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AN OUTPUT SENSITIVITY PROBLEM FOR A CLASS OF FRACTIONAL ORDER DISCRETE-TIME LINEAR SYSTEMS

Youssef BENFATAH*, **Amine EL BHIH***, **Mostafa RACHIK***, **Marouane LAFIF***

*Faculty of Sciences Ben M'Sik, Department of Mathematics and Computer Science, Hassan II University, Casablanca, Sidi Othman BP 7955, Morocco

youssef.benfatah@gmail.com, elbhihamine@gmail.com, m_rachik@yahoo.fr, marouane8@gmail.com,

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Abstract: Consider the linear discrete-time fractional order systems with uncertainty on the initial state $\begin{cases}\Delta^{\alpha}x_{i+1} = Ax_i + Bu_i, & i \geq 0\\ x_0 = \tau_0 + \hat{\tau}_0 \in \mathbb{R}^n, & \hat{\tau}_0 \in \Omega,\\ y_i = Cx_i, & i \geq 0\end{cases}$ where A, B and C are appropriate matrices, x_0 is the initial state, y_i is the signal output, α the order of the derivative, τ_0 and $\hat{\tau}_0$ are the known and unknown part of x_0, respectively, $u_i = Kx_i$ is feedback control and $\Omega \subset \mathbb{R}^n$ is a polytope convex of vertices $w_1, w_2, \dots, w_p$. According to the Krein–Milman theorem, we suppose that $\hat{\tau}_0 = \sum_{j=1}^{p} \alpha_j w_j$ for some unknown coefficients $\alpha_1 \geq 0, \dots, \alpha_p \geq 0$ such that $\sum_{j=1}^{p} \alpha_j = 1$. In this paper, the fractional derivative is defined in the Grünwald–Letnikov sense. We investigate the characterisation of the set $\chi(\hat{\tau}_0, \epsilon)$ of all possible gain matrix K that makes the system insensitive to the unknown part $\hat{\tau}_0$, which means $\chi(\hat{\tau}_0, \epsilon) = \{K \in \mathbb{R}^{m\times n} \,/\, \| \frac{\partial y_i}{\partial \alpha_j} \| \leq \epsilon, \ \forall j = 1, \dots, p, \ \forall i \geq 0\}$, where the inequality $\| \frac{\partial y_i}{\partial \alpha_j} \| \leq \epsilon$ showing the sensitivity of y_i relatively to uncertainties $\{\alpha_j\}_{j=1}^{p}$ will not achieve the specified threshold $\epsilon > 0$. We establish, under certain hypothesis, the finite determination of $\chi(\hat{\tau}_0, \epsilon)$ and we propose an algorithmic approach to made explicit characterisation of such set.

Key words: fractional order systems, output sensitivity, discrete-time systems, maximal output set admissible uncertainty

1. INTRODUCTION

Fractional calculus is an extended version of the traditional integer order calculus in which the definition of derivatives is given to a non-integer order. The non-integer derivative concept is used increasingly for modelling of real systems behaviour in different disciplines of engineering and science (Debnath, 2003). These systems have long-memory transients and hereditary properties that can be more accurately described by fractional-order models. In the recent past, there has been an increasing focus on discrete-time fractional systems (Kaczorek, 2007; Kaczorek, 2008; Sierociuk and Dzieliński, 2006; Ferreira and Torres, 2011). Some important developments of the theory of fractional calculus are presented in Kilbas et al., (2006) and Oldham and Spanier (1974).

On the other hand, undesirable parameters appeared during modelling a system; consequently such parameters could have an impact on various elements of the system including initial conditions, control, dynamic, and observations. To deal with this problem, a variety of approaches have been developed by researchers, including the theory of sentinel (Lions), the detectability in Franklin (2001) and Ogata (1995), identifiability in Thomson (2007); Kauffmann and Bretthawer; and Robert and Graham (2007), the H_∞ control theory in Chi-Tsong) 2008) and the frequency domain and robustness (Rosario, 2005; Gu et al., 2005).

1.1. Related work

Concerning the sensitivity of the system output to the disturbance, the readers can refer to Larrache et al. (2020); Rachik and Lhous (2016); Balatif et al. (2016); and Chraïbi et al. (2006). Larrache et al. (2020) considered an infinite dimensional linear system described as

$$\begin{cases}\dot{x}(t) = Ax(t), & t \geq 0,\\ x(0) = x_0 = \alpha 1_{\omega_1} + \beta 1_{\omega_2}, & \\ y_i = Cx_i + Dv_i, & i \geq 0,\end{cases} \tag{1}$$

where $x(t) \in L^2(\Omega)$, 1_{ω_i} is the indicator function, and Ω is an open bounded of $\mathbb{R}^n$ such that $\Omega = \omega_1 \cup \omega_2$ and $\omega_1 \cap \omega_2 = \varnothing$. The operator A generates a continuous strongly semigroup $\{S(t)\}_{t\geq 0}$ on the space $L^2(\Omega)$, $v_i = Kx_i$, $i \geq 0$ feedback control, $K \in \mathcal{L}(L^2(\Omega), \mathbb{R}^p)$. The initial state x_0 is supposed to be known on ω_1 but not on ω_2. The authors suggest a method to identify within these controls law v_i, $i \geq 0$ making the system insensitive to the impact of these unknown parameters β. The case of the disturbances infecting a linear system's initial state has been investigated in Kolmanovsky and Gilbert (1998) and Namerikawa et al. (2004).

The sentinel theory was initiated and developed by Lions in (Lions, 1992; Lions, 1988). Sawadogo (2020) have used the

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sentinel method to control the migration by studying the dynamics of a single species population and whose initial distribution is unknown.

In the literature, the notion of maximal output set is of great significant in the area of control and analysis of linear and nonlinear systems. Numerous studies have been carried out on the construction of the maximal output set (El Bhih et al., 2020; Yamamoto, 2019; Osorio and Ossareh, 2018; Abdelhak and Rachik, 2019; Gilbert and Tan, 1991). Different algorithms have been included in the research literature to specify the maximal state constraint sets (Gilbert and Tan, 1991; Dórea and Hennet, 1996).

