10.24425/119080

Archives of Control Sciences
Volume 28(LXIV), 2018
No. 1*, pages 119–133*

Reachability and observability of positive discrete-time linear systems with integer positive and negative powers of the state Frobenius matrices

TADEUSZ KACZOREK

The notions of monomial generalized Frobenius matrices is proposed and the reachability and observability of positive discrete-time linear systems with positive and negative integer powers of the state matrices is addressed. Necessary and sufficient conditions for the reachability of the positive systems are established.

Key words: discrete-time, linear, positive, system, monomial Frobenius generalized matrix, reachability, observability

1. Introduction

A dynamical system is called positive if its state variables and outputs take nonnegative values for all nonnegative inputs and nonnegative initial conditions. The positive linear and nonlinear continuous-time and discrete-time systems have been addressed in many papers and books [1–23]. Positive descriptor systems have been analyzed in [1–3, 6–11, 15, 17, 22, 23] and positive nonlinear systems in [18, 19]. The minimum energy control of positive systems has been investigated in [9–12, 14] and the stability of positive systems in [4, 14, 21, 23]. The positive systems consisting of n subsystems with different fractional orders have been introduced in [13, 16].

In this paper the reachability and observability of positive discrete-time linear systems with integer positive and negative powers of state monomial generalized Frobenius matrices will be addressed.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning positive linear systems are recalled. The notion of monomial generalized Frobenius matrices has been introduced and the reachability of positive linear systems with these state matrices has been analyzed in Section 3.

T. Kaczorek is with Białystok University of Technology, Faculty of Electrical Engineering, Wiejska 45D, 15-351 Białystok, Poland. E-mail: kaczorek@ee.pw.edu.pl

This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.
Received 17.07.2017.

The reachability of the positive linear systems with integer positive powers of the state monomial generalized Frobenius matrices has been investigated in Section 4 and with integer negative powers in Section 5. The possibility of an extension of the considerations to the observability of positive discrete-time linear systems is shown in Section 6. Concluding remarks are given in Section 7.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, Z_+ is the set of nonnegative integers, I_n – the $n\times n$ identity matrix.

2. Preliminaries

Consider the positive discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0,1,\ldots\}, \tag{1a}$$

$$y_i = Cx_i, \tag{1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$.

Theorem 1 *[4, 14] The solution of the equation (1a) has the form*

$$x_i = A^i x_0 + \sum_{k=0}^{i-1} A^{i-k-1}Bu_k, \quad i\in Z_+. \tag{2}$$

Definition 1 *[4, 14] The discrete-time linear system (1) is called (internally) positive if $x_i\in\Re_+^n$, $y_i\in\Re_+^p$ for every $x_0\in\Re_+^n$ and all $u_i\in\Re_+^m$ for $i\in Z_+$.*

Theorem 2 *[4, 14] The discrete-time linear system (4) is (internally) positive if and only if*

$$A\in\Re_+^{n\times n}, \qquad B\in\Re_+^{n\times m}, \qquad C\in\Re_+^{p\times n}. \tag{3}$$

3. Reachability of positive discrete-time linear systems with monomial generalized Frobenius matrices

Definition 2 *[14] The positive discrete-time linear system (1) (or the pair (A,B)) is called reachable in q steps ($q\leqslant n$) if there exists for a given final state $x_f\in\Re_+^n$ an input sequence $u_i\in\Re_+^m$, $i=0,1,\ldots,q-1$ which steers the state vector $x_i\in\Re_+^n$ of the system from $x_0=0$ to x_f, i.e. $x_q=x_f$.*

A column is called monomial if it has only one positive entry and the remaining entries are zero.

Theorem 3 *[14] The positive discrete-time linear system (1) is reachable in q steps if and only if the reachability matrix*

$$R_q = [B \quad AB \quad \cdots \quad A^{q-1}B] \in \Re^{n\times mq}, \qquad q \leqslant n. \tag{4}$$

contains n linearly independent monomial columns.

