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REMARKS ON THE OUTER-INDEPENDENT DOUBLE ITALIAN DOMINATION NUMBER

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Abstract. Let G be a graph with vertex set $V(G)$. If $u \in V(G)$, then $N[u]$ is the closed neighborhood of u. An outer-independent double Italian dominating function (OIDIDF) on a graph G is a function $f : V(G) \longrightarrow \{0,1,2,3\}$ such that if $f(v) \in \{0,1\}$ for a vertex $v \in V(G)$, then $\sum_{x\in N[v]} f(x) \geq 3$, and the set $\{u \in V(G) : f(u) = 0\}$ is independent. The weight of an OIDIDF f is the sum $\sum_{v\in V(G)} f(v)$. The outer-independent double Italian domination number $\gamma_{oidI}(G)$ equals the minimum weight of an OIDIDF on G. In this paper we present Nordhaus–Gaddum type bounds on the outer-independent double Italian domination number which improved corresponding results given in [F. Azvin, N. Jafari Rad, L. Volkmann, *Bounds on the outer-independent double Italian domination number*, Commun. Comb. Optim. 6 (2021), 123–136]. Furthermore, we determine the outer-independent double Italian domination number of some families of graphs.

Keywords: double Italian domination number, outer-independent double Italian domination number, Nordhaus–Gaddum bound.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

For definitions and notations not given here we refer to [9]. We consider simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. The *open neighborhood* of a vertex v is the set

$$N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$$

and its *closed neighborhood* is the set

$$N[v] = N_G[v] = N(v) \cup \{v\}.$$

The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The *complement* $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. If D is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $G[D]$ of G *induced* by D is the graph having vertex set D and whose edge set consists of those edges of G incident with two vertices of D. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. An edge adjacent to a leaf is called a *pendant edge*.

A set S of vertices is *independent* if no two vertices in S are adjacent. The maximum cardinality of an independent set in G is called the *independence number* $\alpha(G)$ of G. A *vertex cover* of a graph G is a set S of vertices such that each edge of G has at least one end point in S. The minimum cardinality of a vertex cover is denoted by $\beta(G)$. If $\delta(G) \geq 1$, then the identity $\alpha(G) + \beta(G) = n(G)$, due to Gallai [8], is well-known. We write P_n for the path of order n, C_n for the cycle of length n and K_n for the complete graph of order n. The *complete t-partite graph* $K_{n_1,n_2,\dots,n_t}$ has $n = n_1 + n_2 + \ldots + n_t$ vertices and $V(K_{n_1,n_2,\dots,n_t}) = S_1 \cup S_2 \cup \ldots \cup S_t$, where $|S_i| = n_i$ for $1 \leq i \leq t$, $\{u, v\} \subseteq S_i$ implies u and v are not adjacent, and $u \in S_i$ and $v \in S_j$ with $i < j$ implies u and v are adjacent, and, specifically, $K_{1,n-1}$ is called a *star*. If we add two disjoint pendant edges to a cycle of length three, then we obtain the *bull graph*, denoted by B_5.

Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [7] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations have been studied (see for example, [3–6]).

Mojdeh and Volkmann [10] considered the following variant of Roman domination. A *double Italian dominating function* (DIDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ such that if $f(v) \in \{0, 1\}$ for a vertex $v \in V(G)$, then $\sum_{x \in N[v]} f(x) \geq 3$. The weight of a DIDF f is the sum $w(f) = \sum_{v \in V(G)} f(v)$, and the minimum weight of a DIDF in a graph G is the *double Italian domination number*, denoted by $\gamma_{dI}(G)$. A DIDF f on G of weight $\gamma_{dI}(G)$ is called a $\gamma_{dI}(G)$-function. For a DIDF f, let (V_0, V_1, V_2, V_3) be the ordered partition of $V(G)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2, 3$. There is a 1–1 correspondence between the function f and the ordered partition (V_0, V_1, V_2, V_3). So we will write $f = (V_0, V_1, V_2, V_3)$. This concept was further studied in [1, 12].