1.2. Problem statement

A fundamental requirement for most dynamical systems is to keep a given output function insensitive to the disturbances. In this paper, we suppose that the initial state of the system is composed of two parts: the unknown part noted as $\hat{\tau}_0$ and the known part noted as τ_0. We propose a new technique to characterise the set $\chi(\hat{\tau}_0,\epsilon)$ of all possible gain matrix K, based on the maximal output set $\Upsilon(K,\epsilon)$, so that the sensitivity of the resulting system output would be relatively tolerable, that is, make the system insensitive to the unknown part $\hat{\tau}_0$ of the initial state x_0, of commensurate fractional order discrete-time controlled linear systems which are modelled by equations of fractional state space. To the best of our knowledge, the output sensitivity of such systems has not been treated yet. We propose some new sufficient conditions that ensure the finite determination of the set $\chi(\hat{\tau}_0,\epsilon)$. Moreover, we present an effective algorithm to obtain the maximal output set $\Upsilon(K,\epsilon)$ and then the set $\chi(\hat{\tau}_0,\epsilon)$. The algorithm having theoretical convergence properties are provided in Gilbert and Tan (1991).

Therefore, we study a discrete-time linear control systems of fractional order with uncertainty on the initial state, evolving on $\mathbb{R}^n$. More precisely, we consider the system as

$$\begin{cases}\Delta^\alpha x_{i+1} = Ax_i + Bu_i, & i \geq 0\\ x_0 = \tau_0 + \hat{\tau}_0 \in \mathbb{R}^n\end{cases} \tag{2}$$

The corresponding output is

$$y_i = Cx_i, \quad i \geq 0 \tag{3}$$

where A is a matrix of order $n \times n$, the system dynamics, B is the input matrix of order $n \times m$ and C is the output matrix of order $p \times n$; α is the order of the derivative, τ_0 is the known part, $\hat{\tau}_0$ is the unknown part and $u_i = Kx_i$ is the feedback control.

The remainder of this paper is organised as follows: In Section 2, we recall a fundamental definition of fractional derivatives (Grünwald-Letnikov), then we consider the discrete-time system proposed in Dzieliński and Sierociuk (2005). With uncertainty on the initial state, we recall the Krein–Millman theorem and some definitions. Section 3 deals with the characterisation of the set of all possible gain matrices which make the system insensitive to the unknown part, based on the maximal output set. New sufficient conditions are provided to show the finite determination of such set. In Section 4, we propose an algorithm approach to identify the index of admissibility. We illustrate some examples and numerical simulations in Section 5. We conclude the paper by in Section 6.

Notation: $\mathbb{R}^n$ the set of real vectors with n components, $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ the set of real matrices of order $n \times n$, $\mathbb{N}$ the set of nonnegative integers, $\sigma_s^k = \{s, s+1, \dots, k\} \subset \mathbb{N}$ where $s \leq k$, I_n denotes the identity matrix in $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$. The components of a vector b are noted as $(b)_j$ and the components of a matrix A are noted as $(A)_{ij}$.

2. FRACTIONAL CALCULUS AND DYNAMIC MODELS

To begin our work, we will introduce certain basic notions concerning the fractional calculus that are utilised along the paper. The definition of the discrete fractional derivative in this paper is as follows: Grünwald–Letnikov (Oldham and J. Spanier, 1974; Podlubny, 1999).

Definition 1. The Grünwald–Letnikov (backward) difference of fractional order α of the function $x(.)$ at $k \in [0,+\infty[$ is given as

$$\Delta^\alpha x(k) = \frac{1}{h^\alpha}\sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k-j), \tag{4}$$

where the order of the derivative $\alpha \in]0,1[$, $h \in \mathbb{R}^{*+}$ is a sampling period taken equal to unity in all what follows, and $k \in \mathbb{N}$ is the number of samples for which the approximation of the derivative is calculated.

The term $\binom{\alpha}{j}$ in Definition 2 can be obtained from the following relation:

$$\binom{\alpha}{j} = \begin{cases}1 & \text{for} \quad j = 0,\\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for} \quad j > 0.\end{cases} \tag{5}$$

Let us consider now the discrete-time linear fractional order system as defined in Dzieliński and Sierociuk (2005), described as

$$\begin{cases}\Delta^\alpha x_{i+1} = Ax_i + Bu_i,\\ \\ x_0 = \tau_0 + \hat{\tau}_0 \in \mathbb{R}^n\end{cases} \tag{6}$$

where $A \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ is the system dynamics, $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$ is the input matrix and $C \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^p)$ is the output matrix; α is the order of the derivative, τ_0 is the known part, $\hat{\tau}_0$ is the unknown part and

$$x_i = \begin{pmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^n \end{pmatrix} \in \mathbb{R}^n$$

is the state variable.

The associated output function is

$$y_i = Cx_i \in \mathbb{R}^p$$

Its initial value is denoted by x_0. The control low (feedback control) is

$$u_i = Kx_i \in \mathbb{R}^m. \tag{7}$$

In this system, the differentiation order α is taken as the same for all the state variables x_i^j, $j = 1,2,\dots,n$, that is

$$\Delta^\alpha x_i = \begin{pmatrix} \Delta^\alpha x_i^1 \\ \Delta^\alpha x_i^2 \\ \vdots \\ \Delta^\alpha x_i^n \end{pmatrix}$$

This is referred to as commensurate order. We will propose a technique to determine among these controls as low which makes the system insensitive to the effect of the unknown part $\hat{\tau}_0$.

We replace $\Delta^\alpha x_{i+1}$ by its value; system (6) could be rewritten as

$$\begin{cases} x_{i+1} = \sum_{j=0}^{i} A_j x_{i-j}, \\ x_0 = \tau_0 + \hat{\tau}_0 \in \mathbb{R}^n \end{cases} \quad (8)$$

where

$$A_0 = A + BK + \alpha I_n \quad (9)$$

and

$$A_j = -(-1)^{j+1} \binom{\alpha}{j+1} I_n, \quad \forall j \geq 1. \quad (10)$$

Remark 1. The model described in (8) can be classified as a discrete-time system with a time-delay in the state. For practical use, the number of simple taken into consideration needs to be reduced to the predefined number L called the memory length and $x_i = 0$ for $i < 0$ (Dzieliński and Sierociuk, 2008).