Example 1 Consider the positive system (1a) with the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

The pair (A, B_1) is reachable in three steps $(q = 3)$ since the reachability matrix

$$R_3 = [B_1 \quad AB_1 \quad A^2B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

has three linearly independent monomial columns (is the monomial matrix).

The pair (A, B_2) is unreachable since the reachability matrix

$$R_3 = [B_2 \quad AB_2 \quad A^2B_2] = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ 1 & 3 & 11 \end{bmatrix} \tag{7}$$

has only one monomial column.

Definition 3 *The positive pair (A, B) satisfies the circulation condition if A^kB, $k = 1, \dots, n-1$ adds a new monomial column which is linearly independent of the monomial columns B, AB, ..., $A^{k-1}B$.*

Theorem 4 *The positive pair (A, B) is reachable if and only if the following two conditions are satisfied:*

1) the matrix

$$[A \;\; B] \in \Re_+^{n\times(n+1)} \tag{8}$$

contains n linearly independent monomial columns;

2) the pair (A, B) satisfies the circulation condition.

The condition 1) is the necessary condition for the reachability of the pair (A, B). If the condition (8) is not satisfied then the pair (A, B) is unreachable.

Example 2 The positive pair

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

satisfies the circulation condition since $AB = [0 \;\; 0 \;\; 2]^T$ is a monomial column linearly independent of the monomial column $B = [1 \;\; 0 \;\; 0]^T$ and $A^2B = [0 \;\; 4 \;\; 0]^T$ is a monomial column linearly independent of the monomial columns B and AB.

The positive pair

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

does not satisfy the circulation condition since $AB = [1 \;\; 0 \;\; 0]^T$ is a monomial column linearly dependent of the monomial column B.

Definition 4 *The matrices*

$$F_1 = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_n \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} a_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_{n-1} \\ 0 & 0 & \cdots & a_{n-2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_1 & \cdots & 0 & 0 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 0 & \cdots & a_{n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_2 & \cdots & 0 & 0 \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}, \tag{11}$$

$$(a_k, \quad k = 1, \ldots, n \;\; \textit{are real numbers})$$

are called the monomial generalized Frobenius matrices.

Remark 1 *From (11) it follows that $F_2 = F_1^T$ and F_3^T, F_4^T are also the monomial generalized Frobenius matrices.*

Consider the positive system (1a) with the matrices (11) and

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re^n, \quad B_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re^n \tag{12}$$

Theorem 5 *The pair* (F_1,B_1) *((F_2,B_2)) is reachable if and only if*

$$a_k > 0 \quad for \;\; k = 1,\dots,n \tag{13}$$

and the pair (F_1,B_2) *((F_2,B_1)) is reachable if and only if*

$$a_k > 0 \quad for \;\; k = 1,\dots,n-1. \tag{14}$$

Proof. The reachability matrix (7) of the pair (F_1,B_1) has the form

$$R_{n1} = [B_1 \;\; F_1B_1 \;\; \cdots \;\; F_1^{n-1}B_1] = \begin{bmatrix} 1 & 0 & 0 & \cdots & a_1a_2\dots a_n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_na_{n-1} & \cdots & 0 \\ 0 & a_n & 0 & \cdots & 0 \end{bmatrix}. \tag{15}$$

and of the pair (F_1,B_2)

$$R_{n2} = [B_2 \;\; F_1B_2 \;\; \cdots \;\; F_1^{n-1}B_2] = \begin{bmatrix} 0 & 0 & \cdots & a_1a_2\dots a_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1} & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{16}$$

The matrix (15) is monomial if and only if the condition (13) is satisfied and the matrix (16) if and only if the conditions (14) is satisfied.

