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Design Sensitivity Analysis of Frequency Response Functions and Steady-state Response for Structures with Viscoelastic Dampers

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Abstract

In this paper, the design sensitivity of the frequency response function and amplitudes of the steady-state vibration of planar frames with viscoelastic (VE) dampers mounted on them is considered. The dampers are modeled using a five-parameter rheological model with fractional derivatives. The design sentisivity with respect to change of damper parameter is analyzed in detail. The direct method is used to determine the first and the second order sensitivities. Moreover, the results of typical calculations are presented and discussed.

Keywords: Fractional models of VE dampers, Dynamic characteristics, Design sensitivity

1. Introduction

The design sensitivity analysis of structures and mechanical systems is a very important issue, which is helpful in solving many engineering problems, such as: optimization of structures, parametric identification problems, structural health monitoring problems, model updating problems [1], structural reliability problems, damage detection [2] and others. In the recent years, studies on the analysis of sensitivity for systems with viscoelastic dampers have been started e.g., the eigensensitivity analysis of viscoelastic (VE) structures is presented in [3].

The frequency response function is one of the most important tools of evaluation of the dynamic response of structure. Its design sensitivity analysis has been studied by several authors. For example, the direct differentiation method is presented in paper [4] and both the direct differentiation method and the adjoint variable method is described in [5,6].

In this paper, the direct differentiation method for the design sensitivity analysis of structure with viscoelastic dampers modeled by fraction derivatives is presented. This work is an extension of the previous paper [7], which dealt with the sensitivity analysis of eigenvalues and eigenvectors of structure with fractional dampers.

Firstly, in this paper, the model of damper and the equation of motion of a structure with dampers described by fractional derivatives are presented. Then the method of calculation of the frequency response functions (FRF) and amplitudes of steady-state vibration is presented. Next, the design sensitivity analysis is shown. Finally, the two-storey planar frame is considered. In the example, the sensitivity of FRF with respect to change of parameter of damper is calculated and the correctness of the presented method is proved. At the end, the conclusions are presented.

2. Description of structures with VE dampers

Many rheological models of dampers have been proposed in the literature. The most popular among them are the two classic ones: the Maxwell and the Kelvin models. In order to better describe the damper, so-called fractional models are often used. They describe the rheological properties of dampers more efficiently than the classic ones [8]. A so-called the springpot element, shown as a small diamond in Figure 1, is described by the two constants c and α, where α denotes the order of the fractional derivative.

In this paper, the fractional model of a damper is used (see Figure 1). The damper is described by five parameters: stiffnesses k_0 and k_1, springpot factors c_0 and c_1, and the fractional parameter α ($0 < \alpha < 1$). As special cases, it contains a number of specific models, e.g., the three-parameter Maxwell and Kelvin models, the four-parameter fractional Maxwell model.

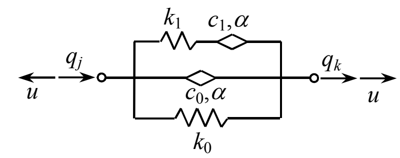

Figure 1. A model of the damper

The force in the considered model of damper is written as:

$$u(t) = u_0(t) + u_1(t) \tag{1}$$

where $u_0(t)$ is the force in the fractional Kelvin element and $u_1(t)$ is the force in the fractional Maxwell element.

Evaluation equations for the Kelvin model can be written as follows:

$$u_0(t) = k_0 \Delta q(t) + c_0 D_t^{\alpha} \Delta q(t) \tag{2}$$

where: $\Delta q_i(t) = q_k - q_j$, q_k and q_j denote the nodal displacements of the considered model of damper. D_t^{α} denotes the Riemann-Liouville fractional derivative of the order α with respect to time t [9,10]. For the Kelvin model, the evaluation equations could be described in the following way:

$$u_1(t) + \frac{c_1}{k_1} D_t^{\alpha} u_1(t) = c_1 D_t^{\alpha} \Delta q(t) \tag{3}$$

After taking the Laplace transform, Equation (1) can be written in the form:

$$\bar{u}(s) = \bar{u}_0(s) + \bar{u}_1(s), \tag{4}$$

and Equations (2) and (3) take the following form:

$$\bar{u}_0(s) = k_0 \Delta\bar{q}(s) + c_0 s^{\alpha} \Delta\bar{q}(s) \tag{5}$$

$$\bar{u}_1(s) + \tau_1 s^{\alpha} \bar{u}_1(s) = c_1 s^{\alpha} \Delta\bar{q}(s) \tag{6}$$

where the quantities with the bar denote the Laplace transform, i.e.: $\Delta\bar{q}(s) = L[\Delta q(t)]$, $\bar{u}_0(s) = L[u_0(t)]$., $\bar{u}_1(s) = L[u_1(t)]$, $s^{\alpha}\bar{u}(s) = L[D_t^{\alpha} u(t)]$, and s is the Laplace variable.