Thus, system (8) becomes

$$\begin{cases} x_{i+1} = \sum_{j=0}^{L} A_j x_{i-j}, \\ x_0 = \tau_0 + \hat{\tau}_0 \in \mathbb{R}^n \end{cases} \quad (11)$$

Definition 2. The system given by (6) could be rewritten as an infinite dimensional system taking the form

$$\begin{pmatrix} x_{i+1} \\ x_i \\ x_{i-1} \\ \vdots \end{pmatrix} = \widetilde{A} \begin{pmatrix} x_i \\ x_{i-1} \\ x_{i-2} \\ \vdots \end{pmatrix} + \widetilde{B} u_i \quad , \quad y_i = \widetilde{C} \begin{pmatrix} x_i \\ x_{i-1} \\ x_{i-2} \\ \vdots \end{pmatrix}$$

where $\widetilde{A} = \begin{pmatrix} A + \alpha I_n & A_1 & A_2 & \cdots \\ I_n & 0 & 0 & \cdots \\ 0 & I_n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$, $\widetilde{B} = \begin{pmatrix} B \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ and $\widetilde{C} = (C \quad 0 \quad 0 \quad \cdots)$.

Theorem 1 (*Dzieliński-Sierociuk, 2008) The system given by definition (2) is asymptotically stable if and only if* $\| \tilde{A} \| < 1$, *where* $\|.\|$ *denotes the matrix norm defined as* $max|\lambda_i|$, *where* λ_i *is the* i^{th} *eigenvalue of the matrix* $\tilde{A}$.

The general solution of (11) (Buslowicz, 1983) is given as

$$x_i = G_i x_0 \quad (12)$$

where

$$G_i = \begin{cases} I_n & \text{if} \quad i = 0 \\ \sum_{j=0}^{L} A_j G_{i-1-j} & \text{if} \quad i \geq 1 \end{cases} \quad (13)$$

with $G_i = 0, \quad \forall i < 0$.

Remark 2 From (12) and (13) for $\alpha = 1$ we have

$$x_i = (A + I + BK)^i x_0 \quad (14)$$

for which the corresponding solution of the linear discrete-time systems is

$$\begin{cases} x_{i+1} = (A + I)x_i + Bu_i, \quad i \geq 0 \\ x_0 \in \mathbb{R}^n \end{cases} \quad (15)$$

Remark 3 In the case of noncommensurate order we have

$$\begin{cases} x_{k+1} &= \sum_{j=0}^{k} A_j x_{k-j} \\ x_0 &= \tau_0 + \hat{\tau}_0 \in \mathbb{R}^n \\ y_k &= Cx_k, \quad k \geq 0 \end{cases} \quad (16)$$

where the matrices A_j are given as

$$A_0 = A + BK + \text{diag}\left(\binom{\alpha_1}{1}, \ldots, \binom{\alpha_n}{1}\right) \quad (17)$$

and for all $j \geq 1$

$$A_j = -(-1)^{j+1} \text{diag} \left\{ \overbrace{\binom{\alpha_1}{j+1}, \ldots, \binom{\alpha_n}{j+1}}^{n-\text{times}} \right\}. \quad (18)$$

If $A \subset \mathbb{R}^n$, then the convex hull of A is the smallest convex set containing A, that is, it consists of all finite convex combinations of elements in A. The closed convex hull of A is the closure of the convex hull of A. Now, we present the theorem of Krein–Milman (see Haim Brezis).

Theorem 2 (*Krein–Milman) Let* $K \subset \mathbb{R}^n$ *be a compact convex set. Then* K *coincides with the closed convex hull of its extreme points. In the following, we will assume that the unknown part* $\hat{\tau}_0 \in \Omega$, *where the set* $\Omega \subset \mathbb{R}^n$ *is a polytope convex of vertices* $w_1, w_2, \ldots, w_p$. *According to Krein–Milman theorem, the unknown part* $\hat{\tau}_0$ *could be written as follows:*

$$\hat{\tau}_0 = \sum_{j=1}^{p} \alpha_j w_j. \quad (19)$$

for some unknown coefficients $\alpha_1 \geq 0, \ldots, \alpha_p \geq 0$, such that $\sum_{j=1}^{p} \alpha_j = 1$.

Definition 3. (Larrache, 2020) An unknown part $\hat{\tau}_0$ is said to be ϵ −tolerable if the corresponding output satisfies the following condition:

$$\| \frac{\partial y_i}{\partial \alpha_j} \| \leq \epsilon, \quad \forall j \in \sigma_1^p, \quad \forall i \geq 0. \quad (20)$$

Otherwise, $\hat{\tau}_0$ is said to be ϵ −intolerable.

Definition 4. For a given $\epsilon > 0$, and a gain matrix $K \in \mathbb{R}^{m \times n}$, the set

$$\Upsilon(K, \epsilon) = \{x \in \mathbb{R}^n / \quad \| y_i \| = \| CG_i x \| \leq \epsilon, \quad \forall i \geq 0\} \quad (21)$$

is called the maximal output set.

Our goal is to characterise the set $\chi(\hat{\tau}_0, \epsilon)$ of all gain matrices K, which makes the systems insensitive to the unknown part $\hat{\tau}_0$, to be explicit as

$$\chi(\hat{\tau}_0, \epsilon) = \{K \in \mathbb{R}^{m \times n} / \| \frac{\partial y_i}{\partial \alpha_j} \| \leq \epsilon, \quad \forall j \in \sigma_1^p, \quad \forall i \geq 0\}. \quad (22)$$

On the other hand, we have

$$\frac{\partial y_i}{\partial \alpha_j} = \frac{\partial CG_i(\tau_0 + \hat{\tau}_0)}{\partial \alpha_j} = CG_i w_j, \quad \forall j \in \sigma_1^p, \quad \forall i \geq 0. \quad (23)$$

This leads to

$$\chi(\hat{\tau}_0, \epsilon) = \{K \in \mathbb{R}^{m \times n} / \| CG_i w_j \| \leq \epsilon, \forall j \in \sigma_1^p, \forall i \geq 0\}. \quad (24)$$

In Remark 4, we will show the interest of introducing the set $\Upsilon(K, \epsilon)$ in the characterisation of $\chi(\hat{\tau}_0, \epsilon)$.