Therefore, by Theorem 3 the pair (F_1,B_1) is reachable if and only if (13) holds and the pair (F_1,B_2) if (14) holds. The proof for the pairs (F_2,B_1), (F_2,B_2) is similar.□

Theorem 6 *The pairs* (F_3,B_1)*,* (F_3,B_2)*,* (F_4,B_1) *and* (F_4,B_2) *are unreachable for any values of* $a_k > 0$*,* $k = 1,\dots,n$*.*

Proof. The reachability matrix of the pair (F_3,B_1) has the form

$$R_{n3} = [B_1 \;\; F_3B_1 \;\; \cdots \;\; F_3^{n-1}B_1] = \begin{bmatrix} 1 & a_n & \cdots & a_n^{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{17}$$

and of the pair (F_3, B_2)

$$R_{n4} = [B_2 \;\; F_3B_2 \;\; \cdots \;\; F_3^{n-1}B_2] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & a_1^{0.5(n-1)} a_{n-1}^{0.5(n-1)} \end{bmatrix}. \tag{18}$$

From (17) and (18) it follows that the matrices are not monomial for any values of a_k, $k = 1, \ldots, n$. Therefore, the pairs (F_3, B_1), (F_3, B_2) are unreachable. The proof for the pairs (F_4, B_1), (F_4, B_2) is similar.□

4. Reachability of the positive discrete-time linear systems with positive integer powers of the state matrices

Consider the positive system (1) with positive integer powers of the state matrices $A = F_j^k$ for $j = 1,2,3,4$ and $k = 2,3,\ldots$.

Theorem 7 *Any integer positive power $k = 2,3,\ldots$ of the monomial generalized Frobenius matrix $F_j \in \Re_+^{n\times n}$, $j = 1,2,3,4$ is also the monomial generalized Frobenius matrix*

$$F_j^k = \begin{cases} \textit{nondiagonal} \;\; \textit{for} \;\; k = 2,\, 3,\, \ldots,\, n-1,\, n+1,\, \ldots \\ \textit{diagonal} \;\; \textit{for} \;\; k = n,\, 2n,\, \ldots\, . \end{cases} \tag{19}$$

Proof. To simplify the notation the proof will be given for $n = 4$ and $j = 1$. It is easy to check that

$$\begin{aligned} F_1^2 &= \begin{bmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \\ a_4 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & a_1a_2 & 0 \\ 0 & 0 & 0 & a_2a_3 \\ a_3a_4 & 0 & 0 & 0 \\ 0 & a_1a_4 & 0 & 0 \end{bmatrix}, \\ F_1^3 &= \begin{bmatrix} 0 & 0 & 0 & a_1a_2a_3 \\ a_2a_3a_4 & 0 & 0 & 0 \\ 0 & a_1a_3a_4 & 0 & 0 \\ 0 & 0 & a_1a_2a_4 & 0 \end{bmatrix}, \qquad F_1^4 = a \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ F_1^5 &= aF_1, \qquad F_1^6 = aF_1^2, \qquad F_1^7 = aF_1^3, \\ F_1^8 &= a^2 I_4, \qquad a = a_1a_2a_3a_4 \end{aligned} \tag{20}$$

and in general case we obtain (19).□

Theorem 8 *The positive pairs* (F_1^k, B_1), (F_1^k, B_2) *and* (F_2^k, B_1), (F_2^k, B_2) *are reachable for* $k = 1, \ldots, n-1, n+1, \ldots, 2n-1, 2n+1, \ldots$ *and unreachable for* $k = jn,\ j = 1, 2, \ldots$.

Proof. The condition 1) of Theorem 4 is satisfied for any B_1 and B_2 since by Theorem 7 F_1^k (F_2^k) is the monomial matrix for any k.

Note that

$$F_1^2 = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & a_1a_2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2}a_{n-1} \\ a_{n-1}a_n & 0 & 0 & \cdots & 0 \\ 0 & a_1a_n & 0 & \cdots & 0 \end{bmatrix},$$

$$F_1^3 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1a_2a_{n-1} \\ a_{n-2}a_{n-1}a_n & 0 & 0 & \cdots & 0 \\ 0 & a_1a_{n-1}a_n & 0 & \cdots & 0 \\ 0 & 0 & a_1a_2a_n & \cdots & 0 \end{bmatrix}, \ \ldots, \tag{21}$$

$$F_1^n = a_1a_2\ldots a_nI_n.$$

Using (21) we obtain

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad F_1B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_n \end{bmatrix},$$

$$F_1^2B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n-2}a_{n-1}a_n \\ 0 \\ 0 \end{bmatrix}, \qquad F_1^{n-1}B_1 = \begin{bmatrix} a_1a_2\ldots a_{n-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{22}$$

Note that (22) are linearly independent monomial columns satisfying the circulation condition. Therefore, the positive pair $(F_1, B_1$ is reachable. The same result we obtain for the positive pair (F_1, B_2). The proof for the positive pairs (F_1^k, B_1), (F_1^k, B_2) and (F_2^k, B_1), (F_2^k, B_2) for $k = 2, 3, \ldots, n-1$ is similar.