Finally:

$$\bar{u}(s) = G(s)\Delta\bar{q}(s) \tag{7}$$

where:

$$G(s) = k_0(1+\tau_0 s^{\alpha}) + k_1 \frac{\tau_1 s^{\alpha}}{1+\tau_1 s^{\alpha}}, \qquad \tau_0 = c_0 / k_0, \qquad \tau_1 = c_1 / k_1 \tag{8}$$

The classic Kelvin and Maxwell models are obtained by introducing $\alpha = 1$.

The equation of motion of structures with VE dampers can be written in the following form:

$$\mathbf{M}_s \ddot{\mathbf{q}}(t) + \mathbf{C}_s \dot{\mathbf{q}}(t) + \mathbf{K}_s \mathbf{q}(t) = \mathbf{p}(t) + \mathbf{f}(t) \tag{9}$$

where: $\mathbf{M}_s$, $\mathbf{C}_s$ and $\mathbf{K}_s$, denote the mass, the damping and the stiffness matrix of structure, respectively. The structure is modeled as a shear frame with mass lumped at the storey level. Moreover, $\mathbf{q}(t) = [q_1 \ ... \ q_n]^T$ is the vector of displacements of the structure, $\mathbf{p}(t) = [p_1 \ ... \ p_n]^T$ is the vector of excitation forces and $\mathbf{f}(t) = [f_1 \ ... \ f_n]^T$ is the vector of the interaction forces between the frame and the dampers (see Figure 2).

Vector $\mathbf{f}(t)$ is a sum of the vectors $\mathbf{f}_i(t)$. Each of them is formed if only the damper i is located on the frame, i.e.:

$$\mathbf{f}(t) = \sum_{i=1}^{m} \mathbf{f}_i(t)\,. \tag{10}$$

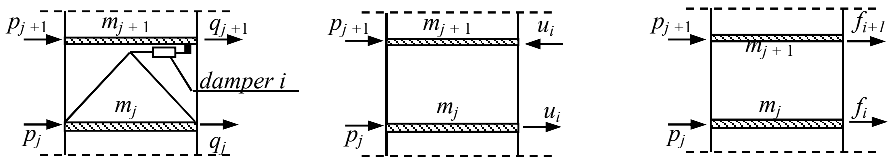

Figure 2. Diagram of frame with VE dampers

For the damper located between the floors j and $j + 1$ (see Fig. 2), the following may be written:

$$\mathbf{f}_i(t) = \mathbf{e}_i u_i(t), \qquad \mathbf{e}_i = [0 \ \ ... \ \ e_j = +1 \ \ e_{j+1} = -1 \ \ ... \ \ 0]^T\,. \tag{11}$$

After taking the Laplace transform, the equation of motion could be written as:

$$(s^2\mathbf{M}_s + s\mathbf{C}_s + \mathbf{K}_s)\,\bar{\mathbf{q}}(s) = \bar{\mathbf{p}}(s) + \bar{\mathbf{f}}(s) \tag{12}$$

where: $\overline{\mathbf{q}}(s) = L[\mathbf{q}(t)]$, $\overline{\mathbf{p}}(s) = L[\mathbf{p}(t)]$, $\overline{\mathbf{f}}(s) = L[\mathbf{f}(t)]$.

