REPORTS ON MATHEMATICAL LOGIC
56 (2021), 75–99
doi:10.4467/20842589RM.21.006.14376

Satoru NIKI and Hitoshi OMORI

A NOTE ON HUMBERSTONE'S CONSTANT Ω

A b s t r a c t. We investigate an expansion of positive intuitionistic logic obtained by adding a constant Ω introduced by Lloyd Humberstone. Our main results include a sound and strongly complete axiomatization, some comparisons to other expansions of intuitionistic logic obtained by adding actuality and empirical negation, and an algebraic semantics. We also briefly discuss its connection to classical logic.

1. Introduction

In [8, §3], Lloyd Humberstone introduced an expansion of the implicational fragment of intuitionistic logic as one of the simple examples of an expansion of intuitionistic logic that lacks the Deduction Theorem. The expansion of Humberstone involves a constant Ω which is different from the more usual

Received 19 February 2021
Keywords and phrases: intuitionistic logic, minimal logic, actuality operator, empirical negation, deduction theorem.
AMS subject classification: 03B20, 03B50.

falsum constant $\bot$. The aim of this note is to explore an expansion of the positive intuitionistic logic obtained by adding the constant Ω. We refer to the resulting expansion as $\mathbf{IPC}^{\Omega}$. Our main observations include an axiomatization, an algebraic semantics, and a comparison with two expansions of the positive intuitionistic logic, one known as **JX** of Krister Segerberg, originally introduced by Ingebrigt Johansson, and another obtained by adding the actuality operator, which is also discussed in [8, §3] by Humberstone.

Our note is structured as follows. After introducing the semantics and proof system for $\mathbf{IPC}^{\Omega}$ in §2, we establish soundness and strong completeness result in §3. We then briefly compare **JX** and $\mathbf{IPC}^{\Omega}$ in §4. This will be followed by another presentation of $\mathbf{IPC}^{\Omega}$ in §5, and in §6 we add the falsum constant $\bot$ and observe some interesting features of the resulting system. We then turn our attention to another expansion of intuitionistic logic, also considered by Humberstone, in terms of actuality operator in §7, and offer a brief comparison. This will be enriched by devising an algebraic semantics for the main systems in §8. The note is then concluded in §9 with a brief discussion on regarding the expansions of intuitionistic logic as adding classical negation to intuitionistic logic, as well as a few directions for further investigations.

2. Semantics and Proof system

The language $\mathcal{L}^{\Omega}$ consists of a finite set $\{\Omega, \wedge, \vee, \to\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}^{\Omega}$. We denote a formula of $\mathcal{L}^{\Omega}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}^{\Omega}$ by Γ, Δ, Σ, etc.

We now present the semantics, and then turn to the proof system.

Definition 2.1 (Humberstone)**.** An $\mathbf{IPC}^{\Omega}$-model for the language $\mathcal{L}^{\Omega}$ is a quadruple $\langle W, g, \leq, V\rangle$, where W is a non-empty set (of states); $g \in W$ (the base state); $\leq$ is a partial order on W with g being the least element; and $V : W \times \mathsf{Prop} \to \{0, 1\}$ an assignment of truth values to state-variable pairs with the condition that $V(w_1, p) = 1$ and $w_1 \leq w_2$ only if $V(w_2, p) = 1$ for all $p \in \mathsf{Prop}$ and all $w_1, w_2 \in W$. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$;
- $I(w, \Omega) = 1$ iff $w \neq g$;
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$;
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$;
- $I(w, A \to B) = 1$ iff for all $x \in W$: if $w \leq x$ and $I(x, A) = 1$ then $I(x, B) = 1$.

Semantic consequence is now defined in terms of truth preservation at g: $\Gamma \models A$ iff for all $\mathbf{IPC}^{\Omega}$-models $\langle W, g, \leq, I\rangle$, $I(g, A) = 1$ if $I(g, B) = 1$ for all $B \in \Gamma$.

Then, as expected, the heredity condition carries over to Form.

Lemma 2.2. *For all $A \in \mathsf{Form}$ and for all $w_1, w_2 \in W$, $I(w_1, A) = 1$ and $w_1 \leq w_2$ only if $I(w_2, A) = 1$.*

Proof. By induction on the complexity of the formula A. We only treat the case with Ω since other cases are standard. Assume that we have $x, y \in W$ with $I(x, \Omega) = 1$, $x \leq y$ and $I(y, \Omega) \neq 1$. Then the last condition is equivalent to $y = g$, and so together with $x \leq y$, we obtain $x \leq g$. But, recall too that g is the least element with respect to $\leq$. Therefore, by the antisymmetry of $\leq$, $x = g$. But, this contradicts the fact that $I(x, \Omega) = 1$ (which is equivalent to $x \neq g$). □

Remark 2.3. Note that we have $\Omega \models A$ for all formulas A, in view of the fact that $I(g, \Omega) \neq 1$. However, $\not\models \Omega \to p$ since if we consider a model with two points g, w with an order $g \leq w$, and an assignment V with $V(g, p) \neq 1$ and $V(w, p) \neq 1$, then we have $I(g, \Omega \to p) \neq 1$. Therefore, we may observe the failure of the Deduction Theorem. This also implies that Ω is not definable in intuitionistic logic since the Deduction Theorem holds in intuitionistic logic.

We also note that the *falsum* constant $\bot$ is not definable in $\mathbf{IPC}^{\Omega}$.

Proposition 2.4. *In $\mathbf{IPC}^{\Omega}$, $\bot$ is not definable.*

Proof. Consider a model $\langle W, g, \leq, V\rangle$ such that $|W| \geq 2$ and $V(w, p) = 1$ for all w and p. Then for $w \neq g$, we can show by induction that $I(w, B) = 1$ for any B. Hence $\bot$, for which $I(w, \bot)$ must be 0, cannot be defined. □

We now present a proof system in terms of a Hilbert-style calculus. Note that Humberstone discusses the proof-theoretical aspect of Ω in terms of sequent calculus.

Definition 2.5. The system $\mathbf{IPC}^{\Omega}$ consists of the following axiom schemata and rules of inference:

$$A \to (B \to A) \quad \text{(Ax1)}$$

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \quad \text{(Ax2)}$$

$$(A \wedge B) \to A \quad \text{(Ax3)}$$

$$(A \wedge B) \to B \quad \text{(Ax4)}$$

$$(C \to A) \to ((C \to B) \to (C \to (A \wedge B))) \quad \text{(Ax5)}$$

$$A \to (A \vee B) \quad \text{(Ax6)}$$

$$B \to (A \vee B) \quad \text{(Ax7)}$$

$$(A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \quad \text{(Ax8)}$$

$$A \vee (A \to \Omega) \quad \text{(Ax9)}$$

$$\frac{A \quad A \to B}{B} \ \text{(MP)} \qquad \frac{\Omega \vee C}{A \vee C} \ \text{(DEOQ)}$$

Finally, we write $\Gamma \vdash A$ if there is a sequence of formulas $B_1, \dots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \dots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of $\mathbf{IPC}^{\Omega}$; (iii) is obtained by (MP) or (DEOQ) from formulas preceding it in sequence.[1]

Although the Deduction Theorem fails, we do have a slightly modified version. We now turn to establishing the result.

Proposition 2.6. *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, *if* $\Gamma, A \vdash B$ *then* $\Gamma \vdash A \to (B \vee \Omega)$.

Proof. By the induction on the length n of the proof of $\Gamma, A \vdash B$. If $n = 1$, then we have the following three cases.

- If B is one of the axioms of $\mathbf{IPC}^{\Omega}$, then we have $\vdash B$. Therefore, by (Ax6) and (Ax1), we obtain $\vdash A \to (B \vee \Omega)$ which implies the desired result.
- If $B \in \Gamma$, we have $\Gamma \vdash B$, and thus we obtain the desired result by (Ax6) and (Ax1).
- If $B = A$, then by (Ax6), we have $A \to (B \vee \Omega)$ which implies the desired result.