A double Italian dominating function $f = (V_0, V_1, V_2, V_3)$ on a graph G is called in [2] an *outer-independent double Italian dominating function* (OIDIDF) if V_0 is an independent set. The *outer-independent double Italian domination number* $\gamma_{oidI}(G)$ equals the minimum weight of an OIDIDF on G. An OIDIDF f on G of weight $\gamma_{oidI}(G)$ is called a $\gamma_{oidI}(G)$-function. Clearly, $\gamma_{dI}(G) \leq \gamma_{oidI}(G)$.

In this paper we present Nordhaus–Gaddum type results on the outer-independent double Italian domination number which improved corresponding results given in [2]. Furthermore, we give different bounds, and we determine the outer-independent double Italian domination number of some special graphs.

We make use of the following results.

Proposition 1.1 ([10]). *If G is a graph of order $n \geq 2$ then $\gamma_{dI}(G) \geq 3$, with equality if and only if $\Delta(G) = n - 1$.*

Proposition 1.2 ([2]). *Let G be a graph of order n. If $\delta(G) \geq 1$, then $\gamma_{oidI}(G) \leq \frac{3n}{2}$, with equality if and only if $G = pK_2$ for an integer $p \geq 1$.*

The equality part in Proposition 1.2 can be found in the proof of Theorem 5 in [2].

Proposition 1.3 ([2]). *If G is a graph of order n with $\delta(G) \geq 2$, then $\gamma_{oidI}(G) \leq n$.*

Proposition 1.4 ([2]). *If C_n is a cycle of length n, then $\gamma_{oidI}(C_n) = n$.*

Proposition 1.5 ([2]). *If P_n is a path of order $n \geq 4$, then $\gamma_{oidI}(P_n) = n + 1$.*

2. BOUNDS

Theorem 2.1. *Let G be a graph of order n with $\delta(G) \geq 1$.*

(1) ([2]) *Then $\beta(G) \leq \gamma_{oidI}(G) \leq 3\beta(G)$.*
(2) ([2]) *If $\delta(G) \geq 2$, then $\gamma_{oidI}(G) \leq 2\beta(G)$.*
(3) *If $\delta(G) \geq 2$, and G is not bipartite, then $\gamma_{oidI}(G) \leq 2\beta(G) - 1$.*
(4) *Let $\delta(G) \geq 3$, and let S be a vertex cover of minimum cardinality. If $\delta(G[S]) \geq 2$, then $\gamma_{oidI}(G) = \beta(G)$.*

Proof. Let S be a vertex cover of minimum cardinality. Then $V(G) \setminus S$ is a maximum independent set.

Item (1). If we define the function f by $f(x) = 3$ for $x \in S$ and $f(x) = 0$ for $x \in V(G) \setminus S$, then f is an OIDIDF on G of weight $3\beta(G)$. Therefore $\gamma_{oidI}(G) \leq 3\beta(G)$, and the upper bound is proved. For the lower bound assume that g is a $\gamma_{oidI}(G)$-function. Then $g(x) = 0$ for at most $\alpha(G) = n - \beta(G)$ vertices x, and therefore $\gamma_{oidI}(G) = w(g) \geq \beta(G)$.

Item (2). We define f by $f(x) = 2$ for $x \in S$ and $f(x) = 0$ for $x \in V(G) \setminus S$. Since $\delta(G) \geq 2$, we see that f is an OIDIDF on G of weight $2\beta(G)$. Therefore $\gamma_{oidI}(G) \leq 2\beta(G)$.

Item (3). Since G is not bipartite and $V(G) \setminus S$ is independent, the induced subgraph $G[S]$ contains an edge uv. Now define the function f by $f(u) = 1$, $f(x) = 2$ for $x \in S \setminus \{u\}$ and $f(x) = 0$ for $x \in V(G) \setminus S$. We deduce that f is an OIDIDF on G of weight $2\beta(G) - 1$, and thus $\gamma_{oidI}(G) \leq 2\beta(G) - 1$.

Item (4). Define f by $f(x) = 1$ for $x \in S$ and $f(x) = 0$ for $x \in V(G) \setminus S$. Since $\delta(G) \geq 3$ and $\delta(G[S]) \geq 2$, we observe that f is an OIDIDF on G of weight $\beta(G)$. So $\gamma_{oidI}(G) \leq \beta(G)$, and Item (1) leads to $\gamma_{oidI}(G) = \beta(G)$. □

Since $\beta(K_n) = n - 1$ for the complete graph of order $n \geq 2$, the next corollary follows from Theorem 2.1 (4) immediately.