For *m* dampers, the following equation is obtained:

$$\overline{\mathbf{f}}(s) = \sum_{i=1}^{m} \mathbf{e}_i \overline{u}_i(s) \tag{13}$$

Substituting Equation (7) written for damper *i* to Equation (13) leads to:

$$\overline{\mathbf{f}}(s) = -\sum_{i=1}^{m} G_i(s)\, \mathbf{L}_i \overline{\mathbf{q}}(s)\ , \tag{14}$$

where: $\mathbf{L}_i = \mathbf{e}_i \mathbf{e}_i^{\mathrm{T}}$. After substituting Equation (14) into (12) the equation of motion of structure with VE dampers could be written as:

$$\mathbf{D}(s)\overline{\mathbf{q}}(s) = \overline{\mathbf{p}}(s) \tag{15}$$

where:

$$\mathbf{D}(s) = s^2\mathbf{M}_s + s\mathbf{C}_s + \mathbf{K}_s + \sum_{i=1}^{m} \mathbf{G}_i(s)\ , \qquad \mathbf{G}_i(s) = G_{\mathrm{i}}(s)\mathbf{L}_i\,. \tag{16}$$

3. Frequency response function (FRF)

The dynamic response of structure can be described by using the frequency response functions. In this context, we assume that

$$\mathbf{p}(t) = \mathbf{P}\exp(i\lambda t) \tag{17}$$

where $\mathbf{P} = [P_1, \dots, P_n]^T$ (compare Figure 2), λ is the excitation frequency. The steady-state solution to the motion equation could be assumed in the two equivalent forms:

$$\mathbf{q}(t) = \mathbf{H}(\lambda)\mathbf{P}\exp(i\lambda t) = \mathbf{a}\exp(i\lambda t) \tag{18}$$

where $\mathbf{H}(\lambda)$ is the matrix of frequency response functions and $\mathbf{a}$ is the vector of amplitudes of steady-state vibration. After substituting Formulae (17) and (18) into (9) we obtain:

$$\mathbf{D}(\lambda)\mathbf{H}(\lambda) = \mathbf{I} \tag{19}$$

hence

$$\mathbf{H}(\lambda) = \mathbf{D}(\lambda)^{-1}\,. \tag{20}$$

We can also obtain the formula describing the matrix $\mathbf{D}(\lambda)$ by substituting relationship $s = i\lambda$ into Equation (16). Hence:

$$\mathbf{D}(\lambda) = \left[-\lambda^2\mathbf{M} + i\lambda\mathbf{C} + \mathbf{K} + \sum_{i=1}^{m}\mathbf{G}_i(\lambda)\right],\ \mathbf{H}(\lambda) = \left[-\lambda^2\mathbf{M} + i\lambda\mathbf{C} + \mathbf{K} + \sum_{i=1}^{m}\mathbf{G}_i(\lambda)\right]^{-1}. \tag{21}$$

Based on Relationship (18) we can also write:

$$\mathbf{a} = \mathbf{H}(\lambda)\mathbf{P} \tag{22}$$

4. Design sensitivity

In order to determine the relationship describing the sensitivity of FRF, it is necessary to use the following obvious equation:

$$\mathbf{H}(\lambda)\mathbf{H}(\lambda)^{-1} = \mathbf{I} \tag{23}$$

Differentiating Equation (23) with respect to the design parameter p leads to:

$$\frac{\partial \mathbf{H}(\lambda,p)}{\partial p} = -\mathbf{H}(\lambda,p)\frac{\partial \mathbf{D}(\lambda,p)}{\partial p}\mathbf{H}(\lambda,p) \tag{24}$$

where:

$$\frac{\partial \mathbf{D}(\lambda,p)}{\partial p} = \left[-\lambda^2\frac{\partial \mathbf{M}(p)}{\partial p} + i\lambda\frac{\partial \mathbf{C}(p)}{\partial \lambda} + \frac{\partial \mathbf{K}(p)}{\partial p} + \sum_{i=1}^{m}\frac{\partial \mathbf{G}_i(\lambda,p)}{\partial p}\right].$$

Differentiating the Equation (23) the second time, we obtain the second order sensitivity:

$$\frac{\partial^2 \mathbf{H}(\lambda,p)}{\partial p^2} = -\mathbf{H}(\lambda,p)\frac{\partial^2 \mathbf{D}(\lambda,p)}{\partial p^2}\mathbf{H}(\lambda,p) - 2\mathbf{H}(\lambda,p)\frac{\partial \mathbf{D}(\lambda,p)}{\partial p}\frac{\partial \mathbf{H}(\lambda,p)}{\partial p} \tag{25}$$

where:

$$\frac{\partial \mathbf{D}^2(\lambda,p)}{\partial p^2} = \left[-\lambda^2\frac{\partial^2 \mathbf{M}(p)}{\partial p^2} + i\lambda\frac{\partial^2 \mathbf{C}(p)}{\partial p^2} + \frac{\partial^2 \mathbf{K}(p)}{\partial p^2} + \sum_{i=1}^{m}\frac{\partial^2 \mathbf{G}_i(\lambda,p)}{\partial p^2}\right].$$

