \to H$. Consequently, $-AC^{-1}$ generates a semigroup $\{S(t); t \geq 0\}$ in H. There exists a positive constant M such that $\|S(t)\| \leq M$, for all $0 \leq t \leq b$.

Let us recall the following known definitions in fractional calculus. For more details, see [1].

Definition 1. The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ t > 0, \alpha > 0,$$

provided the right-hand side is pointwise defined in $[0, \infty)$, where Γ is the gamma function.

Definition 2. The Riemann-Liouville derivative of order α with the lower limit 0 for a function $f : [0, \infty) \to R$ can be written as

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds,$$

$$t > 0, \quad n-1 < \alpha < n.$$

Definition 3. The Caputo derivative of order α for a function $f : [0, \infty) \to R$ can be written as

$$^C D^\alpha f(t) =^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right),$$

$$t > 0, \quad n-1 < \alpha < n.$$

In order to explain our theorem, we need the following assumptions.

(H$_1$) $f : [0, b] \times H \to H$ satisfies the following
(a) $f(t, \cdot) : H \to H$ is continuous for each $t \in [0, b]$ and for each $x \in H$, $f(\cdot, x) : [0, b] \to H$ is strongly measurable;
(b) there is a positive integrable function $L^\infty([0, b], (0, \infty))$ and a continuous nondecreasing function $\Lambda_f : [0, \infty) \to (0, \infty)$ such that for every $(t, x) \in [0, b] \times L_2(\mathfrak{F}, H)$, we have

$$\mathbb{E}\|f(t, x)\|^2 \leq n(t) \Lambda_f\left(\mathbb{E}\|x\|^2\right),$$

$$\lim\inf_{r\to\infty} \frac{\Lambda_f(r)}{r} = \sigma_f < \infty.$$

(H$_2$) $g : [0, b] \times H \to L_2^0$ satisfies the following
(a) $g(t, \cdot) : H \to L_2^0$ is continuous for each $t \in [0, b]$ and for each $x \in H$, $g(\cdot, x) : [0, b] \to L_2^0$ is strongly measurable;
(b) there is a positive integrable function $m \in L^\infty([0, b], (0, \infty))$ and a continuous nondecreasing function $\Lambda_g : [0, \infty) \to (0, \infty)$ such that for every $(t, x) \in [0, b] \times L_2(\mathfrak{F}, H)$, we have

$$\mathbb{E}\|g(t, x)\|_{L_2^0}^2 \leq m(t) \Lambda_g\left(\mathbb{E}\|x\|^2\right),$$

$$\lim\inf_{r\to\infty} \frac{\Lambda_g(r)}{r} = \sigma_g < \infty.$$

(H$_3$) Assume that the following relationship holds:

$$3\frac{M^2\left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha}}{2\alpha-1} \sigma_f \sup_{0\leq s\leq b} n(s)$$

$$+3\frac{M^2\left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \sigma_g \sup_{0\leq s\leq b} m(s) < 1.$$

For $x \in X$ and $0 < \alpha \leq 1$, we define two families $\{\mathcal{S}_C(t) : t \geq 0\}$ and $\{\mathcal{T}_C(t) : t \geq 0\}$ of operators by

$$\mathcal{S}_\alpha(t) = \int_0^\infty \Psi_\alpha(\theta) S(t^\alpha \theta) d\theta,$$

$$\mathcal{T}_\alpha(t) = \alpha \int_0^\infty \theta \Psi_\alpha(\theta) S(t^\alpha \theta) d\theta,$$

$$\mathcal{S}_C(t) = C^{-1}\mathcal{S}_\alpha(t), \quad \mathcal{T}_C(t) = C^{-1}\mathcal{T}_\alpha(t),$$

where

$$\Psi_\alpha(\theta) = \frac{1}{\pi\alpha}\sum_{n=1}^\infty (-1)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),$$

$$\theta \in (0,\infty).$$

is the function of Wright type defined in $(0,\infty)$ which satisfies

$$\Psi_\alpha(\theta) \geq 0,$$

$$\int_0^\infty \Psi_\alpha(\theta) d\theta = 1,$$

$$\int_0^\infty \theta^\zeta \Psi_\alpha(\theta) d\theta = \frac{\Gamma(1+\zeta)}{\Gamma(1+\alpha\zeta)},$$

$$\zeta \in (-1,\infty).$$

Lemma 4 [25]. The operators $\mathcal{S}_C$ and $\mathcal{T}_C$ have the following properties:

(a) For any fixed $t \geq 0$, $\mathcal{S}_C(t)$ and $\mathcal{T}_C(t)$ are linear and bounded operators, and

$$\|\mathcal{S}_C(t)x\| \leq M\left\|C^{-1}\right\|\|x\|$$

$$\text{and} \quad \|\mathcal{T}_C(t)x\| \leq \frac{M\left\|C^{-1}\right\|}{\Gamma(\alpha)}\|x\|.$$

(b) $\{\mathcal{S}_C(t) : t \geq 0\}$ and $\{\mathcal{T}_C(t) : t \geq 0\}$ are compact.

$$\begin{cases} {}^C D_t^\alpha Cx(t) = -Ax(t) + Bu(t) + f(t, x(t)) \\ \qquad\qquad + \sigma(t, x(t))\dfrac{dw(t)}{dt}, \\ x(0) = x_0. \end{cases} \tag{1}$$

We first present the definition of mild solutions for the system.

Definition 5. A stochastic process $x(t) : [0,T] \times \Omega \to H$ is said to be a mild solution of the system (1) if

(a) $x(t)$ is $\mathfrak{F}_t$-adapted, $t \geq 0$,

(b) $x(t)$ is continuous on $[0,T]$ almost surely and the following stochastic integral equation is satisfied:

$$x(t) = \mathcal{S}_C(t)Cx_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{T}_C(t-s) f(s, x(s)) ds + \int_0^t (t-s)^{\alpha-1}\mathcal{T}_C(t-s) g(s, x(s)) dw(s).$$

3. Existence theorem

In the present section, we formulate and prove sufficient conditions for the existence of the mild solution of the system (1). We define the operator $\Theta : \mathfrak{H}_2 \to \mathfrak{H}_2$

$$(\Theta x)(t) := \mathcal{S}_C(t)Cx_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{T}_C(t-s) f(s, x(s)) ds + \int_0^t (t-s)^{\alpha-1}\mathcal{T}_C(t-s) g(s, x(s)) dw(s)$$

and prove that it has a fixed point in $\mathfrak{H}_2$. To do this, we first prove some lemmas.