Corollary 2.2. *If $n \geq 4$, then $\gamma_{oidI}(K_n) = n - 1$.*

The next examples will demonstrate that Theorem 2.1 is sharp.

Example 2.3.

(1) If $Q = pK_2$, then $\gamma_{oidI}(Q) = 3p = 3\beta(Q)$.
(2) Let Q be a graph of order q with vertex set $\{v_1, v_2, \ldots, v_q\}$. If we add $t_i \geq 2$ pendant edges to each vertex v_i for $1 \leq i \leq q$, then let H be the resulting graph. Let g be an OIDIDF on H, and let $a_{i,1}, a_{i,2}, \ldots, a_{i,t_i}$ be the leaves adjacent to v_i for $1 \leq i \leq q$. Then it is straightforward to verify that $g(v_i) + \sum_{j=1}^{t_i} g(a_{i,j}) \geq 3$ for each $1 \leq i \leq q$. Therefore $\gamma_{oidI}(H) \geq 3q = 3\beta(H)$. According to Theorem 2.1 (1), we obtain $\gamma_{oidI}(H) = 3q = 3\beta(H)$. These examples show that Theorem 2.1 (1) is sharp.
(3) If C_{2p} is a cycle of even length, then it follows from Proposition 1.4 that $\gamma_{oidI}(C_{2p}) = 2p = 2\beta(C_{2p})$, and thus Theorem 2.1 (2) is sharp.
(4) If C_{2p+1} is a cycle of odd length, then it follows from Proposition 1.4 that $\gamma_{oidI}(C_{2p+1}) = 2p + 1 = 2\beta(C_{2p+1}) - 1$, and thus Theorem 2.1 (3) is sharp.

Corollary 2.2 shows that Theorem 2.1 (4) is sharp.

Theorem 2.4. *If G is a graph of order $n \geq 2$, then $\gamma_{oidI}(G) \geq 3$, with equality if and only if $G = K_{1,n-1}$, $G = K_{1,1,n-2}$ $(n \geq 3)$ or $G = K_{1,1,1,n-3}$ $(n \geq 4)$.*

Proof. Proposition 1.1 implies $\gamma_{oidI}(G) \geq \gamma_{dI}(G) \geq 3$. If $G = K_{1,n-1}$ is a star, then define f by $f(v) = 3$ for the center v of the star and $f(x) = 0$ for $x \in V(G) \setminus \{v\}$. Then f is an OIDIDF on G of weight 3 and therefore $\gamma_{oidI}(G) \leq 3$, and thus $\gamma_{oidI}(G) = 3$. If $G = K_{1,1,n-2}$ with $S_1 = \{x_1\}$ and $S_2 = \{x_2\}$, then define f by $f(x_1) = 2$, $f(x_2) = 1$ and $f(x) = 0$ for $x \in V(G) \setminus \{x_1, x_2\}$. Then f is an OIDIDF on G of weight 3 and therefore $\gamma_{oidI}(G) = 3$. If $G = K_{1,1,1,n-3}$ with $S_1 = \{x_1\}$, $S_2 = \{x_2\}$ and $S_3 = \{x_3\}$, then define f by $f(x_1) = f(x_2) = f(x_3) = 1$ and $f(x) = 0$ for $x \in V(G) \setminus \{x_1, x_2, x_3\}$. Then f is an OIDIDF on G of weight 3 and thus $\gamma_{oidI}(G) = 3$.

Coversely, assume that $\gamma_{oidI}(G) = 3$. Then there is a vertex v with value 3 such that the remaining $n - 1$ vertices with value 0 are independent and adjacent to v, and therefore G is a star; or there are two adjacent vertices u and v with value 2 and 1, respectively, such that the remaining $n - 2$ vertices with value 0 are independent and adjacent to u and v, and therefore $G = K_{1,1,n-2}$; or there are three mutually adjacent vertices u, v, w with value 1 such that the remaining $n - 3$ vertices with value 0 are independent and adjacent to u, v and w, and therefore $G = K_{1,1,1,n-3}$. □

Theorem 2.4 implies $\gamma_{oidI}(K_{1,1,n-2}) = 3 = 2\beta(K_{1,1,n-2}) - 1$ $(n \geq 3)$ and $\gamma_{oidI}(K_{1,1,1,n-2}) = 3 = \beta(K_{1,1,1,n-2})$ $(n \geq 4)$. These are further examples which show the sharpness of Theorem 2.1 (3) and (4).