[1] (DEOQ) abbreviates *d*isjunctive version of *ex omega quodlibet*.

For $n > 1$, then there are two additional cases to be considered.

- If B is obtained by applying (MP), then we will have $\Gamma, A \vdash C$ and $\Gamma, A \vdash C \to B$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash A \to (C \vee \Omega)$ and $\Gamma \vdash A \to ((C \to B) \vee \Omega)$, and by making use of a thesis in positive intuitionistic logic, we obtain $\Gamma \vdash A \to (B \vee \Omega)$ as desired.
- If B is obtained by applying (DEOQ), then $B = C \vee D$ and we will have $\Gamma, A \vdash \Omega \vee D$ length of the proof of which is less than n. Thus, by induction hypothesis, we have $\Gamma \vdash A \to ((\Omega \vee D) \vee \Omega)$. By (Ax6), *inter alia*, we have $\Gamma \vdash A \to ((C \vee D) \vee \Omega)$ as desired.

This completes the proof. □

Proposition 2.7. *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash A \to (B \vee \Omega)$ *then* $\Gamma, A \vdash B$*.*

Proof. By the assumption $\Gamma \vdash A \to (B \vee \Omega)$. Then, by (MP), we have $\Gamma, A \vdash B \vee \Omega$. Thus, we obtain the desired result by (DEOQ). □

By combining Propositions 2.6 and 2.7, we obtain the following theorem.

Theorem 2.8. *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$*,* $\Gamma, A \vdash B$ *iff* $\Gamma \vdash A \to (B \vee \Omega)$*.*

As a corollary of this variant of the Deduction Theorem, we obtain the following which will prove vital for the completeness theorem.

Proposition 2.9. *For all* $\Gamma \cup \{A, B, C\} \subseteq \mathsf{Form}$*, if* $\Gamma, A \vdash C$ *and* $\Gamma, B \vdash C$*, then* $\Gamma, A \vee B \vdash C$*.*

3. Soundness and Completeness

We now turn to prove the soundness and the strong completeness of the axiomatization given in Definition 2.5. The soundness part is straightforward.

Theorem 3.1. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash A$ *then* $\Gamma \models A$*.*

Proof. By induction on the length of the proof. Here we look at the cases for (DEOQ). So supposing $\Gamma \models \Omega \vee C$, we show $\Gamma \models A \vee C$. Assume $I(g, B) = 1$ for all $B \in \Gamma$. Now if $I(g, C) = 0$, then by I.H. $I(g, \Omega) = 1$; thus $g \neq g$, a contradiction. Hence $I(g, C) = 1$; therefore $I(g, A \vee C) = 1$, as desired. □

For the completeness proof, we introduce some notions, following the presentation in [16].

Definition 3.2. We introduce the following notions.

(i) $\Sigma \vdash_\Pi A$ iff $\Sigma \cup \Pi \vdash A$.
(ii) Σ is a Π-*theory* iff:

  (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$.

  (b) if $\vdash_\Pi A \to B$ then (if $A \in \Sigma$ then $B \in \Sigma$).

(iii) Σ is *prime* iff (if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$).
(iv) $\Sigma \vdash_\Pi \Delta$ iff for some $D_1, \ldots, D_n \in \Delta$, $\Sigma \vdash_\Pi D_1 \vee \ldots \vee D_n$.
(v) $\vdash_\Pi \Sigma \to \Delta$ iff for some $C_1, \ldots, C_n \in \Sigma$ and $D_1, \ldots, D_m \in \Delta$:

$$\vdash_\Pi C_1 \wedge \cdots \wedge C_n \to D_1 \vee \cdots \vee D_m.$$

(vi) Σ is Π-*deductively closed* iff (if $\Sigma \vdash_\Pi A$ then $A \in \Sigma$).
(vii) $\langle \Sigma, \Delta \rangle$ is a Π-*partition* iff:

  (a) $\Sigma \cup \Delta = \mathsf{Form}$

  (b) $\nvdash_\Pi \Sigma \to \Delta$

(viii) Σ is *non-trivial* iff $A \notin \Sigma$ for some formula A.

Lemma 3.3. *If Γ is a non-empty Π-theory, then $\Pi \subseteq \Gamma$.*

Proof. Take $A \in \Pi$. Then, we have $\Pi \vdash A$. Now since Γ is non-empty, take any $C \in \Gamma$. Then, by (Ax1), we obtain $\Pi \vdash C \to A$, i.e. $\vdash_\Pi C \to A$. Thus, combining this together with $C \in \Gamma$ and the assumption that Γ is Π-theory, we conclude that $A \in \Gamma$. □

We now introduce a number of lemmas concerning extensions of sets with various properties. For the proofs, cf. [4, §2] which are based on [16].

Lemma 3.4. *If $\langle \Sigma, \Delta \rangle$ is a Π-partition then Σ is a prime Π-theory.*

Lemma 3.5. *If* $\nvdash_{\Pi} \Sigma \to \Delta$ *then there are* $\Sigma' \supseteq \Sigma$ *and* $\Delta' \supseteq \Delta$ *such that* $\langle \Sigma', \Delta' \rangle$ *is a* Π*-partition.*

Corollary 3.6. *Let* Σ *be a non-empty* Π*-theory,* Δ *be closed under disjunction, and* $\Sigma \cap \Delta = \emptyset$*. Then there is* $\Sigma' \supseteq \Sigma$ *such that* $\Sigma' \cap \Delta = \emptyset$ *and* Σ' *is a prime* Π*-theory.*

Lemma 3.7. *If* $\Sigma \nvdash \Delta$ *then there are* $\Sigma' \supseteq \Sigma$ *and* $\Delta' \supseteq \Delta$ *such that* $\langle \Sigma', \Delta' \rangle$ *is a partition, and* Σ' *is deductively closed.*

Remark 3.8. Note that the proof of this lemma relies on Proposition 2.9.

Corollary 3.9. *If* $\Sigma \nvdash A$ *then there is* $\Pi \supseteq \Sigma$ *such that* $A \notin \Pi$*,* Π *is a prime* Π*-theory and is* Π*-deductively closed.*

Lemma 3.10. *If* Δ *is a* Π*-theory and* $A \to B \notin \Delta$*, then there is a prime* Π*-theory* Γ*, such that* $A \in \Gamma$ *and* $B \notin \Gamma$*.*

Proof. Let $\Sigma = \{C : A \to C \in \Delta\}$. We check that Σ is a Π-theory. First, if $C_1, C_2 \in \Sigma$ then $A \to C_1, A \to C_2 \in \Delta$. Since $\vdash (A \to C_1 \wedge A \to C_2) \to (A \to (C_1 \wedge C_2))$ and Δ a Π-theory, we have $A \to (C_1 \wedge C_2) \in \Delta$. Thus $C_1 \wedge C_2 \in \Sigma$. Now suppose that $\vdash_{\Pi} C \to D$ and $C \in \Sigma$. Then $\vdash_{\Pi} (A \to C) \to (A \to D)$ and $A \to C \in \Delta$; so $A \to D \in \Delta$ and hence $D \in \Sigma$.

Clearly $A \in \Sigma$ and $B \vee \cdots \vee B \notin \Sigma$. Based on this, let Δ' be the closure of $\{B\}$ under disjunction. Then $\Sigma \cap \Delta' = \emptyset$, and the result follows from Corollary 3.6. □

Remark 3.11. Note that, since Σ is non-trivial, the Γ thus obtained is non-trivial as well.

We are now ready to prove completeness.