Lemma 6. Under the assumptions(S_1)–(S_3), (H_1)–(H_3), there exists a positive number r such that $\Theta(B_r) \subset B_r$.

Proof. If it is not true, then for each $r > 0$, there exists a function $x_r \in B_r$, but $\Theta(x_r) \notin B_r$. In other words, there exists $t \in [0,b]$ such that $r < \mathbb{E}\|(\Theta x_r)(t)\|^2$. For such t we find that

$$\begin{aligned} r < \mathbb{E}\|(\Theta x_r)(t)\|^2 &\leq 3\mathbb{E}\|\mathcal{S}_C(t)Cx_0\|^2 \\ &+ 3\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}\mathcal{T}_C(t-s) f(s, x(s)) ds\right\|^2 \\ &+ 3\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}\mathcal{T}_C(t-s) g(s, x(s)) dw(s)\right\|^2 \\ &=: 3(I_1 + I_2 + I_3). \end{aligned} \tag{2}$$

Let us estimate I_i, $i = 1, ..., 5$. By Lemma 4 and assumption (H_1), we have

$$I_1 \leq M^2\left\|C^{-1}\right\|^2\|Cx_0\|^2, \tag{3}$$

$$I_2 \le \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} ds \int_0^t \mathbb{E}\left\|f(s,x(s))\right\|^2 ds \le \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha}}{2\alpha-1} \sup_{0\le s\le b} n(s)\Lambda_f(r). \tag{4}$$

By the well-known property of the stochastic integral, Lemma 4, and the Hölder inequality, we can deduce that

$$\begin{aligned} I_3 &\le \mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} \mathcal{T}_C(t-s) g(s,x(s)) dw(s)\right\|^2 \\ &= \mathbb{E}\int_0^t \left\|(t-s)^{\alpha-1} \mathcal{T}_C(t-s) g(s,x(s))\right\|^2 ds \\ &\le \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} \mathbb{E}\left\|g(s,x(s))\right\|^2 ds \\ &\le \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \sup_{0\le s\le b} m(s)\Lambda_g(r). \end{aligned} \tag{5}$$

Combining the estimates (2)–(5) yields

$$\begin{aligned} r &\le \mathbb{E}\left\|(\Theta z_r)(t)\right\|^2 < 3M^2\left\|C^{-1}\right\|^2 \left\|Cx_0\right\|^2 \\ &+ 3\frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha}}{2\alpha-1} \sup_{0\le s\le b} n(s)\Lambda_f(r) \\ &+ 3\frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \sup_{0\le s\le b} m(s)\Lambda_g(r). \end{aligned} \tag{6}$$

Dividing both sides of (6) by r and taking $r \to \infty$, we obtain that

$$3\frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha}}{2\alpha-1} \sigma_f \sup_{0\le s\le b} n(s) + 3\frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \sigma_g \sup_{0\le s\le b} m(s) \ge 1,$$

which is a contradiction by assumption (H_3). Thus, $\Theta(B_r) \subset B_r$ for some $r > 0$.

Lemma 7. Let assumptions (S_1)–(S_3), (H_1)–(H_3) hold. Then the set $\{\Theta z : z \in B_r\}$ is an equicontinuous family of functions in $[0,b]$.

Proof. Let $0 < \varepsilon < t < b$ and $\delta > 0$ such that

$$\left\|\mathcal{S}_C(s_1) - \mathcal{S}_C(s_2)\right\|^2 < \varepsilon, \qquad \left\|\mathcal{T}_C(s_1) - \mathcal{T}_C(s_2)\right\|^2 < \varepsilon$$

for every $s_1, s_2 \in [0,b]$ with $|s_1 - s_2| < \delta$. For $z \in B_r$, $0 < h < \delta$, $t+h \in [0,b]$, we have

$$\begin{aligned} &\mathbb{E}\left\|(\Theta x)(t+h) - (\Theta x)(t)\right\|^2 \\ &\le 7\left\|\mathcal{S}_C(t+h)Cx_0 - \mathcal{S}_C(t)Cx_0\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^t \left((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right) \mathcal{T}_C(t+h-s) f(s,x(s)) ds\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_t^{t+h} (t+h-s)^{\alpha-1} \mathcal{T}_C(t+h-s) f(s,x(s)) ds\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} \left(\mathcal{T}_C(t+h-s) - \mathcal{T}_C(t-s)\right) f(s,x(s)) ds\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^t \left((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right) \mathcal{T}_C(t+h-s) g(s,x(s)) dw(s)\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_t^{t+h} (t+h-s)^{\alpha-1} \mathcal{T}_C(t+h-s) g(s,x(s)) dw(s)\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} \left(\mathcal{T}_C(t+h-s) - \mathcal{T}_C(t-s)\right) g(s,x(s)) dw(s)\right\|^2 := 7(I_1 + ... I_7). \end{aligned} \tag{7}$$

It is clear that

$$\begin{aligned} I_1 &\le \left\|\mathcal{S}_C(t+h) - \mathcal{S}_C(t)\right\|^2 \left\|Cx_0\right\|^2, \\ I_2 &\le \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} b \int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right|^2 ds \sup_{0\le s\le b} n(s)\Lambda_f(r), \\ I_3 &\le \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} h \int_t^{t+h} (t+h-s)^{2(\alpha-1)} ds \sup_{0\le s\le b} n(s)\Lambda_f(r) \\ &= \frac{h^{2\alpha}}{2\alpha-1} \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \sup_{0\le s\le b} n(s)\Lambda_f(r), \\ I_4 &\le b\int_0^t (t-s)^{2(\alpha-1)} ds \sup_{0\le s\le t} \left\|\mathcal{T}_C(t+h-s) - \mathcal{T}_C(t-s)\right\|^2 \sup_{0\le s\le b} n(s)\Lambda_f(r) \\ &= \varepsilon \frac{b^{2\alpha}}{2\alpha-1} \sup_{0\le s\le b} n(s)\Lambda_f(r). \end{aligned}$$

In a like manner we have

$$I_5 \le \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right|^2 ds \sup_{0\le s\le b} m(s)\,\Lambda_g(r),$$

$$I_6 \le \frac{h^{2\alpha-1}}{2\alpha-1} \frac{M^2 \left\|C^{-1}\right\|^2}{\Gamma^2(\alpha)} \sup_{0\le s\le b} m(s)\,\Lambda_g(r),$$

$$I_7 \le \varepsilon \frac{b^{2\alpha-1}}{2\alpha-1} \sup_{0\le s\le b} m(s)\,\Lambda_g(r).$$

Therefore, for ε sufficiently small, the right-hand side of (7) tends to zero as $h \to 0$. Thus, the set $\{\Theta x : x \in B_r\}$ is equicontinuous.