3. COMPLETE t-PARTITE GRAPHS

Theorem 3.1. *Let $G = K_{n_1,n_2}$ be the complete bipartite graph such that $n_1 \leq n_2$.*

(1) *If $n_1 = 1$, then $\gamma_{oidI}(G) = 3$.*
(2) *If $n_1 = 2$, then $\gamma_{oidI}(G) = 4$.*
(3) *If $n_1 \geq 3$, then $\gamma_{oidI}(G) = n_1 + 2$.*

Proof. Theorem 2.4 implies Item (1).

Item (2). It follows from Theorem 2.1 (2) that $\gamma_{oidI}(G) \leq 2\beta(G) = 4$. Since $\gamma_{oidI}(G) \geq 4$ according to Theorem 2.4, we deduce that $\gamma_{oidI}(G) = 4$.

Item (3). If $n_1 \geq 3$, then let $u \in S_2$. Define the function f by $f(x) = 1$ for $x \in S_1$, $f(u) = 2$ and $f(x) = 0$ for $x \in S_2 \setminus \{u\}$. Then f is an OIDIDF on G of weight $n_1 + 2$ and therefore $\gamma_{oidI}(G) \leq n_1 + 2$. If g is an OIDIDF on G, then we observe that $g(x) = 0$ for at most n_2 vertices. However, if $g(x) = 0$ for all $x \in S_2$, then $g(x) \geq 2$ for all $x \in S_1$ and therefore $w(g) \geq 2n_1$. If $g(x) \geq 2$ for all $x \in S_2$, then $w(g) \geq 2n_2 \geq 2n_1$. Next assume that $g(x) \geq 1$ for all $x \in S_2$ and $g(v) = 1$ for at least one vertex $v \in S_2$. Then we observe that

$$w(g) \geq \sum_{x \in N[v]} g(x) + (n_2 - 1) \geq 3 + (n_2 - 1) \geq n_1 + 2.$$

Finally, we assume that $g(v) = 0$ for at least one vertex $v \in S_2$. Then $g(x) \geq 1$ for all vertices $x \in S_1$. In addition, assume that $g(x) \geq 1$ for at least $r \geq 1$ vertices $x \in S_2$. If $r \geq 2$, then $w(g) \geq n_1 + r$, and if $r = 1$, then we observe that $w(g) \geq n_1 + 2$. Altogether, we deduce that $\gamma_{oidI}(G) \geq n_1 + 2$ and so $\gamma_{oidI}(G) = n_1 + 2$. □

Theorem 3.1 (2) yields $\gamma_{oidI}(K_{2,n-2}) = 4 = 2\beta(K_{2,n-2})$ $(n \geq 4)$. This is a further example which show the sharpness of Theorem 2.1 (2).

Theorem 3.2. *Let $G = K_{n_1,n_2,\ldots,n_t}$ be the complete t-partite graph of order n such that $t \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_t$.*

(1) *If $t \geq 4$ or $t = 3$ and $n_1 \geq 2$, then $\gamma_{oidI}(G) = n - n_t$.*
(2) *If $t = 3$, $n_1 = 1$ and $n_2 = 1$, then $\gamma_{oidI}(G) = 3$.*
(3) *If $t = 3$, $n_1 = 1$ and $n_2 = 2$, then $\gamma_{oidI}(G) = 4$.*
(4) *If $t = 3$, $n_1 = 1$ and $n_2 \geq 3$, then $\gamma_{oidI}(G) = n_2 + 2$.*

Proof. Item (1). If $t \geq 4$ or $t = 3$ and $n_1 \geq 2$, then $\delta(G) \geq 3$ and $S = S_1 \cup S_2 \cup \ldots \cup S_{t-1}$ is a minimum vertex cover of G such that $\delta(G[S]) \geq 2$. Therefore it follows from Theorem 2.1 (4) that $\gamma_{oidI}(G) = \beta(G) = n - n_t$.