For $k = n$ we have $F_1^n = aI_n$, $a = a_1 a_2 \dots a_n$ and the pairs (F_1^n, B_1), (F_1^n, B_2) are unreachable since the reachability matrix

$$[B_1 \;\; F_1^n B_1 \;\; \cdots \;\; (F_1^n)^{n-1} B_1] = \begin{bmatrix} 1 & a & \cdots & a^{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{23}$$

has only one monomial column.□

Example 3 Consider the positive system (1a) with the matrices

$$\begin{aligned} A &= F_1 = \begin{bmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ a_3 & 0 & 0 \end{bmatrix}, \qquad B = B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ B &= B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad a_k > 0 \;\text{ for }\; k = 1, 2, 3. \end{aligned} \tag{24}$$

It is easy to check that

$$\begin{aligned} F_1^2 &= \begin{bmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ a_3 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & a_1 a_2 \\ a_2 a_3 & 0 & 0 \\ 0 & a_1 a_3 & 0 \end{bmatrix}, \\ F_1^3 &= aI_3, \qquad a = a_1 a_2 a_3 . \end{aligned} \tag{25}$$

The pairs (F_1, B_1), (F_1, B_2) are reachable since the monomial vectors

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad F_1 B_1 = \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix}, \qquad F_1^2 B_1 = \begin{bmatrix} 0 \\ a_2 a_3 \\ 0 \end{bmatrix} \tag{26a}$$

and

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad F_1 B_2 = \begin{bmatrix} 0 \\ a_2 \\ 0 \end{bmatrix}, \qquad F_1^2 B_2 = \begin{bmatrix} a_1 a_2 \\ 0 \\ 0 \end{bmatrix} \tag{26b}$$

are linearly independent. Therefore, by Theorem 4 (or 3) the pairs are reachable.

The pairs (F_1^2, B_1) and (F_1^2, B_2) are also reachable since the monomial vectors

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad F_1^2 B_1 = \begin{bmatrix} 0 \\ a_2 a_3 \\ 0 \end{bmatrix}, \qquad F_1^4 B_1 = aF_1 B_1 = a \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix} \tag{27a}$$

and

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad F_1^2 B_2 = \begin{bmatrix} a_1 a_2 \\ 0 \\ 0 \end{bmatrix}, \qquad F_1^4 B_2 = a F_1 B_2 = a \begin{bmatrix} 0 \\ a_2 \\ 0 \end{bmatrix} \tag{27b}$$

are linearly independent.

5. Reachability of the positive discrete-time linear systems with negative integer powers of the state matrices

In this section the positive system (1) with negative integer powers of the state matrices $A = F_j^{-k}$ for $j = 1, 2, 3, 4$ and $k = 2, 3, \ldots$ will be investigated.

Theorem 9 *If the matrices (11) are nonsingular, i.e.*

$$a_k \neq 0 \quad \textit{for} \;\; k = 1, \ldots, n \tag{28}$$

then the inverse matrices F_k^{-1}, $k = 1, 2, 3, 4$ are also monomial generalized Frobenius matrices of the forms

$$F_1^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{a_n} \\ \frac{1}{a_1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{n-1}} & 0 \end{bmatrix}, \quad F_2^{-1} = \begin{bmatrix} 0 & \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{n-1}} \\ \frac{1}{a_n} & 0 & 0 & \cdots & 0 \end{bmatrix},$$
$$F_3^{-1} = \begin{bmatrix} \frac{1}{a_n} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{a_1} \\ 0 & 0 & \cdots & \frac{1}{a_2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{a_{n-1}} & \cdots & 0 & 0 \end{bmatrix}, \quad F_4^{-1} = \begin{bmatrix} 0 & 0 & \cdots & \frac{1}{a_1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{a_{n-2}} & \cdots & 0 & 0 \\ \frac{1}{a_{n-1}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{a_n} \end{bmatrix}. \tag{29}$$