Lemma 8. Let assumptions (S_1)–(S_3), (H_1)–(H_3) hold. For every $t \in [0,b]$ the set $V(t) := \{(\Theta x)(t) : x \in B_r\}$ is relatively compact in $L_2(\mathfrak{F}, H)$.

Proof. Let $0 < t < b$ be fixed. Recall that

$$(\Theta x)(t) = C^{-1}(\Theta_0 x)(t),$$

where

$$(\Theta_0 x)(t) := \mathcal{S}_\alpha(t)\,Cx_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{T}_\alpha(t-s) f(s, x(s))\,ds + \int_0^t (t-s)^{\alpha-1} \mathcal{T}_\alpha(t-s)\, g(s, x(s))\,dw(s).$$

For $x \in B_r$, we derive

$$\mathbb{E}\left\|(\Theta_0 x)(t)\right\|^2 \le 3M^2 \left\|C^{-1}\right\|^2 \left\|Cx_0\right\|^2 + 3\frac{M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha}}{2\alpha-1} \sup_{0\le s\le b} n(s)(1+r) + 3\frac{M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \sup_{0\le s\le b} m(s)(1+r),$$

in other words $\{(\Theta_0 x)(t) : x \in B_r\}$ is bounded in $L_2(\mathfrak{F}, H)$. Since C^{-1} is compact, then $(\Theta(B_r))(t) = C^{-1}\{(\Theta_0 x)(t) : x \in B_r\}$ is relatively compact in $L_2(\mathfrak{F}, H)$. The proof is complete.

Theorem 9. Assume (S_1)–(S_3), (H_1)–(H_3) are satisfied. Then the system (1) has a mild solution in $\mathfrak{H}_2$.

Proof. It follows Lemmas 6–8 and the Arzela-Ascoli theorem that $\Theta(B_r)$ is relatively compact in $\mathfrak{H}_2$. Hence Θ is a completely continuous operator on $\mathfrak{H}_2$. From the Schauder fixed point theorem, Θ has a fixed point in $B_r \subset \mathfrak{H}_2$.

4. Approximate controllability

In this section, first we study relationship between the approximate controllability of the fractional stochastic linear system and the deterministic fractional linear system. Next, we prove sufficient conditions for the approximate controllability of the fractional Sobolev type semilinear stochastic differential equation.

Consider the stochastic linear system

$$\begin{cases} {}^C D_t^\alpha Cx(t) = Ax(t) + Bu(t) + \sigma(t)\dfrac{dw(t)}{dt}, \\ x(0) = x_0, \end{cases} \tag{8}$$

deterministic linear system

$$\begin{cases} {}^C D_t^\alpha Cx(t) = Ax(t) + Bv(t), \\ x(0) = x_0, \end{cases} \tag{9}$$

here $B : U \to H$ is a linear bounded operator, $u \in L_2^{\mathfrak{F}}([0,b], U)$, $v \in L_2([0,b], U)$. Introduce the following operators

$$L_0^b u := \int_0^b (b-s)^{\alpha-1} \mathcal{T}_C(b-s)\,Bv(s)\,ds,$$

$$L_0^b : L_2^{\mathfrak{F}}([0,b], H) \to L_2(\mathfrak{F}_b, H),$$

$$\Pi_0^b h := \int_0^b (b-s)^{2(\alpha-1)} \mathcal{T}_C(b-s)\, BB^* \mathcal{T}_C^*(b-s)\,\mathbb{E}\{h \mid \mathfrak{F}_s\}\,ds,$$

$$\Pi_0^b : L_2(\mathfrak{F}_b, H) \to L_2(\mathfrak{F}_b, H),$$

$$\Gamma_s^b h := \int_s^b (b-r)^{2(\alpha-1)} \mathcal{T}_C(b-r)\,BB^* \mathcal{T}_C^*(b-r)\,h\,dr,$$

$$\Gamma_s^b : H \to H.$$

It is clear these operators are linear bounded for all

$$\frac{1}{2} < \alpha \le 1.$$

To prove our main result in this section we need the following lemmas.

Lemma 10 [2, 15]. The control system (9) is approximately controllable in $[s,b]$ if and only if one of the following conditions hold.

(a) $\Gamma_s^b > 0$.

(b) $\varepsilon\left(\varepsilon I + \Gamma_s^b\right)^{-1}$ converges to the zero operator as $\varepsilon \to 0^+$ in the strong operator topology.

(c) $\varepsilon\left(\varepsilon I + \Gamma_s^b\right)^{-1}$ converges to the zero operator as $\varepsilon \to 0^+$ in the weak operator topology.

Lemma 11. The function $f(s) = \left(\varepsilon I + \Gamma_s^b\right)^{-1} h$ is continuous in $[0,b]$.

Proof. It follows from continuity of Γ_s^b and from the following identity.

$$\left(\varepsilon I+\Gamma_s^b\right)^{-1}h-\left(\varepsilon I+\Gamma_t^b\right)^{-1}h$$
$$=\left(\varepsilon I+\Gamma_t^b\right)^{-1}\left[\left(\varepsilon I+\Gamma_s^b-\Gamma_s^b+\Gamma_t^b\right)\left(\varepsilon I+\Gamma_s^b\right)^{-1}-I\right]h$$
$$=\left(\varepsilon I+\Gamma_t^b\right)^{-1}\left[I+\left(\Gamma_t^b-\Gamma_s^b\right)\left(\varepsilon I+\Gamma_s^b\right)^{-1}-I\right]h$$
$$=\left(\varepsilon I+\Gamma_t^b\right)^{-1}\left(\Gamma_t^b-\Gamma_s^b\right)\left(\varepsilon I+\Gamma_s^b\right)^{-1}h.$$

Lemma 12 [2]. For every $h\in L_2\left(\mathfrak{F}_b,H\right)$ there exists $\varphi\in L_2^{\mathfrak{F}}\left(\left[0,b\right],L_2^0\right)$ such that

$$h=\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\varphi\left(r\right)w\left(r\right),$$

$$\Pi_0^bh=\Gamma_s^b\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\Gamma_\tau^b\varphi\left(\tau\right)dw\left(\tau\right).$$