Theorem 3.12. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models A$ *then* $\Gamma \vdash A$*.*

Proof. We prove the contrapositive. Suppose that $\Gamma \nvdash A$. Then, by Corollary 3.9, there is a $\Pi \supseteq \Gamma$ such that Π is a prime Π-theory, Π-deductively closed and $A \notin \Pi$. Define the interpretation $\mathfrak{A} = \langle X, \Pi, \leq, I \rangle$, where $X = \{\Delta : \Delta$ is a non-empty non-trivial prime Π-theory$\}$, $\Delta \leq \Sigma$ iff

$\Delta \subseteq \Sigma$ and I is defined thus. For every state Σ and propositional parameter p:

$$I(\Sigma, p) = 1 \text{ iff } p \in \Sigma$$

We show by induction on B that $I(\Sigma, B) = 1$ iff $B \in \Sigma$. We concentrate on the cases where B has the form Ω and $C \to D$.

When $B \equiv \Omega$, if $I(\Sigma, \Omega) = 1$ then by definition $\Sigma \neq \Pi$. Since Σ is non-empty, we have $\Sigma \supsetneq \Pi$ by Lemma 3.3, and this means there is $C \in \Sigma \backslash \Pi$. Because Π is prime and deductively closed, $C \vee (C \to \Omega) \in \Pi$ and so either $C \in \Pi$ or $C \to \Omega \in \Pi$. But the former is impossible by our choice of C. Thus $C \to \Omega \in \Pi$. Now as Σ is a Π-theory, $C \to \Omega \vdash C \to \Omega$ and $C \in \Sigma$ impies $\Omega \in \Sigma$. For the other direction, if $\Omega \in \Sigma$ and $\Sigma = \Pi$, then as Π is deductively closed, $\Omega \vee A \in \Pi$ and consequently $A \vee A \in \Pi$. Hence $A \in \Pi$ for all A, a contradiction. Therefore $\Sigma \neq \Pi$, which amounts to $I(\Sigma, \Omega) = 1$.

When $B \equiv C \to D$, by IH $I(\Sigma, C \to D) = 1$ iff for all Δ s.t. $\Sigma \subseteq \Delta$, if $C \in \Delta$ then $D \in \Delta$. Hence it suffices to show that this latter condition is equivalent to $C \to D \in \Sigma$. For the forward direction, we argue by contraposition; so assume $C \to D \notin \Sigma$. Then by Lemma 3.10 we can find a non-empty non-trivial prime Π-theory Σ' such that $C \in \Sigma'$ but $D \notin \Sigma'$. For the backward direction, assume $C \to D \in \Sigma$ and $C \in \Delta$ for any Δ s.t. $\Sigma \subseteq \Delta$. Then $C \to D \in \Delta$ as well, and so $D \in \Delta$ since Δ is a Π-theory.

It now suffices to observe that $B \in \Pi$ for all $B \in \Gamma$ and $A \notin \Pi$, which in view of the above means $\Gamma \not\models A$. This completes the proof. □

4. A comparison to JX of Segerberg

As some readers might have already recognized, $\mathbf{IPC}^{\Omega}$ can be regarded as an extension of **JX** introduced by Johansson in [10] and investigated by Segerberg in [18]. In this section, we will observe a couple of results comparing the two systems. Let us first recall **JX**. We will use the language $\mathcal{L}^{\Omega}$, and thus refer to Segerberg's $\perp$ as Ω.

For the proof system, we may introduce it as a subsystem of $\mathbf{IPC}^{\Omega}$ as follows.

Definition 4.1 (Segerberg)**.** The system **JX** is obtained from $\mathbf{IPC}^{\Omega}$ by eliminating the rule (DEOQ). We refer to the consequence relation as $\vdash_{\mathbf{JX}}$.

For the semantics, we need to add another element to capture the constant Ω.

Definition 4.2 (Segerberg)**.** A **JX**-model for $\mathcal{L}^{\Omega}$ is a quintuple $\langle W, g, \leq, Q, V\rangle$, where W is a non-empty set (of states); $g \in W$; $\leq$ is a partial order on W with g being the least element; Q is an upward closed subset of W and $V : W \times \mathsf{Prop} \to \{0,1\}$ an assignment of truth values to state-variable pairs with the condition that $V(w_1, p) = 1$ and $w_1 \leq w_2$ only if $V(w_2, p) = 1$ for all $p \in \mathsf{Prop}$ and all $w_1, w_2 \in W$. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w,p) = V(w,p)$;
- $I(w,\Omega) = 1$ iff $w \in Q$;
- $I(w, A \wedge B) = 1$ iff $I(w,A) = 1$ and $I(w,B) = 1$;
- $I(w, A \vee B) = 1$ iff $I(w,A) = 1$ or $I(w,B) = 1$;
- $I(w, A \to B) = 1$ iff for all $x \in W$: if $w \leq x$ and $I(x,A) = 1$ then $I(x,B) = 1$.

Furthermore, we require that for all $x \in W$, for all $y \in W \setminus Q$, $x \leq y$ iff $x = y$.[2] We say $\Gamma \models_{\mathbf{JX}} A$ iff for all models $\langle W, g, \leq, Q, I\rangle$, $I(g,A) = 1$ if $I(g,B) = 1$ for all $B \in \Gamma$.

Then, we have the following basic result due to Segerberg.

Theorem 4.3 (Segerberg)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{\mathbf{JX}} A$ *iff* $\Gamma \models_{\mathbf{JX}} A$.

Let us now turn to compare **JX** and $\mathbf{IPC}^{\Omega}$. It should be clear that we immediately obtain the following in view of the definition of the proof systems.

Proposition 4.4. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \vdash_{\mathbf{JX}} A$ *then* $\Gamma \vdash A$.

The other way around, however, does not hold:

Proposition 4.5. $\Omega \vdash A$, *but* $\Omega \nvdash_{\mathbf{JX}} A$.

[2]Segerberg merely states that the accessibility relation $\leq$ is the *identity relation* for elements in $W \setminus Q$. This may be interpreted in a few different ways, but it becomes clear by inspecting his completeness proof that our condition is what he intended. We would like to thank the referee for pressing us to clarify this footnote.

Still, if we focus on the set of theorems, then we do have the following result.

Proposition 4.6. *For all* $A \in \mathsf{Form}$*, if* $\vdash A$ *then* $\vdash_{\mathbf{JX}} A$*.*

Proof. Assume $\vdash A$ and $\nvdash_{\mathbf{JX}} A$. Then by completeness, there is a **JX**-model $\langle W, g, \leq, Q, V\rangle$ such that $I(g, A) = 0$. We may assume $g \notin Q$, because otherwise we may take a new model with a new base state $g' \notin Q$ such that $I(g', p) = 1$ iff $I(g, p) = 1$. Then the original model is a generated submodel of the new model, so $I(g, A) = 0$ in the new model as well. Then $I(g', A) = 0$, i.e. A is refuted in a model with the base state not in Q.

Now, $\langle W, g, \leq, Q, V\rangle$ is nothing but a model of $\mathbf{IPC}^{\Omega}$, expanded by the (redundant) presence of $Q = W \setminus \{g\}$. But then $I(g, \Omega) = 0$ contradicts the assumption that $\vdash A$ by soundness. Therefore $\vdash_{\mathbf{JX}} A$. □

5. Taking 'arrow Ω' as primitive

Given a constant Ω, we may also define a unary operator in terms of $\to$ and Ω. Following the notation of Humberstone, we consider $\neg_{\Omega} A$, defined as $A \to \Omega$, and take this connective as a primitive connective. For this purpose, let $\mathcal{L}^{\neg_{\Omega}}$ be a propositional language consisting of $\{\neg_{\Omega}, \wedge, \vee, \to\}$.

Definition 5.1. A model for the language $\mathcal{L}^{\neg_{\Omega}}$ is a quadruple $\langle W, g, \leq, V\rangle$, defined similarly to $\mathbf{IPC}^{\Omega}$. Valuations V are then extended to interpretations I to state-formula pairs by the same conditions for intuitionistic connectives, and the following condition for $\neg_{\Omega}$:

- $I(w, \neg_{\Omega} A) = 1$ iff for all $x \in W$: if $w \leq x$ and $I(x, A) = 1$ then $x \neq g$.