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Remark 4 The set $\chi(\hat{\tau}_0, \epsilon)$ can be rewritten as

$$\chi(\hat{\tau}_0, \epsilon) = \{K \in \mathbb{R}^{m\times n} \;/\; w_j \in \Upsilon(K, \epsilon), \quad \forall j \in \sigma_1^p\}. \tag{25}$$

Therefore, system (6) is insensitive to the unknown part $\hat{\tau}_0$ if and only if $w_j \in \Upsilon(K, \epsilon), \quad \forall j \in \sigma_1^p$. In order to characterise the set $\Upsilon(K, \epsilon)$ and then our set $\chi(\hat{\tau}_0, \epsilon)$, we introduce the sets $\Upsilon^k(K, \epsilon), \quad k \geq 0$ defined as

$$\Upsilon^k(K, \epsilon) = \{x \in \mathbb{R}^n / \quad \| y_i \| = \| CG_i x \| \leq \epsilon, \quad \forall i \in \sigma_0^k\}. \tag{26}$$

3. CHARACTERISATION OF THE MAXIMAL OUTPUT SET $\Upsilon(K, \epsilon)$

The main purpose of this section is to characterise, under certain hypothesis, the maximal output set $\Upsilon(K, \epsilon)$ and then the set $\chi(\hat{\tau}_0, \epsilon)$. We prove the finite determination of the set $\Upsilon(K, \epsilon)$ and then the set $\chi(\hat{\tau}_0, \epsilon)$, and this leads to the algorithmic procedure for the computation of such set.

Definition 5. (Gilbert, 1991; Rachik, 2002) The set $\Upsilon(K, \epsilon)$ is said to be finitely determined, if there exists an integer k such that $\Upsilon(K, \epsilon) = \Upsilon^k(K, \epsilon)$. Let k^* be the smallest integer such that $\Upsilon(K, \epsilon) = \Upsilon^{k^*}(K, \epsilon)$; we call k^* the admissibility index.

Remark 5 $\{\Upsilon^k(K, \epsilon)\}_{k\geq 0}$ is a decreasing sequence, that is, $\forall k \leq s$ we have

$$\Upsilon(K, \epsilon) \subset \Upsilon^s(K, \epsilon) \subset \Upsilon^k(K, \epsilon). \tag{27}$$

Proposition 1. *The set $\Upsilon(K, \epsilon)$ of some gain matrix K is*
- *(i) Convex,*
- *(ii) Symmetric,*
- *(iii) Contain the origin in its interior,*
- *(iv) Closed.*

Proof. (i), (ii) and (iii) from the definition of $\Upsilon(K, \epsilon)$.

(iv) We define for each $k \in \mathbb{N}$ the function T_k as

$$T_k: \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^p \\ x & \longmapsto & CG_k x. \end{array} \tag{28}$$

Then

$$\Upsilon(K, \epsilon) = \bigcap_{k\geq 0} T_k^{-1}(B(0, \epsilon)) \tag{29}$$

where $B(0, \epsilon) = \{x \in \mathbb{R}^p / \; \|x\| \leq \epsilon\}$.

Since $B(0, \epsilon)$ is closed and $(T_k)_{k\geq 0}, k \in \mathbb{N}$ are continuous functions, then $T_k^{-1}(B(0, \epsilon)), k \in \mathbb{N}$ are closed. Therefore $\Upsilon(K, \epsilon)$ is closed.

We give a necessary condition ensuring the finite determination of the set $\Upsilon(K, \epsilon)$ and then the set $\chi(\hat{\tau}_0, \epsilon)$.

Proposition 2. *If $\Upsilon(K, \epsilon)$ is finitely determined, then there exists an integer k^* such that $\Upsilon^{k^*}(K, \epsilon) = \Upsilon^{k^*+1}(K, \epsilon)$.*

Proof. Suppose $\Upsilon(K, \epsilon)$ is finitely determined. Then

$$\exists k \in \mathbb{N}, \qquad \Upsilon(K, \epsilon) = \Upsilon^k(K, \epsilon). \tag{30}$$

On the other hand

$$\Upsilon^k(K, \epsilon) = \Upsilon(K, \epsilon) \subset \Upsilon^{k+1}(K, \epsilon) \subset \Upsilon^k(K, \epsilon) \tag{31}$$

since $\{\Upsilon^k(K, \epsilon)\}_{k\geq 0}$ is a decreasing sequence.

This leads to

$$\Upsilon^k(K, \epsilon) = \Upsilon^{k+1}(K, \epsilon), \qquad \text{for some } k \geq 0 \tag{32}$$

which completes the proof.

An efficient result is then introduced that permits us to determine the set $\Upsilon(K, \epsilon)$ through a finite number of inequations and then the set $\chi(\hat{\tau}_0, \epsilon)$ leads also to the generation of an algorithmic approach to obtain admissibility index k^*. In our study, we consider two cases:

First case: $p = n$ (i.e. the observation space and the state space have the same dimension).

Second case: $p < n$.

First case, $p = n$. In this case C is an $n \times n$ matrix.

Proposition 3. *Suppose the following assumptions hold:*

(i) $\sum_{j=0}^{L} \| A_j \| \leq 1$ and $\Upsilon^k(K, \epsilon) = \Upsilon^{k+1}(K, \epsilon)$ for some k,

(ii) C commutes with A_j for all $0 \leq j \leq L$.

Then $\Upsilon(K, \epsilon)$ is finitely determined.

Proof. Clearly $\Upsilon(K, \epsilon) \subset \Upsilon^k(K, \epsilon)$. Let $x_0 \in \Upsilon^k(K, \epsilon)$, then

$$\| CG_i x_0 \| \leq \epsilon, \quad \forall i \leq k + 1.$$

But

$$\| CG_{k+2} x_0 \| = \| C\left(\sum_{j=0}^{L} A_j G_{k+1-j}\right) x_0 \|$$

$$= \| \sum_{j=0}^{L} \left(CA_j G_{k+1-j} x_0\right) \|$$

$$= \| \sum_{j=0}^{L} \left(A_j CG_{k+1-j} x_0\right) \|$$

$$\leq \sum_{j=0}^{L} \| A_j \| \| CG_{k+1-j} x_0 \|$$

$$\leq \epsilon \sum_{j=0}^{L} \| A_j \|$$

since $\| CG_{k+1-j} x_0 \| \leq \epsilon, \quad \forall j \in \sigma_0^L$.

Now, using the assumption $\sum_{j=0}^{L} \| A_j \| \leq 1$; it follows that

$$\| CG_{k+2} x_0 \| \leq \epsilon.$$

By iteration, we show that

$$\| CG_{k+j} x_0 \| \leq \epsilon, \qquad \forall j \geq 2$$

That is,

$$\| CG_i x_0 \| \leq \epsilon, \qquad \forall i \geq k + 2.$$

Consequently,

$$\| CG_i x_0 \| \leq \epsilon, \qquad \forall i \geq 0$$

This leads to

$$x_0 \in \Upsilon(K, \epsilon)$$

and complete the proof.