Theorem 2.4 implies Item (2).

Item (3). If $t = 3$, $n_1 = 1$ and $n_2 = 2$, then define the function f by $f(x) = 2$ for $x \in S_1$, $f(x) = 1$ for $x \in S_2$ and $f(x) = 0$ for $x \in S_3$. Then f is an OIDIDF on G of weight 4 and therefore $\gamma_{oidI}(G) \leq 4$. Since $\gamma_{oidI}(G) \geq 4$ according to Theorem 2.4, we deduce that $\gamma_{oidI}(G) = 4$.

Item (4). If $t = 3$, $n_1 = 1$ and $n_2 \geq 3$, then define the function f by $f(x) = 2$ for $x \in S_1$, $f(x) = 1$ for $x \in S_2$ and $f(x) = 0$ for $x \in S_3$. Then f is an OIDIDF on G of weight $n_2 + 2$ and therefore $\gamma_{oidI}(G) \leq n_2 + 2$. If g is an OIDIDF on G, then $g(x) = 0$ for at most n_3 vertices, and therefore $w(g) \geq n_2 + 1$. Next assume, without loss of generality, that $g(w) = 0$ for at least one vertex $w \in S_3$. However, if $g(x) = 0$ for all vertices $x \in S_3$, then $g(x) = 1$ for all vertices $x \in S_1 \cup S_2$ is not possible and therefore $w(g) \geq n_2 + 2$. In addition, if $f(x) \geq 1$ for at least one vertex $x \in S_3$, then we also have $w(g) \geq n_2 + 2$. Consequently, $\gamma_{oidI}(G) \geq n_2 + 2$ and so $\gamma_{oidI}(G) = n_2 + 2$. □

4. NORDHAUS–GADDUM TYPE RESULTS

Results of Nordhaus–Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their classical paper [11], Nordhaus and Gaddum discussed this problem for the chromatic number. We discuss this problem for the outer-independent double Italian domination number.

Theorem 4.1 ([2]). *Let G be a graph G of order $n \geq 3$. Then $\gamma_{oidI}(G)+\gamma_{oidI}(\overline{G}) \leq 3n$, with equality if and only if $G \in \{K_3, \overline{K_3}\}$.*

Next we improve Theorem 4.1. In the following let $K_n - e$ be the complete graph minus an edge e.

Theorem 4.2. *Let $G \notin \{K_n, \overline{K_n}\}$ be a graph G of order $n \geq 5$. Then*

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 3n - 3,$$

with equality if and only if

$$G \in \{K_n - e, \overline{K_n - e}, K_{1,2,2}, \overline{K_{1,2,2}}, B_5\}.$$

Proof. First assume that $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$. Assume next that $\delta(G) = 1$ or $\delta(\overline{G}) = 1$, say $\delta(G) = 1$. Let $d_G(v) = 1$ and let w be a neighbor of v in G. Then v is in $\overline{G}$ adjacent to all vertices of $V(G) \setminus \{v, w\}$, and since $\delta(\overline{G}) \geq 1$, w is in $\overline{G}$ adjacent to a vertex $u \neq v, w$. If we define the function f by $f(v) = f(w) = 2$, $f(u) = 0$ and $f(x) = 1$ for $x \in V(G) \setminus \{u, v, w\}$, then f is an OIDIDF on $\overline{G}$ and therefore $\gamma_{oidI}(\overline{G}) \leq n + 1$.

If $\delta(\overline{G}) = 1$, then we obtain analogously $\gamma_{oidI}(G) \leq n + 1$ and therefore

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq n + 1 + n + 1 = 2n + 2 \leq 3n - 4$$

for $n \geq 6$. Let now $n = 5$. First we observe that G and $\overline{G}$ are connected and $\Delta(G), \Delta(\overline{G}) = 3$. Let w be a vertex with $d_G(w) = 3$, let x, y, z be the neighbors of w in G, and let u be the remaining vertex. If $\{x, y, z\}$ is an independent set in G, then the function f with $f(w) = 3$, $f(u) = 2$ and $f(x) = f(y) = f(z) = 0$ is an OIDIDF on G of weight 5, and we deduce that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 5 + 6 = 11 = 3n - 4.$$