We set up the following system $\mathbf{IPC}^{\neg_{\Omega}}$ corresponding to the above semantics, and refer to the resulting semantic consequence relation as $\models_N$.

Definition 5.2. The system $\mathbf{IPC}^{\neg_{\Omega}}$ consists of (Ax1)-(Ax8), (MP) and the following axiom schemata and a rule of inference:

$$A \vee \neg_{\Omega} A \quad \text{(N1)}$$
$$(A \wedge \neg_{\Omega} A) \to (B \to \neg_{\Omega} B) \quad \text{(N2)}$$
$$\frac{A \vee B \quad \neg_{\Omega} A}{B} \quad \text{(DS)}$$

We shall use $\vdash_N$ for the derivability in $\mathbf{IPC}^{\neg_{\Omega}}$.

Remark 5.3. The above rule (DS) is included in view of an interesting observation by Sergei Odintsov in [14, pp.87–88].

Proposition 5.4. *For all formulas A, B, the following formulas are derivable in* $\mathbf{IPC}^{\neg\Omega}$.

$$(A \to \neg_\Omega A) \to \neg_\Omega A \tag{An}$$

$$(A \to B) \to (\neg_\Omega B \to \neg_\Omega A) \tag{C}$$

Proof. (An) follows from (N1). Then (C) follows from (N2), (An) and $\vdash_N (A \wedge (A \to B)) \to (A \wedge B)$. □

Remark 5.5. It is known, thanks to [1], that (C) and (An) defines the negation of minimal logic (with primitive negation). It therefore follows that $\mathbf{IPC}^{\neg\Omega}$ strictly contains minimal logic.

Again the Deduction Theorem fails, as expected, but we do have a slightly modified version. We now turn to establish the result.

Proposition 5.6. *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, *if* $\Gamma, A \vdash_N B$ *then* $\Gamma \vdash_N \neg_\Omega A \vee B$.

Proof. By the induction on the length n of the proof of $\Gamma, A \vdash_N B$. If $n = 1$, then we have the following three cases.

- If B is one of the axioms of $\mathbf{IPC}^{\neg\Omega}$, then we have $\vdash_N B$. Therefore, by (Ax7), we obtain $\vdash_N \neg_\Omega A \vee B$ which implies the desired result.
- If $B \in \Gamma$, we have $\Gamma \vdash_N B$, and thus we obtain the desired result by (Ax7).
- If $B = A$, then by (N1), we have $\vdash_N \neg_\Omega A \vee B$ which implies the desired result.

For $n > 1$, then there are two additional cases to be considered.

- If B is obtained by applying (MP), then we will have $\Gamma, A \vdash_N C$ and $\Gamma, A \vdash_N C \to B$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_N \neg_\Omega A \vee C$ and $\Gamma \vdash_N \neg_\Omega A \vee (C \to B)$, and by making use of a thesis in positive intuitionistic logic, we obtain $\Gamma \vdash_N \neg_\Omega A \vee B$ as desired.

- If B is obtained by applying (DS), then we will have $\Gamma, A \vdash_N C \vee B$ and $\Gamma, A \vdash_N \neg_\Omega C$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_N \neg_\Omega A \vee (C \vee B)$ and $\Gamma \vdash_N \neg_\Omega A \vee \neg_\Omega C$, therefore $\Gamma \vdash_N (A \wedge C) \vee (\neg_\Omega A \vee B)$, by making use of (N1), and $\Gamma \vdash_N \neg_\Omega(A \wedge C)$, by making use of a thesis in minimal logic. By (DS), we have $\Gamma \vdash_N \neg_\Omega A \vee B$ as desired.

This completes the proof. □

Proposition 5.7. *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, *if* $\Gamma \vdash_N \neg_\Omega A \vee B$ *then* $\Gamma, A \vdash_N B$.

Proof. By the assumption $\Gamma \vdash_N \neg_\Omega A \vee B$. Therefore, $\Gamma, A \vdash_N \neg_\Omega A \vee B$. Moreover, by making use of a thesis in minimal logic, we have $\Gamma, A \vdash_N \neg_\Omega \neg_\Omega A$. Thus, we obtain $\Gamma, A \vdash_N B$ by (DS), as desired. □

By combining Propositions 5.6 and 5.7, we obtain the following theorem.

Theorem 5.8. *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, $\Gamma, A \vdash_N B$ *iff* $\Gamma \vdash_N \neg_\Omega A \vee B$.

As a corollary of this form of the Deduction Theorem, we obtain the following which will prove vital for the completeness theorem.

Lemma 5.9. *For all* $\Gamma \cup \{A, B, C\} \subseteq \mathsf{Form}$, *if* $\Gamma, A \vdash_N C$ *and* $\Gamma, B \vdash_N C$ *then* $\Gamma, A \vee B \vdash_N C$.

We now check the soundness and the completeness of $\mathbf{IPC}^{\neg_\Omega}$ with respect to the models for $\mathcal{L}^{\neg_\Omega}$. The basic outline is identical to that of $\mathbf{IPC}^{\neg_\Omega}$. Since we have the previous lemma, the proofs for other lemmas are identical to the case for $\mathbf{IPC}^{\Omega}$. Finally we show:

Theorem 5.10. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_N A$ *iff* $\Gamma \models_N A$.

Proof. For the soundness direction, we need to check (N1),(N2) and (DS) are valid in the models. For (N2), if $I(w, A \wedge \neg_\Omega A)=1$, then $w \neq g$; so $I(w, \neg_\Omega B)=1$ and thus, $I(g, (A \wedge \neg_\Omega A) \rightarrow (B \rightarrow \neg_\Omega B))=1$.

For the completeness direction, if $\Gamma \nvdash_N A$, then we construct a countermodel as in Theorem 3.12. We have to check the following:

$$I(\Sigma, \neg_\Omega A) = 1 \text{ iff } \neg_\Omega A \in \Sigma.$$

For the left-to-right direction, we show the contrapositive. If $\neg_\Omega A \notin \Sigma$, then $A \to (\neg_\Omega B \wedge \neg_\Omega \neg_\Omega B) \notin \Sigma$, because Σ is a Π-theory and $\vdash_N (A \to (\neg_\Omega B \wedge \neg_\Omega \neg_\Omega B)) \to \neg_\Omega A$ by (N2) and (An). Hence by (the analogue of) Lemma 3.10, there is a non-trivial prime Π-theory $\Sigma' \supseteq \Sigma$ such that $A \in \Sigma'$ and $\neg_\Omega B \wedge \neg_\Omega \neg_\Omega B \notin \Sigma'$. Now if $\Sigma' \neq \Pi$, then there is $C \in \Sigma'/\Pi$. Since Π is deductively closed, it holds by (N1) that $C \vee \neg_\Omega C \in \Pi$. Then either $C \in \Pi$ or $\neg_\Omega C \in \Pi$ by the primeness of Π. Since the former contradicts our assumption for C, it has to be that $\neg_\Omega C \in \Pi$. But $\neg_\Omega C \vdash_N C \to (\neg_\Omega B \wedge \neg_\Omega \neg_\Omega B)$ by (N2) and (An). Thus $\neg_\Omega B \wedge \neg_\Omega \neg_\Omega B \in \Sigma'$ by Σ' being a Π-theory, which is a contradiction. Therefore $\Sigma' = \Pi$. Consequently $I(\Sigma, \neg_\Omega A) = 0$.