Proof. First formula is proved in [2]. The second one follows from the definition of the operator Π_s^b and the stochastic Fubini Theorem:

$$\Pi_s^bh=\int_s^b\left(b-r\right)^{2(\alpha-1)}\mathcal{T}_C\left(b-r\right)BB^*\mathcal{T}_C^*\left(b-r\right)\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}dr$$
$$+\int_s^b\left(b-r\right)^{2(\alpha-1)}\mathcal{T}_C\left(b-r\right)BB^*\mathcal{T}_C^*\left(b-r\right)\int_s^r\varphi\left(\tau\right)dw\left(\tau\right)dr$$
$$=\Gamma_s^b\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\int_\tau^b\left(b-r\right)^{2(\alpha-1)}\mathcal{T}_C\left(b-r\right)BB^*\mathcal{T}_C^*\left(b-r\right)dr\varphi\left(\tau\right)dw\left(\tau\right)$$
$$=\Gamma_s^b\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\Gamma_\tau^b\varphi\left(\tau\right)dw\left(\tau\right).$$

Lemma 13. For every $h\in L_2\left(\mathfrak{F}_b,H\right)$ there exists $\varphi\in L_2^{\mathfrak{F}}\left(\left[0,b\right],L_2^0\right)$ such that

$$\left(\varepsilon I+\Pi_s^b\right)^{-1}h=\left(\varepsilon I+\Gamma_s^b\right)^{-1}\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\left(\varepsilon I+\Gamma_r^b\right)^{-1}\varphi\left(r\right)w\left(r\right).\tag{10}$$

Proof. From Lemma 4 it follows that for every $h,z\in L_2\left(\mathfrak{F}_b,H\right)$ there exist $\varphi,\psi\in L_2^{\mathfrak{F}}\left(\left[0,b\right],L_2^0\right)$ such that

$$h=\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\varphi\left(\tau\right)dw\left(\tau\right),$$
$$z=\mathbb{E}\left\{z\mid\mathfrak{F}_s\right\}+\int_s^b\psi\left(\tau\right)dw\left(\tau\right),\tag{11}$$
$$\Pi_0^bh=\Gamma_s^b\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\Gamma_\tau^b\varphi\left(\tau\right)dw\left(\tau\right).$$

Let $h=\left(\varepsilon I+\Pi_s^b\right)^{-1}z$. It follows that

$$z=\varepsilon h+\Pi_s^bh=\varepsilon\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\varepsilon\int_s^b\varphi\left(\tau\right)w\left(\tau\right)$$
$$+\Gamma_s^b\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\Gamma_\tau^b\varphi\left(\tau\right)dw\left(\tau\right)\tag{12}$$
$$=\left(\varepsilon I+\Gamma_s^b\right)\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\left(\varepsilon I+\Gamma_\tau^b\right)\varphi\left(\tau\right)dw\left(\tau\right).$$

Since the stochastic integral $\int_s^b\varphi\left(\tau\right)w\left(\tau\right)$ is independent of $\mathfrak{F}_s$, from (11) and (12) one has

$$\mathbb{E}\left\{z-\left(\varepsilon I+\Gamma_s^b\right)h\mid\mathfrak{F}_s\right\}+\int_s^b\left[\psi\left(\tau\right)-\left(\varepsilon I+\Gamma_\tau^b\right)\varphi\left(\tau\right)\right]dw\left(\tau\right)=0,$$

$$\mathbb{E}\left\|\mathbb{E}\left\{z-\left(\varepsilon I+\Gamma_s^b\right)h\mid\mathfrak{F}_s\right\}\right\|^2+\mathbb{E}\left\|\int_s^b\left[\psi\left(\tau\right)-\left(\varepsilon I+\Gamma_\tau^b\right)\varphi\left(\tau\right)\right]dw\left(\tau\right)\right\|^2=$$
$$\mathbb{E}\left\|\mathbb{E}\left\{z-\left(\varepsilon I+\Gamma_s^b\right)h\mid\mathfrak{F}_s\right\}\right\|^2+\mathbb{E}\int_s^b\left\|\psi\left(\tau\right)-\left(\varepsilon I+\Gamma_\tau^b\right)\varphi\left(\tau\right)\right\|^2d\tau=0.$$

It follows that

$$\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}=\left(\varepsilon I+\Gamma_s^b\right)^{-1}\mathbb{E}\left\{z\mid\mathfrak{F}_s\right\},$$
$$\varphi\left(\tau\right)=\left(\varepsilon I+\Gamma_\tau^b\right)^{-1}\psi\left(\tau\right).$$

Thus

$$\left(\varepsilon I+\Pi_s^b\right)^{-1}z=h=\mathbb{E}\left\{h\mid\mathfrak{F}_s\right\}+\int_s^b\varphi\left(\tau\right)dw\left(\tau\right)$$
$$=\left(\varepsilon I+\Gamma_s^b\right)^{-1}\mathbb{E}\left\{z\mid\mathfrak{F}_s\right\}+\int_s^b\left(\varepsilon I+\Gamma_\tau^b\right)^{-1}\varphi\left(\tau\right)dw\left(\tau\right).$$

Theorem 14. The stochastic system (8) is approximately controllable in $[0,b]$ if and only if the deterministic system (9) is approximately controllable in every $[s,b]$, $0\leq s<b$.

Proof. Assume that the stochastic system (8) is approximately controllable in $[0,b]$. Then for any $z \in L_2(\mathfrak{F}_b, H)$

$$\mathbb{E}\left\|\varepsilon\left(\varepsilon I+\Pi_0^b\right)^{-1} z\right\|^2 \rightarrow 0 \qquad \text{as} \qquad \varepsilon \rightarrow 0^+.$$

Let us choose z as follows.

$$z=h+\sum_{n=1}^{\infty} \sqrt{\lambda_n} \int_s^b h\left\langle e, e_n\right\rangle \beta_n(r), \qquad \forall h \in H.$$

From this and (10) we have

$$\begin{aligned} & \mathbb{E}\left\|\varepsilon\left(\varepsilon I+\Pi_0^b\right)^{-1} z\right\|^2 \\ & =\left\|\varepsilon\left(\varepsilon I+\Gamma_0^b\right)^{-1} h\right\|^2 \\ & +\sum_{n=1}^{\infty} \lambda_n\left\langle e, e_n\right\rangle^2 \int_0^b\left\|\varepsilon\left(\varepsilon I+\Gamma_r^b\right)^{-1} h\right\|^2 dr. \end{aligned} \tag{13}$$

It follows that there is a subsequence $\{\varepsilon_k\}$ such that for all $h \in H$

$$\left\|\varepsilon_k\left(\varepsilon_k I+\Gamma_s^b\right)^{-1} h\right\| \rightarrow 0 \qquad \text{as} \qquad \varepsilon_k \rightarrow 0^+$$

almost everywhere in $[0,b]$. Because of the continuity of $\left(\varepsilon I+\Gamma_s^b\right)^{-1} h$ this property holds for all $0 \leq s<b$. The latter means that the deterministic system (9) is approximately controllable on every $[s,b]$, $0 \leq s<b$.