So assume, without loss of generality, that $xy \in E(G)$. If $uz \in E(G)$, then $\delta(G) = 1$ implies $uw, ux, uy \in E(\overline{G})$. Then the function f with $f(u) = 3$, $f(z) = 2$ and $f(x) = f(y) = f(w) = 0$ is an OIDIDF on $\overline{G}$ of weight 5, and we obtain

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 6 + 5 = 11 = 3n - 4.$$

If $uz \notin E(G)$, then assume, without loss of generality, that $ux \in E(G)$. If there is no further edge in G, then G is the bull graph B_5 and $\overline{G}$ is the bull graph too. It is easy to see that

$$\gamma_{oidI}(B_5) + \gamma_{oidI}(\overline{B_5}) = 6 + 6 = 12 = 3n - 3,$$

as desired. If G contains a further edge, for example uy, then the function f defined by $f(z) = 3$, $f(w) = 2$ and $f(u) = f(x) = f(y) = 0$ is an OIDIDF on $\overline{G}$ of weight 5, and we have

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 11 = 3n - 4.$$

If $\delta(\overline{G}) \geq 2$, then Proposition 1.3 leads to $\gamma_{oidI}(\overline{G}) \leq n$, and we obtain according to Proposition 1.2 that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq \frac{3n}{2} + n < 3n - 3$$

for $n \geq 7$ and so

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 3n - 4$$

for $n \geq 7$. Let next $n = 6$. If $G = 3K_2$, then Theorem 3.2 (i) leads to

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) = 9 + \gamma_{oidI}(K_{2,2,2}) = 9 + 4 = 13 = 3n - 5.$$

If $G \neq 3K_2$, then Proposition 1.2 implies

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 8 + 6 = 14 = 3n - 4.$$

If $n = 5$, then we distinguish two cases. If G is not connected, then G consists of two components of order two and three, respectively. It follows that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 6 + 5 = 11 = 3n - 4.$$

If G is connected, then the condition $\delta(\overline{G}) \geq 2$ yields to $\Delta(G) \leq 2$ and thus $G = P_5$. Now it follows from Proposition 1.5 that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 6 + 5 = 11 = 3n - 4.$$

Second assume that $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$. According to Proposition 1.3, we deduce that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n \leq 3n - 4.$$

Finally assume that $\delta(G) = 0$ or $\delta(\overline{G}) = 0$, say $\delta(G) = 0$. Let I be the set of isolated vertices of G, $w \in I$ and $F = G - I$. We deduce from Proposition 1.2 that

$$\gamma_{oidI}(G) \leq 2|I| + \frac{3n(F)}{2} = 2|I| + 2n(F) - \frac{n(F)}{2} = 2n - \frac{n(F)}{2}.$$

Since $\overline{G} \neq K_n$, there exist two vertices $u \neq w$ and $v \neq w$ which are not adjacent in $\overline{G}$. Now we see that the function f with $f(w) = 3$, $f(u) = f(v) = 0$ and $f(x) = 1$ for $x \in V(G) \setminus \{u, v, w\}$ is an OIDIDF on $\overline{G}$ of weight n, and therefore $\gamma_{oidI}(\overline{G}) \leq n$. If $n(F) \geq 7$, then it follows that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n - \frac{n(F)}{2} + n \leq 3n - \frac{7}{2}$$

and thus $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 3n - 4$.

Let now $n(F) = 2$. Then $\overline{G} = K_n - e$, and Theorem 2.1 (4) implies

$$\gamma_{oidI}(\overline{G}) = \beta(\overline{G}) = n - 2.$$

Since $\gamma_{oidI}(G) = 2n - 1$, we obtain

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) = 2n - 1 + n - 2 = 3n - 3,$$

as desired.