For the right-to-left direction, assume $\neg_\Omega A \in \Sigma$. If for $\Sigma' \supseteq \Sigma$ it holds that $I(\Sigma', A) = 1$, then by I.H. $A \in \Sigma'$. Now if $\Sigma' = \Pi$, then for any C, $\neg_\Omega A \vee C \in \Sigma'$, as Σ' is a Π-theory. Hence $C \in \Pi$ for all C, as $\Pi (= \Sigma')$ is deductively closed. This contradicts the non-triviality of Π. Thus $\Sigma' \neq \Pi$. So $I(\Sigma, \neg_\Omega A) = 1$. □

6. An expansion by ⊥

We noted earlier in Proposition 2.4 that $\bot$ is not definable in $\mathbf{IPC}^\Omega$. However, if we add $\bot$, then it is observed by Humberstone in [8, p.67] that $\bot$ and Ω become equivalent in the sense that $\bot \vdash \Omega$ and $\Omega \vdash \bot$, while $A \to \Omega \vdash A \to \bot$ does *not* generally hold. Here we shall also consider the addition of $\bot$ to the system. We shall use the language $\mathcal{L}^\Omega_\bot$ which consists of a set $\{\Omega, \bot, \wedge, \vee, \to\}$ of propositional connectives. Then we add one clause and one axiom to the semantics and the proof system, respectively, in the following manner.

Definition 6.1. A model for the language $\mathcal{L}^\Omega_\bot$ is defined as in Definition 2.1. We add the following clause for the interpretation of $\bot$.

- $I(w, \bot) = 0$.

Correspondingly, we have the next axiomatization.

Definition 6.2. The system $\mathbf{IPC}^{\Omega}_{\bot}$ is defined by adding the following axiom scheme to $\mathbf{IPC}^{\Omega}$:

$$\bot \to A. \tag{Ax10}$$

We shall use $\neg A$ as the abbreviation for $A \to \bot$. Let us use $\vdash_{\bot}$ and $\models_{\bot}$ for the consequences in the above semantics and proof system. Then we can demonstrate the soundness and completeness in the same way as $\mathbf{IPC}^{\Omega}$.

Theorem 6.3. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \vdash_{\bot} A$ *iff* $\Gamma \models_{\bot} A$*.*

The addition of $\bot$ to the language gives an interesting view of the extensions of $\mathbf{IPC}^{\Omega}_{\bot}$. For this purpose, we first observe a theorem of the system.

Proposition 6.4. $\vdash_{\bot} \neg\Omega \vee \neg\neg\Omega$.

Proof. By (Ax9), $\vdash_{\bot} (\neg\Omega \vee \neg\neg\Omega) \vee ((\neg\Omega \vee \neg\neg\Omega) \to \Omega)$. From the latter disjunct, it follows that $\neg\Omega \to \Omega$ and $\neg\neg\Omega \to \Omega$. Now since $(\neg A \to A) \to \neg\neg A$ holds intuitionistically, we obtain $\neg\neg\Omega$ and thus Ω. Hence $\vdash_{\bot} (\neg\Omega \vee \neg\neg\Omega) \vee \Omega$. Therefore by (DEOQ), we conclude $\vdash_{\bot} \neg\Omega \vee \neg\neg\Omega$. □

On the other hand, we have the following.

Proposition 6.5. *We have both* $\nvdash_{\bot} \neg\Omega$ *and* $\nvdash_{\bot} \neg\neg\Omega$*.*

Proof. For the former, consider a model with at least two states. For the latter, consider a model with a single state. □

This suggests two ways of extending $\mathbf{IPC}^{\Omega}_{\bot}$, one by $\neg\Omega$ and another by $\neg\neg\Omega$. The former extension collapses $\bot$ and Ω and thus the resulting logic becomes classical. The latter extension, on the other hand, is more interesting in that it cannot be extended to classical logic, for Ω would be derivable if that were possible. Therefore the situation is quite different from intermediate logics, which are all contained in classical logic. For the rest of the section, we establish the frame condition for $\neg\neg\Omega$ and thereby the completeness of the extension.

Lemma 6.6. *Let* I *be an interpretation of a model of* $\mathbf{IPC}^{\Omega}_{\bot}$*.*
Then $I(g, \neg\neg\Omega) = 1$ *iff for some* $w \in W$*,* $w \neq g$*.*

Proof. For the left-to-right direction, if $I(g, \neg\neg\Omega) = 1$, then for all $w \geq g$ there is $w' \geq w$ such that $I(w', \Omega) = 1$. In particular, for some $w \in W, w \neq g$. For the right-to-left direction, assume $w \neq g$ for some $w \in W$. Then if $I(u, \neg\Omega) = 1$ for some $u \geq g$, then $\forall u' \geq u(u' = g)$ This implies $W = \{g\}$, a contradiction. Therefore $I(g, \neg\neg\Omega) = 1$. □

Let $\vdash_{\perp+}$ and $\models_{\perp+}$ denote the consequences of $\mathbf{IPC}^{\Omega}_{\perp}$+$\neg\neg\Omega$ and the class of models of $\mathbf{IPC}^{\Omega}_{\perp}$ satisfying $\exists w(w \neq g)$, respectively. Then, we obtain the following result.

Theorem 6.7. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{\perp+} A$ *iff* $\Gamma \models_{\perp+} A$.

Proof. Soundness follows from the last lemma. For completeness, we have to check that the counter-model constructed in Theorem 3.12 satisfies the frame condition. For this, it is sufficient to observe that $\vdash_{\perp+} \neg\neg\Omega$ and the base state Π in the counter-model is deductively closed. Then $\neg\neg\Omega \in \Pi$ and so $I(\Pi, \neg\neg\Omega) = 1$. Hence by the last lemma, $\exists w(w \neq g)$. □

Before moving ahead, let us note that we can state something stronger against the law of excluded middle in $\mathbf{IPC}^{\Omega}_{\perp}$+$\neg\neg\Omega$.

Proposition 6.8. $\Omega \vee \neg\Omega \vdash_{\perp+} \perp$.

Proof. It is immediate that $\Omega \vdash_{\perp+} \perp$ and moreover the additional axiom $\neg\neg\Omega$ implies that $\neg\Omega \vdash_{\perp+} \perp$. Then use the analogue of Proposition 2.9. □

Contrast this with the fact that $A \vee \neg A \vdash \perp$ is *not* provable in intuitionistic logic for any A.

7. A comparison to other expansions of intuitionistic logic

7.1 Actuality and empirical negation

As another variation on the failure of the Deduction Theorem, Humberstone, in [8], turns his attention to a different connective immediately after the discussion of Ω. More specifically, Humberstone discusses another operator R, which represents a persistent notion of 'actuality'.[3] The formulation

[3] One of the seminal papers in this topic, of course, is [2].

of R uses the same class of Kripke frames as $\mathbf{IPC}^{\Omega}$, i.e. with the base state g. The interpretation of R is then given by the following clauses[4]:

- $I(w, RA) = 1$ iff $I(g, A) = 1$.

This expansion of intuitionistic logic has a very close connection to the system $\mathbf{IPC}^{\sim}$, introduced by Michael De in [3], which is obtained by adding *empirical negation*, denoted by $\sim$. Very roughly put, empirical negation expresses that a proposition is currently unverified. This means that the base state is interpreted to be the present moment, and the interpretation of empirical negation is given as the falsity at the base state. In order to make the connection precise, let $\mathcal{L}^{\sim}$ be the language consisting of the connectives $\{\sim, \wedge, \vee, \rightarrow\}$. Then, the semantics introduced by De goes as follows.

Definition 7.1 (De)**.** An $\mathbf{IPC}^{\sim}$-model for the language $\mathcal{L}^{\sim}$ is a quadruple $\langle W, g, \leq, V\rangle$, defined similarly to $\mathbf{IPC}^{\Omega}$. Valuations V are then extended to interpretations I to state-formula pairs by the same conditions for intuitionistic connectives, and the following condition for $\sim$:

- $I(w, \sim A) = 1$ iff $I(g, A) = 0$.

We shall denote the semantic consequence by $\models_e$ (e for empirical negation).