Proof. If the condition (28) is satisfied then

$$\det F_k = a_1 a_2 \ldots a_n \neq 0 \quad \text{for} \quad k = 1,2,3,4 \tag{30}$$

and there exist the inverse matrices F_k^{-1} for $k = 1,2,3,4$.

It is easy to verify that

$$F_k F_k^{-1} = F_k^{-1} F_k = I_n \quad \text{for} \quad k = 1,2,3,4. \square \tag{31}$$

Theorem 10 *Any integer power $j = 2,3,\ldots$ of the inverse matrices (29) are also monomial generalized Frobenius matrices and*

$$(F_k^{-1})^j = F_k^{-j} \quad \textit{for} \quad j = 2,3,\ldots \quad \textit{and} \quad k = 1,2,3,4. \tag{32}$$

Proof. Proof follows immediately from Theorems 9 and 7. $\square$

Note that to investigation of the reachability of the positive systems with negative integer powers of the state matrices (or equivalently of the pairs (F_k^{-j}, B_1), (F_k^{-j}, B_2)) theorems 4, 8 and 6 can be used.

Example 4 (Continuation of Example 3) Consider the positive system with

$$A = F_1^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{a_3} \\ \frac{1}{a_1} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \end{bmatrix}, \quad B = B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B = B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{33}$$

$$a_k > 0 \quad \text{for} \quad k = 1,2,3.$$

Taking into account that

$$F_1^{-2} = (F_1^{-1})^2 = \begin{bmatrix} 0 & 0 & \frac{1}{a_3} \\ \frac{1}{a_1} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & \frac{1}{a_2 a_3} & 0 \\ 0 & 0 & \frac{1}{a_1 a_3} \\ \frac{1}{a_1 a_2} & 0 & 0 \end{bmatrix} \tag{34}$$

we obtain for the pair (F_1^{-2}, B_1) the following linearly independent monomial columns

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad (F_1^{-1})^2 B_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a_1 a_2} \end{bmatrix}, \quad (F_1^{-1})^4 B_1 = \begin{bmatrix} 0 \\ \frac{1}{a_1^2 a_2 a_3} \\ 0 \end{bmatrix}. \tag{35}$$

Therefore, by Theorem 4 the positive pair (F_1^{-2}, B_1) is reachable. Similar result we obtain for the positive pair (F_2^{-2}, B_1) since the monomial columns

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad (F_1^{-1})^2 B_2 = \begin{bmatrix} 0 \\ \dfrac{1}{a_1 a_3} \\ 0 \end{bmatrix}, \qquad (F_1^{-1})^4 B_2 = \begin{bmatrix} \dfrac{1}{a_1 a_2 a_3^2} \\ 0 \\ 0 \end{bmatrix} \tag{36}$$

are also linearly independent.

Example 5 Consider the positive system with

$$A = F_3^{-1} = \begin{bmatrix} \dfrac{1}{a_3} & 0 & 0 \\ 0 & 0 & \dfrac{1}{a_1} \\ 0 & \dfrac{1}{a_2} & 0 \end{bmatrix}, \qquad B = B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$B = B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad a_k > 0 \text{ for } k = 1,2,3. \tag{37}$$