In the calculation of sensitivity, the first and the second order with respect to the chosen parameter of structure or damper, only the matrices $\partial \mathbf{D}(\lambda,p)/\partial p$ and $\partial^2 \mathbf{D}(\lambda,p)/\partial p^2$ change and can be reduced to a much simpler form.

After calculating the sensitivity of FRF, it is possible to determine the sensitivity of amplitudes of steady-state vibration in a simple way. Differentiating Equation (22) with respect to the design parameter leads to:

$$\frac{\partial \mathbf{a}}{\partial p} = \frac{\partial \mathbf{H}(\lambda,p)}{\partial p}\mathbf{P} \tag{26}$$

where sensitivity of FRF is described by Equation (24).

5. Example

In order to illustrate the presented method, a two-storey building with a three-parameter Maxwell damper situated on the second storey is considered (see Fig. 3). The following data are adopted: the mass of every floor m = 1000kg, the storey stiffness k_s = 100000N/m and the damper parameters: k_1 = 50000N/m, c_1 = 8000Ns^{α}/m and α = 0.6. The damping properties of the structure are neglected.

In this example, the frequency response matrix $\mathbf{H}(\lambda)$is determined for the excitation frequency taken from the range $\lambda \in (0, 20\text{rad/sec})$. The calculation results are presented in Figure 4, where the real and the imaginary parts of the function $H_{11}(\lambda)$ are shown.

The sensitivity of $H_{11}(\lambda)$ with respect to the change of the stiffness parameter k_1 of the damper is calculated and the frequency taken from the range $\lambda \in (0, 20\text{rad/sec})$. The results are presented in Figure 5.

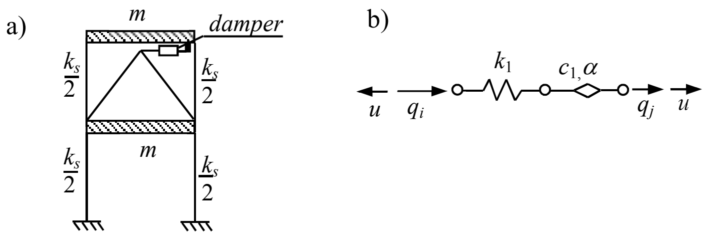

Figure 3. a) Diagram of the considered frame, b) Maxwell model of damper

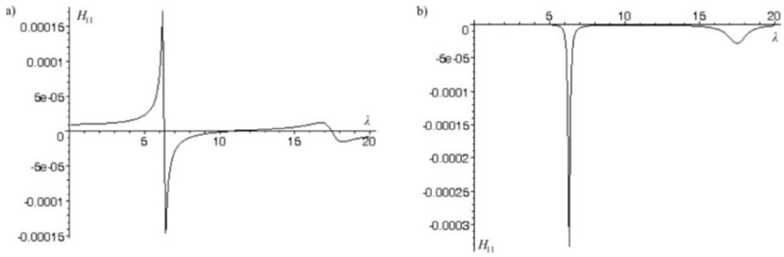

Figure 4. The real and the imaginary parts of the function $H_{11}(\lambda)$: (a) real, b) imaginary)

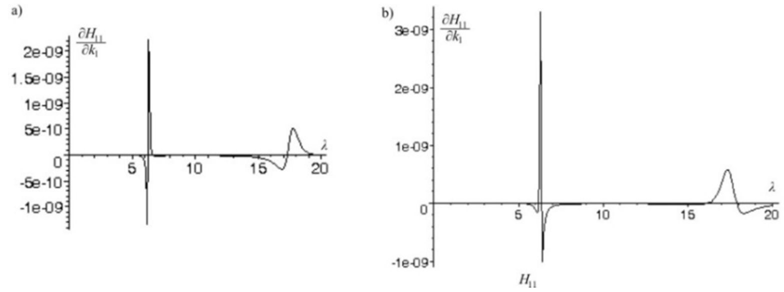

Figure 5. The real and the imaginary parts of sensitivity of $H_{11}(\lambda)$: (a) real, b) imaginary)