For the proof system, the following system is introduced in [12], building on an axiomatization presented in [4].

Definition 7.2. The system $\mathbf{IPC}^{\sim}$ consists of (Ax1)-(Ax8), (MP) and the following axiom schemata and a rule of inference:

$$A \vee \sim A \qquad \text{(E1)} \qquad\qquad (\sim A \wedge \sim B) \rightarrow \sim(A \vee B) \qquad \text{(E3)}$$

$$\sim A \rightarrow (\sim\sim A \rightarrow B) \qquad \text{(E2)} \qquad\qquad \frac{A \rightarrow B}{\sim B \rightarrow \sim A} \qquad \text{(RC)}$$

We shall use $\vdash_e$ for the derivability in $\mathbf{IPC}^{\sim}$.

Then, the following result is established in [4].

Theorem 7.3 (De & Omori)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \vdash_e A$ *iff* $\Gamma \models_e A$*.*

[4]The details of this notion of actuality is investigated in [13], along with its relationship with similar operators.

Now, as observed in [13, §4.1], we have that $I(w, \neg RA) = 1$ iff $I(g, A) = 0$ iff $I(w, {\sim}A) = 1$. That is, if R is added on top of intuitionistic logic, then two expansions obtained by adding R and $\sim$ will be equivalent. Since Ω also very naturally gives rise to $\neg_\Omega$, let us compare Humberstone's additions of Ω and R in terms of $\neg_\Omega$ and $\sim$.

7.2 A quick comparison between $\mathbf{IPC}^{\neg_\Omega}$ and $\mathbf{IPC}^{\sim}$

By looking at the truth conditions for $\neg_\Omega$ and $\sim$, they are identical at the base state:

Fact 7.4. For all $A \in \mathsf{Form}$, $I(g, \neg_\Omega A) = 1$ iff $I(g, A) = 0$ iff $I(g, {\sim}A) = 1$.

In other words, in the privileged 'present' world, $\neg_\Omega A$ is capable of asserting that A does not hold at the moment. Then it appears that $\neg_\Omega$ can also be a contender for the formalisation of empirical negation. Note, however, that we do *not* have the following: $I(w, \neg_\Omega A) = 1$ iff $I(w, {\sim}A) = 1$. Therefore, it will be a different kind of empirical negation. What then is the intuitive reading of $\neg_\Omega$? Given the assumption that the base state represents the present moment, Ω can then be interpreted as a proposition stating that 'It is in the future'. Thus $\neg_\Omega A$ should mean 'If A is verified, it is in the future.' This is arguably a natural interpretation of empirical statements like 'Goldbach's conjecture is not proved'. However, we will leave the further philosophical comparisons for another occasion. Instead, let us turn to observe a few more differences between $\mathbf{IPC}^{\neg_\Omega}$ and $\mathbf{IPC}^{\sim}$.

Lemma 7.5. *$\models (A \land \neg_\Omega A) \to \neg_\Omega B$ but $\not\models_e (p \land {\sim}p) \to {\sim}q$.*

Proof. The former is an easy consequence of (C). For the latter, take $W{=}\{g, w\}$ such that $V(w, p) = V(g, q){=}1$ and $V(g, p){=}0$. Then $I(w, p{\land}{\sim}p){=}1$ but $I(w, {\sim}q){=}0$. Therefore $I(g, (p{\land}{\sim}p){\to}{\sim}q){=}0$. □

Lemma 7.6. *$\models_e {\sim}{\sim}A \to A$ but $\not\models \neg_\Omega\neg_\Omega p \to p$.*

Proof. For the former, we have $I(w, {\sim}{\sim}A){=}1$ if and only if $I(g, A){=}1$; so $I(w, A)$ holds by persistence. Therefore $I(g, {\sim}{\sim}A \to A) = 1$ in any model. On the other hand, in a model with more than two worlds, and for $w \neq g$, it always hold that $I(w, \neg_\Omega\neg_\Omega p) = 1$. But if $V(w, p) = 0$, then $I(g, \neg_\Omega\neg_\Omega p \to p) = 0$. □

Based on these lemmas, we obtain the following result.

Theorem 7.7. *After identifying the connectives* $\neg_\Omega$ *and* $\sim$*,* $\mathbf{IPC}^{\neg_\Omega}$ *and* $\mathbf{IPC}^{\sim}$ *are incomparable with respect to inclusion.*

Note also that the Deduction Theorem again fails for $\vdash_e$, but a slightly modified version of the Deduction Theorem holds for $\mathbf{IPC}^{\sim}$, namely $\Gamma, A \vdash_e B$ iff $\Gamma \vdash_e {\sim\sim} A \to B$. This obtained by combining $\Gamma, A \vdash_e B$ iff $\Gamma \vdash_e {\sim} A \vee B$, just like Theorem 5.8, and the equivalence $\vdash_e ({\sim} A \vee B) \leftrightarrow ({\sim\sim} A \to B)$.

Finally, we noted above that ${\sim} A$ can be defined as $RA{\to}\bot$. Then it seems possible to formulate another unary operator by $RA{\to}\Omega$. This operator, however, turns out to be identical to $A{\to}\Omega$. At the base world, A is true iff RA is true; and at other worlds, the two implications are both always true.

8. Algebraic Semantics for $\mathbf{IPC}^{\Omega}$ and $\mathbf{IPC}^{\sim}$

We now turn to look at the algebraic semantics for the systems $\mathbf{IPC}^{\Omega}$ and $\mathbf{IPC}^{\sim}$, as well as other related systems. We refer to [15] for the basics of algebraic semantics. First, here are some of the basics of Heyting algebras.

Definition 8.1 (Heyting algebra)**.** A *Heyting algebra* $\mathbf{A}$ is a quintuple $\langle A, \vee, \wedge, \to, 0\rangle$, where $\langle A. \vee, \wedge\rangle$ is lattice with the least element 0, and the operation $\to$ satisfies the following *law of residuation*:

$$a \wedge b \leq c \text{ if and only if } a \leq b \to c$$

for all $a, b, c \in A$.

We shall use 1 for the element $0 \to 0$ and $\neg a$ for $a \to 0$. Note 1 is the greatest element of a Heyting algebra. If $0 = 1$, then the algebra is called *degenerate*. If a Heyting algebra has the second greatest element, it is called a *subdirectly irreducible algebra* (cf. [15, 17] for more precise details). We shall use this class of algebras for the semantics of $\mathbf{IPC}^{\Omega}$ and $\mathbf{IPC}^{\sim}$.

8.1 An algebraic semantics for $\mathbf{IPC}^{\Omega}$

We first deal with $\mathbf{IPC}^{\Omega}$.

Definition 8.2 (Humberstone algebra)**.** A *Humberstone algebra* $\mathbf{A}$ is a sextuple $\langle A, \vee, \wedge, \rightarrow, 0, \omega\rangle$, where $\langle A, \vee, \wedge, \rightarrow, 0\rangle$ is a subdirectly irreducible Heyting algebra, and ω is its second greatest element.[5]

Note that the idea of adding an constant corresponding to the second greatest element is not new, as one may observe in [9, 19], where a propositional variable is assigned for each element of a subdirectly irreducible Heyting algebra.

For the purpose of defining an algebraic model, we introduce the notion of assignments.

Definition 8.3 (Assignment)**.** Let A be a Humberstone algebra. An *assignment* h is mapping which assigns each propositional variable to an element $h(p)$ of A. h is then extended with the following clauses.

- $h(A \wedge B) = h(A) \wedge h(B)$;
- $h(A \vee B) = h(A) \vee h(B)$;
- $h(A \rightarrow B) = h(A) \rightarrow h(B)$;
- $h(\Omega) = \omega$.

We shall write $\Gamma \models_o A$ if for every assignment h of any non-degenerate Humberstone algebra, $h(B) = 1$ for all $B \in \Gamma$ implies $h(A) = 1$. Then we observe that $\mathbf{IPC}^{\Omega}$ is sound with respect to Humberstone algebras.