Taking into account that

$$F_3^{-2} = (F_3^{-1})^2 = \begin{bmatrix} \dfrac{1}{a_3} & 0 & 0 \\ 0 & 0 & \dfrac{1}{a_1} \\ 0 & \dfrac{1}{a_2} & 0 \end{bmatrix}^2 = \begin{bmatrix} \dfrac{1}{a_3^2} & 0 & 0 \\ 0 & \dfrac{1}{a_1 a_2} & 0 \\ 0 & 0 & \dfrac{1}{a_1 a_2} \end{bmatrix} \tag{38}$$

we obtain for the pair (F_3^{-2}, B_1) the following linearly dependent columns

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad (F_3^{-1})^2 B_1 = \begin{bmatrix} \dfrac{1}{a_3^2} \\ 0 \\ 0 \end{bmatrix}, \qquad (F_3^{-1})^4 B_1 = \begin{bmatrix} \dfrac{1}{a_3^4} \\ 0 \\ 0 \end{bmatrix}. \tag{39}$$

Therefore, the positive pair (F_3^{-2}, B_1) is unreachable.

Similar result we obtain for the positive pair (F_3^{-2}, B_2) since the monomial columns

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad F_3^{-2}B_2 = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{a_1a_2} \end{bmatrix}, \qquad F_3^{-4}B_2 = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{(a_1a_2)^2} \end{bmatrix} \tag{40}$$

are also linearly dependent.

6. Controllability of positive discrete-time linear systems

Consider the positive discrete-time linear system (1).

Definition 5 *The positive linear system (1) is called observable in q steps if knowing the input sequence u_0, u_1, ..., u_{q-1} and the corresponding output sequence y_0, y_1, ..., y_{q-1} we are able to find the unique initial state of the system $x_0 \in \Re_+^n$.*

To simplify the notation we assume without loss of generality the input sequence $u_i = 0$, $i \in Z_+$. In this case for scalar output y_i, $i = 1,...,q$ from (1b) and (2) we obtain

$$y_{0,q-1} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} x_0 \,. \tag{41}$$

The initial state $x_0 \in \Re_+^n$ can be found from (41) if and only if $q = n$ and the observability matrix

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{42}$$

has n linearly independent monomial rows. In this case $O_n^{-1} \in \Re_+^{n \times n}$ and

$$x_0 = O_n^{-1} y_{0,n-1} \in \Re_+^n \tag{43}$$

if and only if $y_{0,n-1} \in \Re_+^n$.

Therefore, the observability of the positive system (1) is the dual notion to the reachability. It is easy to show [14] that the positive system (1) (or equivalently the pair (A,C)) is observable if and only if the dual system

$$x_{i+1} = A^T x_i + C^T u_i, \qquad i \in Z_+ = \{0,1,\ldots\}, \tag{44a}$$

$$y_i = B^T x_i \tag{44b}$$

(the pair (A^T, C^T)) is reachable.

Therefore, all results obtained for the reachability of the positive system (for the positive pair (A,B)) can be applied to the observability of the dual positive system (the pair (A,C)).

Example 6 Consider the positive system (1) with the matrices

$$A = F_1 = \begin{bmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ a_3 & 0 & 0 \end{bmatrix}, \quad C = C_1 = [1 \ \ 0 \ \ 0], \quad C = C_2 = [0 \ \ 0 \ \ 1], \tag{45}$$
$$a_k > 0 \ \text{ for } k = 1,2,3.$$

Using (45) we obtain

$$O_1 = \begin{bmatrix} C_1 \\ C_1 F_1 \\ C_1 F_1^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ a_1 & a_2 a_3 & 0 \end{bmatrix}. \tag{46}$$

The pair (F_1, C_1) is unobservable since the matrix (46) has only two linearly independent monomial rows.

The pair (F_1, C_2) is observable since the observability matrix

$$O_2 = \begin{bmatrix} C_2 \\ C_2 F_1 \\ C_2 F_1^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ a_3 & 0 & 0 \\ 0 & a_1 a_3 & 0 \end{bmatrix} \tag{47}$$

has three linearly independent monomial rows.

The considerations presented in Sections 4 and 5 for the reachability and in Section 6 for the observability can be easily extended to positive linear discrete-time linear systems with many inputs and many outputs.