Contrary, if the deterministic system (9). is approximately controllable on every $[s,b]$, $0 \leq s<b$, then

$$\lim_{\varepsilon \rightarrow 0^+}\left\|\varepsilon\left(\varepsilon I+\Gamma_s^b\right)^{-1} \phi\right\|=0$$

for all $\phi \in H$. Since

$$\begin{aligned} & \left\|\varepsilon\left(\varepsilon I+\Gamma_r^b\right)^{-1} \varphi(r)\right\|_{L_2^0}^2 \\ & =\sum_{n=1}^{\infty} \lambda_n\left\|\varepsilon\left(\varepsilon I+\Gamma_r^b\right)^{-1} \varphi(r) e_n\right\|^2 \\ & \leq \sum_{n=1}^{\infty} \lambda_n\left\|\varphi(r) e_n\right\|^2=\|\varphi(r)\|_{L_2^0}^2, \end{aligned}$$

by the Lebesgue dominated convergence theorem from (10), we get

$$\lim_{\varepsilon \rightarrow 0^+} \mathbb{E}\left\|\varepsilon\left(\varepsilon I+\Pi_0^b\right)^{-1} h\right\|^2=0, \qquad h \in L_2(\mathfrak{F}_b, H),$$

that is, the stochastic system (8). is approximately controllable in $[0,b]$.

Next for any $u \in L_2^{\mathfrak{F}}([0,b], U)$ we consider the semilinear fractional stochastic system

$$\begin{aligned} x(t)= & \mathcal{S}_C(t) C x_0+\int_0^t(t-s)^{\alpha-1} \mathcal{T}_C(t-s) B u(s) ds \\ & +\int_0^t(t-s)^{\alpha-1} \mathcal{T}_C(t-s) f(s, x(s)) ds \\ & +\int_0^t(t-s)^{\alpha-1} \mathcal{T}_C(t-s) g(s, x(s)) dw(s). \end{aligned} \tag{14}$$

For any $\varepsilon>0$ and $h \in L_2(\mathfrak{F}_b, H)$, we define a control u_ε and an operator Θ_ε as follows.

$$\begin{aligned} u_\varepsilon(t, x)= & (b-t)^{\alpha-1} B^* \mathcal{T}_C^*(b-t)\left(\varepsilon I+\Gamma_0^b\right)^{-1} \\ & \left(C h-\mathcal{S}_C(b) C x_0\right) \\ & -(b-t)^{\alpha-1} B^* \mathcal{T}_C^*(b-t) \int_0^b\left(\varepsilon I+\Gamma_s^b\right)^{-1}(b-s)^{\alpha-1} \\ & \mathcal{T}_C(b-s) f(s, x(s)) ds \\ & -(b-t)^{\alpha-1} B^* \mathcal{T}_C^*(b-t) \int_0^b\left(\varepsilon I+\Gamma_s^b\right)^{-1} \\ & \Big((b-s)^{\alpha-1} \mathcal{T}_C(b-s) g(s, x(s))-\varphi(s)\Big) dw(s) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \left(\Theta_\varepsilon x\right)(t):= & \mathcal{S}_C(t) C x_0 \\ & +\int_0^t(t-s)^{\alpha-1} \mathcal{T}_C(t-s)\left[B u_\varepsilon(s, x)+f(s, x(s))\right] ds \\ & +\int_0^t(t-s)^{\alpha-1} \mathcal{T}_C(t-s) g(s, x(s)) dw(s). \end{aligned}$$

Lemma 15. The solution of (14) corresponding to $u_\varepsilon(t,x)$ satisfies the following identity

$$\begin{aligned} x_\varepsilon(b)= & h-\varepsilon\left(\varepsilon I+\Gamma_0^b\right)^{-1}\left(C \mathbb{E} h-\mathcal{S}_C(b) C x_0\right) \\ & +\varepsilon \int_0^b(b-r)^{\alpha-1}\left(\varepsilon I+\Gamma_r^b\right)^{-1} \mathcal{T}_C(b-r) f\left(r, x_\varepsilon(r)\right) dr \\ & +\varepsilon \int_0^b(b-r)^{\alpha-1}\left(\varepsilon I+\Gamma_r^b\right)^{-1} \mathcal{T}_C(b-r) g\left(r, x_\varepsilon(r)\right) dw(r) \\ & -\varepsilon \int_0^b\left(\varepsilon I+\Gamma_r^b\right)^{-1} \varphi(r) dw(r). \end{aligned}$$

Proof. Inserting the expression of the control (15) into the Eq. (14) we get

$$x_\varepsilon(b) = \mathcal{S}_C(b)Cx_0 + \int_0^b (b-s)^{\alpha-1}\mathcal{T}_C(b-s)\left[Bu_\varepsilon(s,x_\varepsilon) + f(s,x_\varepsilon(s))\right]ds + \int_0^b (b-s)^{\alpha-1}\mathcal{T}_C(b-s)g(s,x_\varepsilon(s))dw(s)$$

$$= \mathcal{S}_C(b)Cx_0 + \int_0^b (b-s)^{\alpha-1}\mathcal{T}_C(b-s)f(s,x_\varepsilon(s))ds + \int_0^b (b-s)^{\alpha-1}\mathcal{T}_C(b-s)g(s,x_\varepsilon(s))dw(s)$$

$$+ \int_0^b (b-s)^{2(\alpha-1)}\mathcal{T}_C(b-s)BB^*\mathcal{T}_C^*(b-s)\left(\varepsilon I + \Gamma_0^b\right)^{-1}\left\{\mathbb{E}h - \mathcal{S}_C(b)x_0\right\}ds$$