Theorem 8.4 (Soundness)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash A$ *then* $\Gamma \models_o A$*.*

Proof. We argue by induction on the length of proof. It suffices to consider the cases related to Ω. For (Ax9), we must show $h(A \vee (A \rightarrow \Omega)) = 1$ for any assignment h. If $h(A) < 1$, then $h(A) \leq \omega$, so $1 \wedge h(A) \leq \omega$. Thus by the law of residuation, $1 \leq h(A) \rightarrow \omega$. Thus $h(A) \vee h(A \rightarrow \Omega) = 1$; consequently $h(A \vee (A \rightarrow \Omega)) = 1$. For (DEOQ),

[5] Note that there is already a notion of Humberstone algebra introduced in [11], but we will use the same label here since there should be no room for confusion.

suppose $\models_o \Omega \vee C$. Then $h(\Omega \vee C) = 1$ for any h in any Humberstone algebra $\mathbf{A}$. Now since $\mathbf{A}$ is subdirectly irreducible, it is *well-connected* [15, p.125], which in this case means either $h(\Omega) = 1$ or $h(C) = 1$. But as $\omega \neq 1$, it has to be that $h(C) = 1$; so $h(A \vee C) = 1$ as well. Thus $\models_o A \vee C$. □

For the algebraic completeness, we shall argue via Kripke completeness. To this end, we introduce the notion of *dual Humberstone algebra*. Given a Kripke frame $\langle W, g, \leq \rangle$ for $\mathbf{IPC}^{\Omega}$, let $\mathcal{U}(W)$ be the set of all upward closed subsets of the frame. Note in particular that $W \setminus \{g\} \in \mathcal{U}(W)$.

Definition 8.5 (dual Humberstone algebra)**.** Let $\mathcal{F} = \langle W, g, \leq \rangle$ be a Kripke frame for $\mathbf{IPC}^{\Omega}$. Then the *dual Humberstone algebra* $\mathcal{U}(\mathcal{F})$ of $\mathcal{F}$ is $\langle \mathcal{U}(W), \cup, \cap, \Rightarrow, \emptyset, W \setminus \{g\} \rangle$ where

$$U_1 \Rightarrow U_2 := \{a \in W : \forall c \geq a (c \in U_1 \text{ implies } c \in U_2)\}$$

for all $U_1, U_2 \in \mathcal{U}(W)$.

Lemma 8.6. *A dual Humberstone algebra is indeed a Humberstone algebra.*

Proof. It is immediate from the fact that a dual Heyting algebra is a Heyting algebra, and that the rootedness of the frame corresponds with the dual algebra being subdirectly irreducible. Clearly, in particular, $W \setminus \{g\}$ is the second greatest element of the dual algebra. Cf. [15, p. 107, 124]. □

We are now ready to establish the completeness direction.

Theorem 8.7 (Algebraic completeness)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models_o A$ *then* $\Gamma \vdash A$.

Proof. Assume $\Gamma \models_o A$. Let $\langle W, g, \leq, V \rangle$ be a Kripke model for $\mathbf{IPC}^{\Omega}$ with the frame $\mathcal{F} = \langle W, g, \leq \rangle$. Assume $I(g, B) = 1$ for all $B \in \Gamma$. Let $\mathcal{U}(\mathcal{F})$ be its dual Humberstone algebra. Choose an assignment h such that $h(p) = \{w : V(w, p) = 1\}$. Then as in the case for Heyting algebra, we can show $h(B) = \{w : I(w, B) = 1\}$ for all B; in particular, $h(\Omega) = W \setminus \{g\} = \{w : I(w, \Omega) = 1\}$. Now by assumption, for any $B \in \Gamma$ it holds $h(B) = W$ and so $h(A) = W$. Thus $I(g, A) = 1$. Therefore $\Gamma \models A$, and consequently $\Gamma \vdash A$ by Theorem 3.12. □

Remark 8.8. We can similarly obtain the algebraic semantics for $\mathbf{IPC}^{\Omega}_{\perp}$ and its extension by the axiom $\neg\neg\Omega$. In the latter case, we have to take the class of Humberstone algebras such that $0 \neq \omega$, because then

$$a \wedge \omega \leq 0 \text{ iff } a \leq 0$$

and so $\omega \to 0 = 0$; consequently $h(\neg\neg\Omega) = 1$. It should also be clear that we can devise an algebraic semantics for $\mathbf{IPC}^{\neg\Omega}$.

8.2 An algebraic semantics for $\mathbf{IPC}^{\sim}$

Let us now turn our attention to the algebraic counterpart of $\mathbf{IPC}^{\sim}$.

Definition 8.9 (De algebra)**.** A *De algebra* $\mathbf{A}$ is a sextuple $\langle A, \vee, \wedge, \to, \sim, 0\rangle$, where $\langle A, \vee, \wedge, \to, 0\rangle$ is a subdirectly irreducible Heyting algebra, and $\sim$ is an unary operator satisfying the next conditions:

- $a \vee {\sim}a = 1$;
- ${\sim}a \wedge {\sim}{\sim}a = 0$;
- $a \leq b \Rightarrow {\sim}b \leq {\sim}a$;

for $a, b \in A$.

We define an assignment h for a De algebra in much the same way as a Humberstone algebra, except that $h({\sim}A) = {\sim}h(A)$. We shall use $\models_d$ for the validity.

Lemma 8.10. *In any De algebra,* ${\sim}1 = 0$*.*

Proof. First, we have ${\sim}0 = 0 \vee {\sim}0 = 1$. Thus ${\sim}1 = {\sim}{\sim}0$. Then since ${\sim}{\sim}0 = 1 \wedge {\sim}{\sim}0 = {\sim}0 \wedge {\sim}{\sim}0 = 0$, we conclude ${\sim}1 = 0$. □

Theorem 8.11 (Soundness)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash_e A$ *then* $\Gamma \models_d A$*.*

Proof. We show by induction on the length of the proof. Here we treat the case for (E3); other cases are straightforward. Let $\mathbf{A}$ be a non-degenerate De algebra and h be an assignment. For any proposition A, we know $h(A \vee {\sim}A) = h(A) \vee {\sim}h(A) = 1$. Thus either $h(A) = 1$ or ${\sim}h(A) = 1$.

In particular, when A is of the form $A \vee B$, we have either $h(A \vee B) = 1$ or $\sim h(A \vee B) = 1$. In the latter case, $h(\sim A \wedge \sim B) \leq \sim h(A \vee B)$; so (E3) follows by the law of residuation. In the former case, by well-connectedness again, either $h(A) = 1$ or $h(B) = 1$. Hence by one of the conditions for De algebra, we infer $\sim h(A) \leq \sim 1$ or $\sim h(B) \leq \sim 1$. Then by Lemma 8.10, $\sim h(A) = 0$ or $\sim h(B) = 0$. Thus $h(\sim A \wedge \sim B) = 0$; so (E3) holds for this case as well. □

For the completeness, we again use dual algebras.

Definition 8.12 (dual De algebra)**.** Let $\mathcal{F} = \langle W, g, \leq \rangle$ be a Kripke frame for $\mathbf{IPC}^{\sim}$. Then the *dual De algebra* $\mathcal{U}(\mathcal{F})$ of $\mathcal{F}$ is $\langle \mathcal{U}(W), \cup, \cap, \Rightarrow, \sim, \emptyset \rangle$ where

$$\sim U := \begin{cases} \emptyset & \text{if } U = W. \\ W & \text{if } U \neq W. \end{cases}$$

for all $U \in \mathcal{U}(W)$.

Proposition 8.13. *A dual De algebra is indeed a De algebra.*

Proof. It suffices to check that a dual De algebra satisfies the conditions related to $\sim$.