Second case: $\dim B(0, \epsilon) = p < n$.

Since the matrix $C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, we define $\hat{C}$ and $\hat{B}(0, \epsilon)$ as

$$\hat{C} = \begin{pmatrix} C \\ 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$$

$\widehat{B}(0,\epsilon) = B(0,\epsilon) \times \{0_{\mathbb{R}^{n-p}}\} \subset \mathbb{R}^n$.

Now considering the new observation $\hat{y}_i = \hat{C}x_i$, we easily verify that, for every integer i

$y_i \in B(0,\epsilon) \Leftrightarrow \hat{y}_i \in \widehat{B}(0,\epsilon)$.

Remark 6 The set $\Upsilon(K,\epsilon)$ associated to C and $B(0,\epsilon)$ is equal to the set $\Upsilon(K,\epsilon)$ associated to $\hat{C}$ and $\widehat{B}(0,\epsilon)$.
Since $\dim\widehat{B}(0,\epsilon) = n$, then the result of the first case can be applied to deduce the following proposition.
Proposition 4. *Suppose the following assumptions hold:*
(i) $\sum_{j=0}^{L} \| A_j \| \leq 1$ *and* $\Upsilon^k(K,\epsilon) = \Upsilon^{k+1}(K,\epsilon)$ *for some* k
(ii) $\hat{C}$ *commutes with* A_j *for all* $0 \leq j \leq L$.
Then $\Upsilon(K,\epsilon)$ *is finitely determined.*
Proposition 5. *If* $\| G_k \| \leq \eta_k, \quad \forall k \geq 0$ *with* $\eta_k \longrightarrow 0$ *as* $k \longrightarrow \infty$ *then* $\Upsilon(K,\epsilon)$ *is finitely determined*.
Proof. Let $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Then

$\| CG_k x \| \leq \| C \| \| G_k \| \| x \|$

$\leq \eta_k \| C \| \| x \|$

And since η_k converges to zero when $k \longrightarrow \infty$ we deduce that

$$\| CG_k x \| \leq \epsilon, \quad \forall k \geq k_0 \quad \text{for certain} \quad k_0 \geq 0. \qquad (33)$$

Let $x_0 \in \Upsilon^{k_0}(K,\epsilon)$. Then

$\| CG_i x_0 \| \leq \epsilon, \quad \forall i \in \sigma_0^{k_0}$

using this time Eq. (33) we obtain

$\| CG_{k_0+1} x_0 \| \leq \epsilon \quad \text{since} \quad k_0 + 1 \geq k_0$.

Hence

$x_0 \in \Upsilon^{k_0+1}(K,\epsilon)$

which completes the proof.

4. ALGORITHMIC DETERMINATION

As a direct consequence of Propositions 3 and 4, we propose in this section a procedure to determine k^*, index of admissibility, and consequently the sets $\Upsilon(K,\epsilon)$ and $\chi(\hat{\tau}_0,\epsilon)$.
Let $\mathbb{R}^p$ be endowed with the following norm:

$$\|x\|_\infty = \max_{1\leq i\leq p} |(x)_i|, \qquad \forall x \in \mathbb{R}^p.$$

We remark that

$\Upsilon^k(K,\epsilon) = \Upsilon^{k+1}(K) \Leftrightarrow \Upsilon^k(K,\epsilon) \subset \Upsilon^{k+1}(K,\epsilon)$

since $\Upsilon^{k+1}(K,\epsilon) \subset \Upsilon^k(K,\epsilon)$.
Thus

$$\Upsilon^k(K,\epsilon) = \Upsilon^{k+1}(K,\epsilon)$$
$$\Leftrightarrow \forall x \in \Upsilon^k(K,\epsilon), \quad \|CG_{k+1}x\|_\infty \leq \epsilon$$
$$\Leftrightarrow \forall x \in \Upsilon^k(K,\epsilon), \quad \left|(CG_{k+1}x)_j\right| - \epsilon \leq 0, \forall j \in \sigma_1^p$$
$$\Leftrightarrow \sup_{|(CG_ix)_l|-\epsilon\leq 0, \forall l\in\sigma_1^p, \forall i\in\sigma_1^k} \left|(CG_{k+1}x)_j\right| - \epsilon \leq 0, \quad \forall j \in \sigma_1^p.$$

This leads to the following algorithm.
Algorithm: Determination of k^*
Require $n, p, L \in \mathbb{N}^*, C, G_i, \epsilon > 0$
$k \leftarrow 0$
for j=1, ...,p **do**
Maximise $J_j(x) = |(CG_{k+1}x)_j| - \epsilon$

Subjet to the constraints $\begin{cases} |(CG_ix)_l| - \epsilon \leq 0 \\ \forall l \in \sigma_1^p, \quad \forall i \in \sigma_0^k. \end{cases}$

end for
$J_j^* \leftarrow \max\{J_j(x)\}$
if $J_j^* \leq 0, \forall j = 1,2,\dots,p$ **then**
$k^* \leftarrow k$
else
$k \leftarrow k + 1$ and return to for
end else
Remark 7 The hypothesis of Proposition 5 in section (3) is sufficient but not necessary. If this condition is not provided, the stopping of the algorithm is not certain. The maximal output set $\Upsilon(K,\epsilon)$ is finitely determined and then the set $\chi(\hat{\tau}_0,\epsilon)$ if the algorithm converges, otherwise it is not.

5. NUMERICAL EXAMPLES

To demonstrate our achieved results, we present in the following section some examples in the two-dimensional case. We will determine the set $\Upsilon_\epsilon(K)$ and then the set $\chi(\hat{\tau}_0,\epsilon)$ as a finite number of inequations using our algorithm. Assuming

$$\textstyle\sum_{j=0}^{L} \| A_j \| < 1 \qquad (34)$$

is checked in all the introduced examples, we will select the gain matrix K such that this condition (34) is verified.
Using the property that (Hilfer, 2000)

$$\sum_{j=0}^{L} (-1)^j \binom{\alpha}{j} = \frac{\Gamma(L+1-\alpha)}{\Gamma(1-\alpha)\Gamma(L+1)}$$

and the fact that

$$\textstyle\sum_{j=0}^{L} \| A_j \| = \| A + BK + \alpha I_n \| - \sum_{j=2}^{L} (-1)^j \binom{\alpha}{j} + \left| \binom{\alpha}{L+1} \right|$$

we deduce that the condition $\sum_{j=0}^{L} \| A_j \| < 1$ can be rewritten as follows:

$$\| A + BK + \alpha I_n \| - \sum_{j=2}^{L} (-1)^j \binom{\alpha}{j} + \left| \binom{\alpha}{L+1} \right| < 1$$

For example, for $L = 50$ and $\alpha = 0.4$, we have

$\sum_{j=0}^{L} \| A_j \| = \| A + BK + \alpha I_n \| + 0.4610$.