$$- \int_0^b (b-s)^{2(\alpha-1)}\mathcal{T}_C(b-s)BB^*\mathcal{T}_C^*(b-s) \times \int_0^s \left(\varepsilon I + \Gamma_r^b\right)^{-1}(b-r)^{\alpha-1}\mathcal{T}_C(b-r)f(r,x(r))drds$$

$$- \int_0^b (b-s)^{2(\alpha-1)}\mathcal{T}_C(b-s)BB^*\mathcal{T}_C^*(b-s) \times \int_0^s \left(\varepsilon I + \Gamma_r^b\right)^{-1}(b-r)^{\alpha-1}\mathcal{T}_C(b-r)g(r,x(r))dw(r)ds$$

$$+ \int_0^b (b-s)^{2(\alpha-1)}\mathcal{T}_C(b-s)BB^*\mathcal{T}_C^*(b-s) \int_0^s \left(\varepsilon I + \Gamma_r^b\right)^{-1}\varphi(r)dw(r)ds,$$

consequently

$$x_\varepsilon(b) = \mathcal{S}_C(b)Cx_0 + \int_0^b (b-s)^{\alpha-1}\mathcal{T}_C(b-s)f(s,x_\varepsilon(s))ds + \int_0^b (b-s)^{\alpha-1}\mathcal{T}_C(b-s)g(s,x_\varepsilon(s))dw(s)$$

$$+\Gamma_0^b\left(\varepsilon I + \Gamma_0^b\right)^{-1}\left\{\mathbb{E}h - \mathcal{S}_C(b)x\right\}$$

$$- \int_0^b \Gamma_r^b\left(\varepsilon I + \Gamma_r^b\right)^{-1}(b-r)^{\alpha-1}\mathcal{T}_C(b-r)f(r,x_\varepsilon(r))dr$$

$$- \int_0^b \Gamma_r^b\left(\varepsilon I + \Gamma_r^b\right)^{-1}(b-r)^{\alpha-1}\mathcal{T}_C(b-r)g(r,x_\varepsilon(r))dw(r)$$

$$+ \int_0^b \Gamma_r^b\left(\varepsilon I + \Gamma_r^b\right)^{-1}\varphi(r)dw(r)$$

$$= h - \varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1}\left(\mathbb{E}h - \mathcal{S}_C(b)Cx_0\right)$$

$$+\varepsilon\int_0^b (b-r)^{\alpha-1}\left(\varepsilon I + \Gamma_r^b\right)^{-1}\mathcal{T}_C(b-r)f(r,x_\varepsilon(r))dr$$

$$+\varepsilon\int_0^b (b-r)^{\alpha-1}\left(\varepsilon I + \Gamma_r^b\right)^{-1}\mathcal{T}_C(b-r)g(r,x_\varepsilon(r))dw(r)$$

$$-\varepsilon\int_0^b \left(\varepsilon I + \Gamma_r^b\right)^{-1}\varphi(r)dw(r).$$

Now, we present and prove our second main result.

Theorem 16. Assume that the assumptions (S_1)–(S_3), (H_1), (H_2) hold. Further, if the functions f and g are uniformly bounded and the linear system (8) is approximately controllable, then the system (14) is approximately controllable in $[0, b]$.

Proof. It is easy to see that under the above conditions the operator Θ_ε has a fixed point in $\mathfrak{H}_2$. In other words, the equation (14) has a solution corresponding to $u_\varepsilon(t,x)$. Let x_ε be a fixed point of Θ_ε. By Lemma 15 we have

$$x_\varepsilon(b) = h - \varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1}\left(\mathbb{E}h - \mathcal{S}_C(b)\,Cx_0\right)$$
$$+\varepsilon\int_0^b (b-r)^{\alpha-1}\left(\varepsilon I + \Gamma_r^b\right)^{-1}\mathcal{T}_C(b-r)\,f(r,x_\varepsilon(r))\,dr$$
$$+\varepsilon\int_0^b (b-r)^{\alpha-1}\left(\varepsilon I + \Gamma_r^b\right)^{-1}\mathcal{T}_C(b-r)\,g(r,x_\varepsilon(r))\,dw(r)$$
$$-\varepsilon\int_0^b \left(\varepsilon I + \Gamma_r^b\right)^{-1}\varphi(r)\,dw(r).$$

Since the functions f and g are uniformly bounded there exists $N > 0$ such that

$$\|f(r,x_\varepsilon(r))\|^p + \|g(r,x_\varepsilon(r))\|^p_{L_2^0} \le N \quad \text{in} \quad [0,T]\times\Omega.$$

Then there is a subsequence, still denoted by $\{f(r,x_\varepsilon(r)), g(r,x_\varepsilon(r))\}$, weakly converging to, say, $(f(r,\omega), g(r,\omega))$ in $H\times L_2^0$. The compactness of $\mathcal{T}_C(t)$, $t > 0$ implies that

$$\begin{cases}\mathcal{T}_C(b-r)\,f(r,x_\varepsilon(r)) \to \mathcal{T}_C(b-r)\,f(r),\\ \mathcal{T}_C(b-r)\,g(r,x_\varepsilon(r)) \to \mathcal{T}_C(b-r)\,g(r)\\ \qquad\text{a.e. in } [0,T]\times\Omega.\end{cases} \tag{16}$$

On the other hand, by the assumption on approximate controllability of (8) and Theorem 14, for all $0 \le r < b$

$$\varepsilon\left(\varepsilon I + \Gamma_r^b\right)^{-1} \to 0 \qquad \text{strongly as} \qquad \varepsilon \to 0^+, \tag{17}$$

and moreover

$$\left\|\varepsilon\left(\varepsilon I + \Gamma_r^b\right)^{-1}\right\| \le 1. \tag{18}$$