- If $U = W$, then $U \cup \sim U = W$. Otherwise $\sim U = W$; so $U \cup \sim U = W$.
- If $U = W$, then $\sim U \cap \sim\sim U = \emptyset$. If $U \neq W$, then $\sim U = W$, so $\sim\sim U = \emptyset$. Hence $\sim U \cap \sim\sim U = \emptyset$.
- If $U \subseteq V$, then if $U = W$, then $V = W$ and so $\sim V = \emptyset$. Thus $\sim V \subseteq \sim U$. On the other hand, if $U \neq W$, then $\sim V \subseteq W = \sim U$.

This completes the proof. □

Theorem 8.14 (Algebraic completeness)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models_d A$ *then* $\Gamma \vdash_e A$*.*

Proof. The argument is analogous to that of Theorem 8.7. It suffices to check $h(\sim A) = \{w : I(w, \sim A) = 1\}$. The latter set is an emptyset if $I(w, A) = 1$ for all w; otherwise it equals W. Since $h(\sim A) = \sim h(A)$ and $h(A) = \{w : I(w, A) = 1\}$ by I.H., the desired equality therefore follows. □

9. Concluding remarks

As a final remark, let us consider a Humberstone algebra and a De algebra with three elements $1, i, 0$ with the order $0 \leq i \leq 1$. Equivalently, we may also take respective Kripke models with two states. Then, by focusing on $\neg_\Omega$ and $\sim$, we obtain the following truth table.

A	$\neg_\Omega A$	$\sim A$	Ω
1	i	0	i
i	1	1	i
0	1	1	i

Given that 1 will be the only designated value in defining the corresponding semantic consequence relations, we may observe that both $\neg_\Omega$ and $\sim$ satisfy the condition that negation is designated iff the negand is undesignated. In fact, this is the condition discussed in [5, §2] as a condition for *classical negation* in expansions of the strong Kleene logic $\mathbf{K}_3$. Moreover, as observed in [5, §2.2] via a combinatorial argument, these are the only two classical negations in the context of $\mathbf{K}_3$. However, $\neg_\Omega$ was dismissed because of its unusual behavior with respect to de Morgan negation which is present in $\mathbf{K}_3$.[6] But, in the present context in which de Morgan negation is not available, then we need not be dismissive.[7] We would like to rather make use of this fact to conclude that we may regard $\neg_\Omega$ as classical negation, and that $\neg_\Omega$ together with empirical negation $\sim$ are two natural options for the purpose of adding classical negation to intuitionistic logic.

For further investigations, we may think of a few directions. First, one may explore further into algebraic semantics and along the systematic investigations into extensions of minimal logic carried out by Odinstov in [14]. Second, there are natural questions concerning systems with slightly different versions of (DEOQ) and (DS) without disjunction. Are they strict subsystems of $\mathbf{IPC}^\Omega$ and $\mathbf{IPC}^{\neg_\Omega}$, respectively? If so, can we provide semantics that characterize these systems? Finally, there is a direction to add Ω or $\neg_\Omega$ to subintuitionistic logic **SJ**. This is carried out for empirical negation in [6], and given the tight connections between $\mathbf{IPC}^\Omega$ and $\mathbf{IPC}^\sim$

[6]More specifically, letting $\neg_d$ to stand for de Morgan negation, we obtain that $\neg_\Omega\neg_d\neg_\Omega A$ is valid for *all* A. This will therefore produce another *contra-classical* logic (cf. [7]).

[7]Of course, if we add Ω to Nelson logic **N3**, then the so-called strong negation will correspond to de Morgan negation.

we observed in this paper, it will be interesting to explore the details of the resulting expansions of **SJ**. For the case of adding Ω, note that the axiom (Ax9) and the rule (DEOQ) remain valid in the models for **SJ**.

Acknowledgements

We would like to thank Lloyd Humberstone for directing our attention to his paper [8] after an online presentation by S.N. of our [13] in August, 2020 at *Advances in Modal Logic 2020*. S.N. would also like to thank Hiroakira Ono for his helpful comments on the algebraic semantics. Moreover, we would like to thank Michael De for this helpful comments on an earlier version. Finally, but not the least, we would like to thank the anonymous referee for an extremely detailed, encouraging, and helpful report with a number of comments, corrections, and suggestions. The research of H.O. was supported by a Sofja Kovalevskaja Award of the Alexander von Humboldt-Foundation, funded by the German Ministry for Education and Research. Moreover, for the preparation of the revised version of this note, S.N. was also supported by a Sofja Kovalevskaja Award of the Alexander von Humboldt-Foundation, funded by the German Ministry for Education and Research.

References

[1] A. Colacito, D. de Jongh, and A.L. Vargas, Subminimal negation, *Soft Computing* **21** (2017), 165–174.

[2] J.N. Crossley and L. Humberstone, The logic of “actually”, *Reports on Mathematical Logic* **8** (1977), 11–29.

[3] M. De, Empirical Negation. *Acta Analytica* **28** (2013), 49–69.

[4] M. De and H. Omori, More on empirical negation, In Rajeev Goré, Barteld Kooi, and Agi Kurucz, editors, Advances in modal logic, volume 10, pp. 114–133. College Publications, 2014.

[5] M. De and H. Omori, Classical negation and expansions of Belnap-Dunn logic, *Studia Logica* **103** (2015), 825–851.

[6] M. De and H. Omori, Classical and empirical negation in subintuitionistic logic, in: Lev Beklemishev, Stéphane Demri, and András Máté, editors, Advances in Modal Logic, volume 11, pages 217–235. College Publications, 2016.

[7] L. Humberstone, Contra-classical logics, *Australasian Journal of Philosophy* **78:4** (2000), 438–474.

[8] L. Humberstone, Extensions of intuitionistic logic without the deduction theorem: Some simple examples, *Reports on Mathematical Logic* **40** (2006), 45–82.

[9] V.A. Jankov, Conjunctively indecomposable formulas in propositional calculi, *Mathematics of the USSR-Izvestiya* **3(1):17** (1969).

[10] I. Johansson, Der minimalkalkül, ein reduzierter intuitionistischer formalismus, *Compositio Mathematica* **4** (1937), 119–136.

[11] Y. Komori, On Komori algebras, *Bulletin of the Section of Logic* **30:2** (2001), 67–70.

[12] S. Niki, Empirical negation, co-negation and contraposition rule I: Semantical investigations, *Bulletin of the Section of Logic* **49:3** (2020), 231–253.

[13] S. Niki and H. Omori, Actuality in intuitionistic logic, in: Nicola Olivetti, Rineke Verbrugge, Sara Negri, and Gabriel Sandu, editors, Advances in Modal Logic, volume 13, pp. 459–479. College Publications, 2020.

[14] S. Odintsov, Constructive negations and paraconsistency, volume 26 of Trends in Logic, Springer Science & Business Media, 2008.

[15] H. Ono, Proof Theory and Algebra in Logic, Short Textbooks in Logic, Springer, 2019.

[16] G. Restall, Subintuitionistic Logics, *Notre Dame Journal of Formal Logic* **35:1** (1994), 116–126.

[17] H.P. Sankappanavar and S. Burris, A course in universal algebra, volume 78 of Graduate Texts in Mathematics, Springer, 1981.

[18] K. Segerberg, Propositional logics related to Heyting's and Johansson's, *Theoria* **34:1** (1968), 26–61.

[19] Y. Tanaka, An infinitary extension of Jankov's theorem, *Studia Logica* **86:1** (2007), 111–131.

Satoru Niki

Department of Philosophy I, Ruhr University,
Universitätsstraße 150, D-44780 Bochum, Germany.

`Satoru.Niki@rub.de`

Hitoshi Omori

Department of Philosophy I, Ruhr University,
Universitätsstraße 150, D-44780 Bochum, Germany.

`Hitoshi.Omori@rub.de`
https://www.ruhr-uni-bochum.de/philosophy/nklogik/