Then the matrix K must select such that

$\| A + BK + \alpha I_n \| < 0.5390$.

The dotted region will indicate the set $\Upsilon_\epsilon(K,\epsilon)$.
Example 1. *Let us consider the following system:*

$$\begin{cases} x_{i+1} = \sum_{j=0}^{i} A_j x_{i-j} \\ x_0 = \tau_0 + \hat{\tau}_0 \in \mathbb{R}^2 \end{cases}$$

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where τ_0 and $\hat{\tau}_0$ are the known and unknown parts of the initial state, respectively.

Let $A, B, C, \alpha, \epsilon$ and L be defined as

$$A = \begin{pmatrix} -1.25 & \frac{11}{24} \\ \frac{5}{12} & -\frac{8}{7} \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -\frac{1}{3} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$C = (1 \quad 2), \qquad \alpha = 0.7, \qquad \epsilon = 0.7, \qquad L = 20.$$

We have

$$A_j = -(-1)^{j+1} \binom{0.7}{j+1} I_2, \quad \forall j \in \{1, 2, \dots, L\}.$$

We select the gain matrix K such that $\| A_0 \| < 0.7395$ (since $\sum_{j=1}^{20} \| A_j \| = 0.2605$). For $K = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\widetilde{A} = A + BK = \begin{pmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & -\frac{1}{7} \end{pmatrix}$$

$$A_0 = \widetilde{A} + \alpha I_2 = \begin{pmatrix} \frac{9}{20} & \frac{1}{8} \\ \frac{1}{6} & \frac{39}{70} \end{pmatrix}$$

and

$$\sum_{j=0}^{20} \| A_j \| = \| A_0 \| + \sum_{j=1}^{20} \| A_j \| = 0.9426 < 1$$

where $\| A_0 \| = \max_{1 \le j \le 2} \sum_{i=1}^{2} |(A_0)_{ij}|$.

In this example, we take τ_0 that belongs to a hexagon (a polygon with six sides), see Fig. 1.

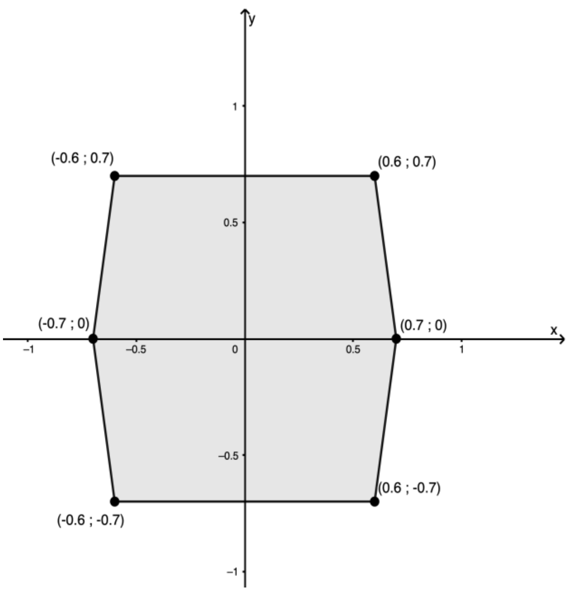

Fig. 1. The set Ω with vertices $w_1, w_2, \dots, w_6$.

$CG_0\binom{x}{y}, \dots, CG_5\binom{x}{y}$ are given by:

$$CG_0\binom{x}{y} = x + 2y$$

$$CG_1\binom{x}{y} = \frac{47}{60}x + \frac{347}{280}y$$

$$CG_2\binom{x}{y} = \frac{2789}{4200}x + \frac{117\,409}{117\,600}y$$

$$CG_3\binom{x}{y} = \frac{2091\,989}{3528\,000}x + \frac{7082\,557}{8232\,000}y$$

$$CG_4\binom{x}{y} = \frac{267\,585\,763}{493\,920\,000}x + \frac{5303\,791\,039}{6914\,880\,000}y$$

$$CG_5\binom{x}{y} = \frac{260\,274\,628\,301}{518\,616\,000\,000}x + \frac{3377\,530\,607\,827}{4840\,416\,000\,000}y.$$

Using our algorithm we obtain $k^* = 4$ and then

$$\Upsilon(K, \epsilon) = \left\{ \binom{x}{y} \in \mathbb{R}^2 \backslash \begin{array}{l} |x + 2y| \le 2 \\ \left|\frac{47}{60}x + \frac{347}{280}y\right| \le 2 \\ \left|\frac{2789}{4200}x + \frac{117\,409}{117\,600}y\right| \le 2, \\ \left|\frac{2091\,989}{3528\,000}x + \frac{7082\,557}{8232\,000}y\right| \le 2 \\ \left|\frac{267\,585\,763}{493\,920\,000}x + \frac{5303\,791\,039}{6914\,880\,000}y\right| \le 2 \end{array} \right\}.$$

We can see that $w_1 = \binom{0.7}{0}, w_2 = \binom{0.6}{0.7}, w_3 = \binom{-0.6}{0.7}, w_4 = \binom{-0.7}{0}, w_5 = \binom{-0.6}{-0.7}, w_6 = \binom{0.6}{-0.7} \in \Upsilon(K, \epsilon)$. Hence, the system is insensitive to the unknown part $\hat{\tau}_0$. Consequently, the gain matrix $K \in \chi(\hat{\tau}_0, \epsilon)$.