Thus from (16)–(18) by the Lebesgue dominated convergence theorem it follows that

$$\mathbb{E}\|x_\varepsilon(b) - h\|^2 \le 6\left\|\varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1}\left(\mathbb{E}h - \mathcal{S}_C(b)\,Cx_0\right)\right\|^2$$
$$+6\mathbb{E}\left(\int_0^b (b-r)^{\alpha-1}\left\|\varepsilon\left(\varepsilon I + \Gamma_r^b\right)^{-1}\right\| \left\|\mathcal{T}_C(b-r)\left[f(r,x_\varepsilon(r)) - f(r)\right]\right\| dr\right)^2$$
$$+6\mathbb{E}\left(\int_0^b (b-r)^{\alpha-1}\left\|\varepsilon\left(\varepsilon I + \Gamma_r^b\right)^{-1}\mathcal{T}_C(b-r)\,f(r)\right\| dr\right)^2$$
$$+6\mathbb{E}\int_0^b (b-r)^{2(\alpha-1)}\left\|\varepsilon\left(\varepsilon I + \Gamma_r^b\right)^{-1}\right\|^2 \left\|\mathcal{T}_C(b-r)\left[g(r,x_\varepsilon(r)) - g(r)\right]\right\|^2_{L_2^0} dr$$
$$+6\mathbb{E}\int_0^b (b-r)^{2(\alpha-1)}\left\|\varepsilon\left(\varepsilon I + \Gamma_r^b\right)^{-1}\mathcal{T}_C(b-r)\,g(r)\right\|^2_{L_2^0} dr$$
$$+6\mathbb{E}\int_0^b \left\|\varepsilon\left(\varepsilon I + \Gamma_r^b\right)^{-1}\varphi(r)\right\|^2_{L_2^0} dr$$
$$\to 0 \ \text{ as } \ \varepsilon \to 0^+.$$

This gives the approximate controllability. Theorem is proved.

5. Application

Consider a control system governed by the semilinear heat equation

$$\begin{aligned}&{}^cD_t^{\frac{3}{4}}\left(x(t,\theta) - x_{\theta\theta}(t,\theta)\right)\\ &= x_{\theta\theta}(t,\theta) + Bu(t,\theta) + f(t,x(t,\theta)) + \frac{dw(t)}{dt},\end{aligned} \tag{19}$$
$$x(t,0) = x(t,\pi) = 0, \qquad x(0,\theta) = \phi(\theta),$$
$$0 \le t \le b, \qquad 0 < \theta < \pi.$$

Let $H = L_2[0,\pi]$ Define $A : D(A) \subset H \to H$ by $A := -x_{\theta\theta}$ and $C : D(C) \subset H \to H$ by $Cx := x - x_{\theta\theta}$, where each domain $D(A), D(C)$ is given by

$$\{x \in H : x, x_\theta \text{ are absolutely continuous}, \ x_{\theta\theta} \in H,\ x(t,0) = x(t,\pi) = 0\}.$$

A and C can be written as follows

$$Ax := \sum_{n=1}^{\infty} n^2\langle x, e_n\rangle e_n, \qquad x \in D(A),$$
$$Cx = \sum_{n=1}^{\infty}\left(1+n^2\right)\langle x, e_n\rangle e_n, \qquad x \in D(C),$$

respectively, where $x_n(\theta) := \sqrt{\frac{2}{\pi}}\sin n\theta$, $n = 1,2,\ldots$ is the orthonormal set of eigenvalues of A. Moreover, for any $x \in H$ we have

$$C^{-1}x = \sum_{n=1}^{\infty}\frac{1}{1+n^2}\langle x, e_n\rangle e_n,$$
$$-AC^{-1}x = \sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}\langle x, e_n\rangle e_n$$
$$S(t)\,x = \sum_{n=1}^{\infty}\exp\left(\frac{-n^2}{1+n^2}t\right)\langle x, e_n\rangle e_n$$
$$\mathcal{T}_C(t) = \frac{3}{4}\int_0^\infty C^{-1}\theta\xi_{\frac{3}{4}}(\theta)\,T\left(t^{\frac{3}{4}}\theta\right)d\theta$$
$$\mathcal{T}_C(t)\,x = \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{1+n^2}\int_0^\infty \theta\xi_{\frac{3}{4}}(\theta) \exp\left(\frac{-n^2}{1+n^2}t^{\frac{3}{4}}\theta\right)d\theta\,\langle x, e_n\rangle e_n$$
$$= -\sum_{n=1}^{\infty}\frac{1}{n^2}t^{\frac{1}{4}}\int_0^\infty \xi_{\frac{3}{4}}(\theta)\,\frac{d}{dt}\exp\left(\frac{-n^2}{1+n^2}t^{\frac{3}{4}}\theta\right)d\theta\,\langle x, e_n\rangle e_n$$

Define an infinite dimensional space U by

$$U = \left\{\sum_{n=2}^{\infty}u_n e_n \mid \sum_{n=2}^{\infty}u_n^2 < \infty\right\}.$$

The norm in U is defined by $\|u\|_U = \left(\sum_{n=2}^{\infty} u_n^2\right)^{1/2}$. Now define a linear continuous mapping from U to X as follows:

$$Bu = 2u_2 + \sum_{n=2}^{\infty} u_n e_n$$

for $u = \sum_{n=2}^{\infty} u_n e_n \in U$.

It is easy to see that

$$\frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{1+n^2} \int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) \exp\left(\frac{-n^2}{1+n^2} t^{\frac{3}{4}} \theta\right) d\theta \langle x, e_n \rangle e_n$$

$$B^* v = (2v_1 + v_2) e_2 + \sum_{n=3}^{\infty} v_n e_n,$$

$$B^* \mathcal{T}_C^*(t) x = \left(\frac{3}{4}\int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) \exp\left(\frac{-1}{2} t^{\frac{3}{4}} \theta\right) d\theta \langle x, e_1 \rangle \right.$$

$$\left. + \frac{3}{20}\int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) \exp\left(\frac{-4}{5} t^{\frac{3}{4}} \theta\right) d\theta \langle x, e_2 \rangle \right) e_2$$

$$+ \frac{3}{4}\sum_{n=3}^{\infty} \frac{1}{1+n^2} \int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) \exp\left(\frac{-n^2}{1+n^2} t^{\frac{3}{4}} \theta\right) d\theta \langle x, e_n \rangle e_n,$$

$$\|B^* \mathcal{T}_C^*(t) x\| = 0, \quad t \in [s,b]$$

$$\Rightarrow \left(\frac{3}{4}\int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) \exp\left(\frac{-1}{2} t^{\frac{3}{4}} \theta\right) d\theta \langle x, e_1 \rangle \right.$$

$$\left. + \frac{3}{20}\int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) \exp\left(\frac{-4}{5} t^{\frac{3}{4}} \theta\right) d\theta \langle x, e_2 \rangle \right)^2$$

$$+ \frac{9}{16}\sum_{n=3}^{\infty} \left(\frac{1}{1+n^2} \int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) \exp\left(\frac{-n^2}{1+n^2} t^{\frac{3}{4}} \theta\right) d\theta \right)^2$$

$$\langle x, e_n \rangle^2 = 0, \quad t \in [s,b]$$

$$\Rightarrow \langle x, e_n \rangle = 0, \quad n = 1, 2, ... \Rightarrow x = 0.$$

Thus the deterministic linear system corresponding to (19) is approximately controllable in every $[s,b]$. Thus if f is bounded, then by Theorem 4 the fractional Sobolev type stochastic system (19) is approximately controllable in $[0,b]$.