If $n(F) = 3$, then $F = P_3$ or $F = C_3$. In both cases it is easy to see that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 3n - 4.$$

Next let $n(F) = 4$. Assume that $F = 2K_2$. If $n = 5$, then $\gamma_{oidI}(G) = 8$, $\overline{G} = K_{1,2,2}$ and therefore it follows by Theorem 3.2 (3) that $\gamma_{oidI}(\overline{G}) = 4$. This leads to

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) = 12 = 3n - 3,$$

as desired. If $n \geq 6$, then $\gamma_{oidI}(G) = 2n - 2$ and by Theorem 3.2 (i) we have $\gamma_{oidI}(\overline{G}) = n - 2$ and so

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) = 2n - 4.$$

If $F \neq 2K_2$, then F is connected. If $\delta(F) \geq 2$, then Proposition 1.3 implies $\gamma_{oidI}(F) \leq 4$ and it follows that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2(n - 4) + 4 + n = 3n - 4.$$

If $\delta(F) = 1$, then let u be a vertex of degree 1, v be a neighbor of u and $a \neq u$ be a neighbor of v in F. If v has a further neighbor $b \neq u, a$, then the function f with $f(v) = 3$, $f(u) = f(a) = 0$, $f(b) = 1$ and $f(x) = 2$ for $x \in V(G) \setminus V(F)$ is an OIDIDF on G of weight $2n - 4$, and so

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n - 4 + n = 3n - 4.$$

In the remaining case a has a neighbor b and hence $F = P_4$. Applying Proposition 1.5, we obtain $\gamma_{oidI}(G) = 2n - 3$. If we define on $\overline{G}$ the function $f(w) = 2$, $f(u) = f(v) = 0$ and $f(x) = 1$ for $x \in V(G) \setminus \{u, v, w\}$, then f is an OIDIDF on $\overline{G}$ of weight $n - 1$. Consequently,

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n - 3 + n - 1 = 3n - 4.$$

If $n(F) = 5$, then $n \geq 6$. If F is not connected, then F consists of two components of order two and three, respectively. We observe that $\gamma_{oidI}(G) \leq 2(n-5) + 6 = 2n - 4$, and this leads to the desired result. Now let F be connected. If $\delta(F) \geq 2$, then we obtain as above

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2(n - 5) + 5 + n = 3n - 5.$$

Let now $\delta(F) = 1$. If $\Delta(F) = 2$, then $F = P_5$ and Proposition 1.5 yields

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2(n - 5) + 6 + n = 3n - 4.$$

If $\Delta(F) = 4$, then let v be a vertex of degree 4 and a, b, c, d be the neighbors of v in F. If we define the function f with $f(v) = 3$, $f(a) = 0$, $f(b) = f(c) = f(d) = 1$ and $f(x) = 2$ for $x \in V(G) \setminus V(F)$, then f is an OIDIDF on G of weight $2n - 4$, and the desired result follows as before. If $\Delta(F) = 3$, then let v be a vertex of degree 3 and a, b, c be the neighbors of v in F. Assume, without loss of generality, that the remaining vertex d is adjacent to a in F. If we define the function f with $f(v) = f(d) = 2$, $f(a) = 0$, $f(b) = f(c) = 1$ and $f(x) = 2$ for $x \in V(G) \setminus V(F)$, then f is an OIDIDF on G of weight $2n - 4$, and the desired result follows as above.

Finally, let $n(F) = 6$. If $F = 3K_2$, then $\gamma_{oidI}(G) = 2n - 3$ and we deduce from Theorem 3.2 (i) that $\gamma_{oidI}(\overline{G}) = n - 2$ and so

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) = 2n - 5.$$

If $F \neq 3K_2$, then Proposition 1.2 implies $\gamma_{oidI}(F) \leq 8$ and thus

$$\gamma_{oidI}(G) \leq 2(n - 6) + 8 = 2n - 4.$$

Hence

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n - 4 + n = 3n - 4. \qquad \square$$

For completeness note that Corollary 2.2 implies

$$\gamma_{oidI}(K_n) + \gamma_{oidI}(\overline{K_n}) = 3n - 1 \quad \text{for} \quad n \geq 4.$$

Furthermore

$$\gamma_{oidI}(K_1) + \gamma_{oidI}(\overline{K_1}) = 4 = 3n + 1 \quad \text{for} \quad n = 1,$$
$$\gamma_{oidI}(K_2) + \gamma_{oidI}(\overline{K_2}) = 7 = 3n + 1 \quad \text{for} \quad n = 2$$

and

$$\gamma_{oidI}(K_3) + \gamma_{oidI}(\overline{K_3}) = 9 = 3n \quad \text{for} \quad n = 3.$$

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