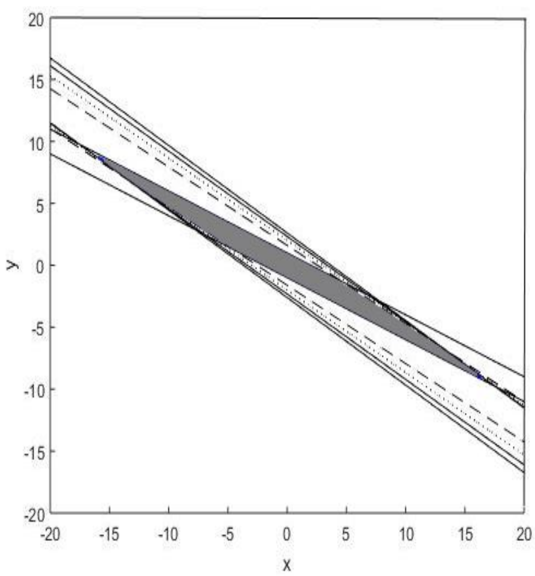

Fig. 2. The set $\Upsilon(K, \epsilon)$ corresponding to $K = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha = 0.7$

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Example 2. *Consider the following system:*

$$\begin{cases} x_{i+1} = \sum_{j=0}^{i} A_j x_{i-j} \\ x_0 = \tau_0 + \hat{\tau}_0 \in \mathbb{R}^2 \end{cases}$$

where τ_0 and $\hat{\tau}_0$ are the known and unknown parts of the initial state, respectively, and $A, B, K, C, \alpha, \epsilon$ are described as follows:

$$C = (2 \quad -1), \quad A = \begin{pmatrix} \frac{8}{7} & \frac{2}{3} \\ -\frac{17}{30} & -\frac{2}{9} \end{pmatrix}$$

$$B = \begin{pmatrix} -0.5 \\ \frac{1}{3} \end{pmatrix}, \qquad \alpha = 0.2, \qquad \epsilon = 0.8.$$

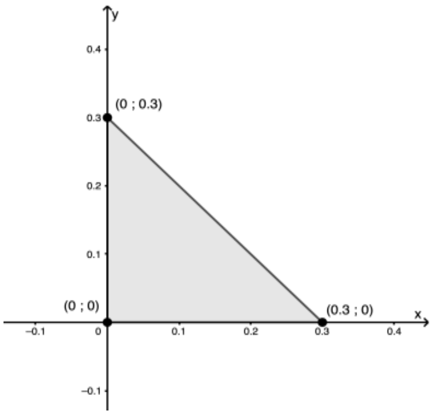

Fig. 3. The set Ω with vertice $w_1 = (0; 0), w_2 = (0.3; 0)$, $w_3 = (0; 0.3)$.

In this example, the memory length L is equal to 30 and we assume τ_0 to belong to a triangle (see Fig. 3). The matrices A_j are given as

$$A_0 = \widetilde{A} + \alpha I_2 = A + BK + \alpha I_2$$

and

$$A_j = -(-1)^{j+1} \binom{\alpha}{j+1} I_3, \qquad j = 1, \dots, L.$$

We select the gain matrix K such that $\| A_0 \| < 0.6311$ (since $\sum_{j=1}^{30} \| A_j \| = 0.3689$). For $K = (2 \quad 1)$ we have

$$A_0 = \begin{pmatrix} \frac{12}{35} & \frac{1}{6} \\ \frac{1}{10} & \frac{14}{45} \end{pmatrix}$$

and

$$\sum_{j=0}^{30} \| A_j \| = \| A_0 \| + \sum_{j=1}^{30} \| A_j \| = 0.8467 < 1$$

where $\| A_0 \| = \max_{1 \le j \le 2} \sum_{i=1}^{2} |(A_0)_{ij}|$.

On the other hand, we have

$$CG_0 \begin{pmatrix} x \\ y \end{pmatrix} = (2 \quad -1) \begin{pmatrix} x \\ y \end{pmatrix} = 2x - y$$

$$CG_1 \begin{pmatrix} x \\ y \end{pmatrix} = (2 \quad -1) \begin{pmatrix} \frac{12}{35} & \frac{1}{6} \\ \frac{1}{10} & \frac{14}{45} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{41}{70} x + \frac{1}{45} y$$

$$CG_2 \begin{pmatrix} x \\ y \end{pmatrix} = (2 \quad -1) \begin{pmatrix} \frac{3149}{14\,700} & \frac{103}{945} \\ \frac{103}{1575} & \frac{1567}{8100} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1601}{4410} x + \frac{1391}{56\,700} y$$

$$CG_3 \begin{pmatrix} x \\ y \end{pmatrix} = (2 \quad -1) \begin{pmatrix} \frac{369\,917}{2315\,250} & \frac{197\,527}{2381\,400} \\ \frac{197\,527}{3969\,000} & \frac{91\,838}{637\,875} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =$$
$$\frac{7495\,319}{27\,783\,000} x + \frac{391\,441}{17\,860\,500} y.$$

We have used the relation $G_k = \sum_{j=0}^{L} A_j G_{k-1-j}, k \ge 1$ to find the matrices G_k. Using our algorithm we obtain $k^* = 1$ and then the set $\Upsilon(K, \epsilon)$

$$\Upsilon(K, \epsilon) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \setminus \begin{matrix} |2x - y| \le 0.8 \\ \left| \frac{41}{70} x + \frac{1}{45} y \right| \le 0.8 \end{matrix} \right\}.$$

Since $w_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0.3 \\ 0 \end{pmatrix}, w_1 = \begin{pmatrix} 0 \\ 0.3 \end{pmatrix} \in \Upsilon(K, \epsilon)$, we deduce that the unknown part $\hat{\tau}_0$ does not influence the associated output function. In this case, the chosen matrix K belongs to $\chi(\hat{\tau}_0, \epsilon)$, so it is useful.

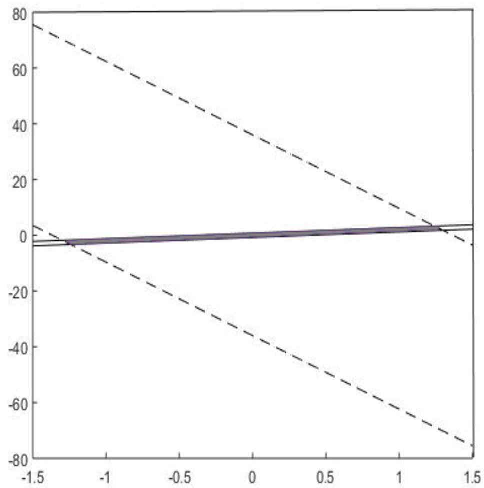

Fig. 4. The set $\Upsilon(K, \epsilon)$ is associated to $K = (2 \quad 1)$ and $\alpha = 0.2$.