REFERENCES

[1] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, London, 1993.

[2] N.I. Mahmudov, "Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces", *SIAM J. on Control and Optimization* 42, 1604–1622 (2003).

[3] J. Klamka, "Constrained approximate controllability", *IEEE Trans. on Automatic Control* 45, 1745–1749 (2000).

[4] J. Klamka, "Stochastic controllability of systems with multiple delays in control", *Int. J. Appl. Math. Comput. Sci*. 19 (1), 39–47 (2009).

[5] J. Klamka, "Stochastic controllability and minimum energy control of systems with multiple delays in control", *Appl. Math. Comput.* 206 (2), 704–715 (2008).

[6] J. Klamka, "Stochastic controllability of linear systems with state delays", *Int. J. Appl. Math. Comput. Sci.* 17 (1), 5–13 (2007).

[7] X. Fu and K. Mei, "Approximate controllability of semilinear partial functional differential systems", *J. Dynamical and Control Systems* 15, 425–443 (2009).

[8] R. Sakthivel, Y. Ren, and N.I. Mahmudov, "On the approximate controllability of semilinear fractional differential systems", *Computers and Mathematics with Applications* 62, 1451–1459 (2011).

[9] N.I. Mahmudov and A. Denker, "On controllability of linear stochastic systems", *Int. J. Control* 73, 144–151 (2000).

[10] J. Klamka, "Stochastic controllability of linear systems with delay in control", *Bull. Pol. Ac.: Tech.* 55, 23–29 (2007).

[11] R. Sakthivel, Y. Ren, and N.I. Mahmudov, "Approximate controllability of second-order stochastic differential equations with impulsive effects", *Modern Physics Letters* B 24, 1559–1572 (2010).

[12] R. Sakthivel, J.J. Nieto, and N.I. Mahmudov, "Approximate controllability of nonlinear deterministic and stochastic systems with unbounded delay", *Taiwanese J. Mathematics* 14, 1777–1797 (2010).

[13] R. Sakthivel, Y. Ren, and N.I. Mahmudov, "On the approximate controllability of semilinear fractional differential systems", *Comput. Math. Appl.* 62, 1451–1459 (2011).

[14] R. Sakthivel, N.I. Mahmudov, and J.J. Nieto, "Controllability for a class of fractional-order neutral evolution control systems", *Appl. Math. and Comp.* 218, 10334–10340 (2012).

[15] A.E. Bashirov and N.I. Mahmudov, "On concepts of controllability for deterministic and stochastic systems", *SIAM J. Control and Optimization* 37, 1808–1821 (1999).

[16] K. Balachandran and J.P. Dauer, "Controllability of functional differential systems of Sobolev type in Banach spaces", *Kybernetika* 34, 349–357 (1998).

[17] J.P. Dauer and N.I. Mahmudov, "Approximate controllability of semilinear functional equations in Hilbert spaces", *J. Math. Anal. Appl.* 273, 310–327 (2002).

[18] N.I. Mahmudov, "Approximate controllability of evolution systems with nonlocal conditions", *Nonlinear Anal.* 68, 536–546 (2008).

[19] N.I. Mahmudov and S. Zorlu, "Controllability of semilinear stochastic systems", *Int. J. Control* 78, 997–1004 (2005).

[20] R. Sakthivel, S. Suganya, and S.M. Anthoni, "Approximate controllability of fractional stochastic evolution equations", *Comput. Math. Appl.* 63 (3), 660–668 (2012).

[21] R. Sakthivel, R. Ganesh, and S. Suganya, "Approximate controllability of fractional neutral stochastic system with infinite delay", *Rep. Math. Phys.* 70 (3), 291–311 (2012).

[22] R. Sakthivel, R. Yong, and N.I. Mahmudov, "Approximate controllability of second-order stochastic differential equations with impulsive effects", *Modern Phys. Lett.* B 24, 1559–1572 (2010).

[23] J. Wang and Y. Zhou, "Existence and controllability results for fractional semilinear differential inclusions", *Nonlinear Anal., Real World Appl.* 12, 3642–3653 (2011).

[24] H. Ahmed, "Controllability for Sobolev type fractional integro-differential systems in a Banach space", *Advances in Difference Equations* 2012, 2012:167 (2012).

[25] M. Feckan, J.R. Wang, and Y. Zhou, “Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators”, *J. Optim. Theory Appl.* 156 (1), 79–95 (2013).

[26] K. Balachandran and J.P. Dauer, “Controllability of Sobolev type integrodifferential systems in Banach spaces”, *J. Mathematical Analysis and Applications* 217, 35–348 (1998).

[27] K. Balachandran and S. Karunanithi, “Regularity of solutions of Sobolev type semilinear integrodifferential equations in Banach spaces”, *Electronic J. Differential Equations* 114, 1–8 (2003).

[28] K. Balachandran and J.Y. Park, “Sobolev type integrodifferential equation with nonlocal condition in Banach spaces”, *Taiwanese J. Mathematics* 7, 155–163 (2003).

[29] K. Balachandran and R. Sakthivel, “Controllability of Sobolev type semilinear integrodifferential systems in Banach spaces”, *Applied Mathematics Letters* 12, 63–71 (1999).

[30] N.I. Mahmudov, “Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces”, *Abstr. Appl. Anal.* Art. ID 502839, 1–9 (2013).

[31] Y. Zhou and F. Jiao, “Existence of mild solutions for fractional neutral evolution equations”, *Comput. Math. Appl*. 59, 1063–1077 (2010).

[32] Y. Zhou and F. Jiao, “Nonlocal Cauchy problem for fractional evolution equations”, *Nonlinear Anal., Real World Appl.* 11, 4465–4475 (2010).

[33] H. Brill, “A semilinear Sobolev evolution equation in Banach space”, *J. Differential Equations* 24, 412–425 (1977).

[34] R.E. Showalter, “Existence and representation theorem for a semilinear Sobolev equation in Banach space”, *SIAM J. on Mathematical Analysis* 3, 527–543 (1972).

[35] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, No. 45, Series: Encyclopedia of Mathematics and its Applications, London, 1992.