In order to verify the correctness of the calculation, the values of FRF after change of the parameter are determined according to the formula:

$$\mathbf{H}(\lambda, p+\Delta p) \approx \mathbf{H}(\lambda, p) + \frac{\partial \mathbf{H}(\lambda)}{\partial p}\Delta p \tag{27}$$

where Δp denotes a change of the design parameter. The obtained values were compared with the exact solution, when the design parameter changed its own value by 1%. The calculation is carried out for the selected frequencies and the obtained results are presented in Table 1. The results proved the presented method to be correct.

Table 1. A comparison of $H_{11}(\lambda, k_1)$

Frequency λ [rad/sec]	$H_{11}(\lambda,k_1)+\frac{\partial H_{11}(\lambda,k_1)}{\partial k_1}\Delta k_1$	Exact value of $H_{11}(\lambda,k_1+\Delta k_1)$
4.0	$0.153048\cdot10^{-4}-0.624154\cdot10^{-7}i$	$0.153048\cdot10^{-4}-0.624135\cdot10^{-7}i$
6.0	$0.824470\cdot10^{-4}-0.158672\cdot10^{-4}i$	$0.824476\cdot10^{-4}-0.158668\cdot10^{-4}i$
6.5	$-0.983082\cdot10^{-4}-0.418210\cdot10^{-4}i$	$-0.983076\cdot10^{-4}-0.418190\cdot10^{-4}i$
7.0	$-0.291213\cdot10^{-4}-0.522328\cdot10^{-5}i$	$-0.291211\cdot10^{-4}-0.522308\cdot10^{-5}i$
7.5	$-0.153077\cdot10^{-4}-0.235479\cdot10^{-5}i$	$-0.153076\cdot10^{-4}-0.235470\cdot10^{-5}i$

Moreover, a comparison was made by using first-order sensitivity values according to Equation (27) and second-order sensitivity values according to the following equation:

$$\mathbf{H}(\lambda, p+\Delta p) \approx \mathbf{H}(\lambda, p) + \frac{\partial \mathbf{H}(\lambda,p)}{\partial p}\Delta p + \frac{1}{2}\frac{\partial^2 \mathbf{H}(\lambda,p)}{\partial p^2}\Delta p^2 \,. \tag{28}$$

The calculations are carried out for a change of parameter k_1, taken from the range 1% – 50% and presented in Figure 6. Now, we can conclude that the second order sensitivity gives results which are very close to an exact solution if the change of parameter k_1 is smaller than 20% .

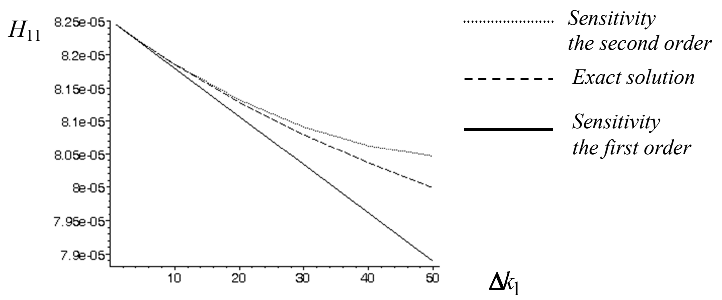

Figure 6. The comparison of $H_{11}(\lambda, k_1 + \Delta k_1)$

6. Conclusions

In this paper the design sensitivity analysis of FRF and amplitudes of the steady-state vibration of structures with VE dampers is presented. The formulae are calculated by using the direct method. The obtained eqations enable determination of the sensitivity of the dynamic characteristics of structures with VE dampers with respect to a chosen design parameter. The considered five-parameter damper model can be used for an analysis of structures with different dampers, described by selected classic and fractional rheological models. In the example, the correctness of the present method is proved.

The method used to calculate the sensitivities of FRF and amplitudes of the steady-state vibration of structures with VE dampers is easy to formulate, systematic to apply, simple to code, and it agrees well with the exact results. Such an analysis has been carried out for the first time.

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