Example 3. *For*

$$A = \begin{pmatrix} \frac{37}{12} & -\frac{15}{8} \\ \frac{1}{10} & -\frac{15}{8} \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad C = (1 \quad -1)$$

$$\varepsilon = 0.1, \quad \alpha = 0.1, \quad L = 8$$

we obtain

$$CG_0 \begin{pmatrix} x \\ y \end{pmatrix} = x - y$$

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$$CG_1\begin{pmatrix}x\\y\end{pmatrix} = (1 \quad -1)\begin{pmatrix}\frac{11}{60} & \frac{1}{8}\\ \frac{1}{10} & \frac{9}{40}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{12}x - \frac{1}{10}y$$

$$CG_2\begin{pmatrix}x\\y\end{pmatrix} = (1 \quad -1)\begin{pmatrix}\frac{41}{450} & \frac{49}{960}\\ \frac{49}{1200} & \frac{173}{1600}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{181}{3600}x - \frac{137}{2400}y$$

$$CG_3\begin{pmatrix}x\\y\end{pmatrix} = C\begin{pmatrix}\frac{25\,297}{432\,000} & \frac{3283}{115\,200}\\ \frac{3283}{144\,000} & \frac{13\,067}{192\,000}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1931}{54\,000}x - \frac{11\,393}{288\,000}y$$

$$CG_4\begin{pmatrix}x\\y\end{pmatrix} = C\begin{pmatrix}\frac{56\,471\,173}{1296\,000\,000} & \frac{270\,829}{13\,824\,000}\\ \frac{270\,829}{17\,280\,000} & \frac{28\,859\,813}{576\,000\,000}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{18\,079\,499}{648\,000\,000}x - \frac{26\,362\,907}{864\,000\,000}y$$

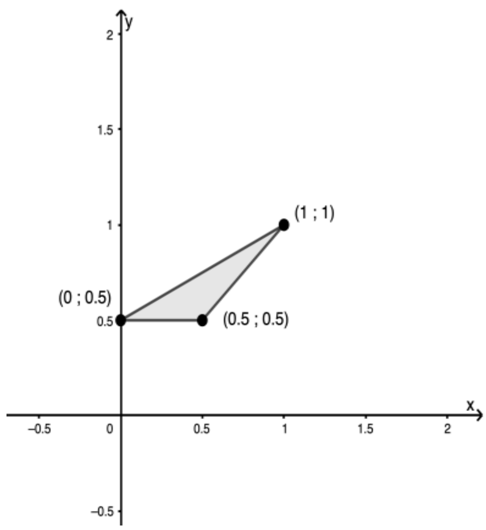

Fig. 5. The set Ω with vertices
$w_1 = (0.5; 0.5), w_2 = (0; 0.5), w_3 = (1; 1)$.

In this example, we assume τ_0 belongs to a triangle (see Fig. 4). Using our algorithm, we obtain $k^* = 3$, and then the set

$$\Upsilon(K,\epsilon) = \left\{\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 / \begin{array}{l} |x-y| \le 0.1 \\ \left|\frac{1}{12}x - \frac{1}{10}y\right| \le 0.1 \\ \left|\frac{181}{3600}x - \frac{137}{2400}y\right| \le 0.1 \\ \left|\frac{1931}{54\,000}x - \frac{11\,393}{288\,000}y\right| \le 0.1 \end{array}\right\}$$

Since $w_1 = \begin{pmatrix}0.5\\0.5\end{pmatrix}, w_2 = \begin{pmatrix}1\\1\end{pmatrix} \in \Upsilon(K,\epsilon)$ and $w_3 = \begin{pmatrix}0\\0.5\end{pmatrix} \notin \Upsilon(K,\epsilon)$, we conclude that the system is influenced by the unknown part $\hat{\tau}_0$. Thus $K = \begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix} \notin \chi(\hat{\tau}_0,\epsilon)$.

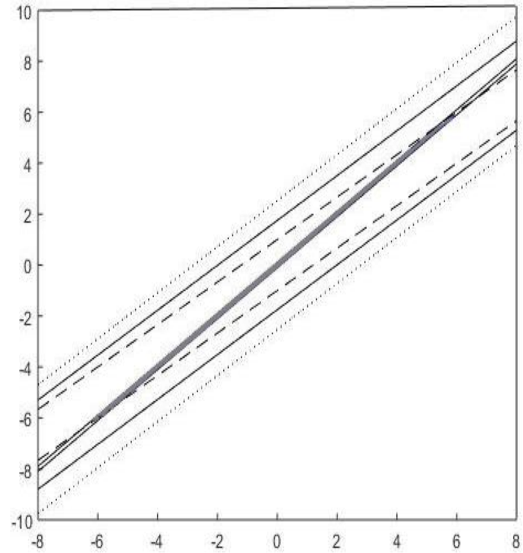

Fig. 6. The set $\Upsilon(K,\epsilon)$ corresponding to $K = \begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix}$ and $\alpha = 0.1$.

In examples 1–3, we have identified the set of all possible gain matrices which make the system insensitive to the unknown part $\hat{\tau}_0$ of the initial state x_0 via a finite number of inequations using Algorithm 1 based on the simplex method, which allow solving problems of maximisation that arise in such algorithm. In Figs. 1–3, we have traced the constraints constituting the sets $\Upsilon(K,\epsilon)$.

6. CONCLUSION

In this paper, we have studied the problem of fractional order discrete-time controlled linear systems with unknown part of the initial state using the Grünwald–Letnikov fractional derivative. We have investigated the characterisation of the set $\chi(\hat{\tau}_0,\epsilon)$ of all possible gain matrices so that the sensitivity of the resulting system output would be relatively tolerable based on the study of maximal output set. Some new sufficient conditions to ensure the finite determination of $\chi(\hat{\tau}_0,\epsilon)$ are given. Furthermore, a useful algorithm is produced to identify the index of admissibility k^* and then the set $\chi(\hat{\tau}_0,\epsilon)$. The theoretical results are shown by various examples and numerical simulation. As a natural continuation of this work, we are studying the following problem.

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Youssef Benfatah: https://orcid.org/0000-0001-9497-003X

Amine El Bhih: https://orcid.org/0000-0003-0877-5882

Mostafa Rachik: https://orcid.org/0000-0002-5118-2786