7. Concluding remarks

The reachability and observability of the positive discrete-time linear systems with integer positive and negative powers of the monomial generalized Frobenius state matrices have been addressed. A new notion of monomial generalized

Frobenius matrices has been introduced. The properties of these matrices have been investigated (Theorems 7, 9 and 10). New necessary and sufficient conditions for the reachability of the positive systems have been established (Theorems 4 and 8). It has been shown that the results concerning the reachability can be easily extended to the observability of the positive linear discrete-time systems.

The considerations have been illustrated by numerical examples. The considerations can be easily extended to the fractional discrete-time linear systems. An open problem is an extension of these considerations to continuous-time linear standard and positive systems.

References

[1] Bru R., Coll C., Romero-Vivo S. and Sanchez E.: Some problems about structural properties of positive descriptor systems, Positive systems (Rome, 2003), *Lecture Notes in Control and Inform. Sci.*, **294**, Springer, Berlin, 2003, 233–240.

[2] Bru R., Coll C. and Sanchez E.: About positively discrete-time singular systems, *System and Control: Theory and Applications*, Electr. Comput. Eng. Ser., World Sci. Eng. Soc. Press, Athens, 2000, 44–48.

[3] Bru R., Coll C. and Sanchez E.: Structural properties of positive linear time-invariant difference-algebraic equations, *Linear Algebra Appl.*, **349**, (2002), 1–10.

[4] Farina L. and Rinaldi S.: Positive Linear Systems; Theory and Applications, *J. Wiley*, New York, 2000.

[5] Kaczorek T.: Analysis of positivity and stability of discrete-time and continuous-time nonlinear systems, *Computational Problems of Electrical Engineering*, **5**(1), (2015).

[6] Kaczorek T.: Checking of the positivity of descriptor linear systems by the use of the shuffle algorithm, *Archives of Control Sciences*, vol. **21**(3), (2011), 287–298.

[7] Kaczorek T.: Descriptor positive discrete-time and continuous-time nonlinear systems, *Proc. of SPIE*, **9290**, doi:10.1117/12.2074558.

[8] Kaczorek T.: Descriptor standard and positive discrete-time nonlinear systems, *Automatyzacja Procesów Dyskretnych*, **1**, (2014), 113–120.

[9] Kaczorek T.: Minimum energy control of descriptor positive discrete-time linear systems, *COMPEL*, **33**(3), (2014), 976–988.

[10] KACZOREK T.: Minimum energy control of fractional descriptor positive discrete-time linear systems, *Int. J. Appl. Math. Comput. Sci.*, **24**(4), (2014), 735–743.

[11] KACZOREK T.: Minimum energy control of positive fractional descriptor continuous-time linear systems, *Control Theory & Applications IET*, **8**(4), (2014), 215–225.

[12] KACZOREK T.: Necessary and sufficient conditions for minimum energy control of positive discrete-time linear systems with bounded inputs, *Bull. Pol. Acad. Sci. Tech.*, **62**(1), (2014), 85–89.

[13] KACZOREK T.: Positive linear systems with different fractional orders, *Bull. Pol. Acad. Sci. Tech.*, **58**(3), (2010), 453–458.

[14] KACZOREK T.: Positive 1D and 2D Systems, Springer-Verlag, London, 2002.

[15] KACZOREK T.: Positive descriptor discrete-time linear systems, *Problems of Nonlinear Analysis in Engineering Systems*, **1**(7), (1998), 38–54.

[16] KACZOREK T.: Positive linear systems consisting of n subsystems with different fractional orders, *IEEE Trans. Circuits and Systems*, **58**(6), (2011), 1203–1210.

[17] KACZOREK T.: Positive singular discrete-time linear systems, *Bull. Pol. Acad. Sci. Tech.*, **45**(4), (1997), 619–631.

[18] KACZOREK T.: Positivity and linearization of a class of nonlinear discrete-time systems by state feedbacks, *Logistyka*, **6**, (2014), 5078–5083.

[19] KACZOREK T.: Positivity and stability of discrete-time nonlinear systems, *IEEE 2nd International Conference on Cybernetics*, 2015, 156–159.

[20] KACZOREK T. and BORAWSKI K.: Stability of continuous-time and discrete-time linear systems with inverse state matrices, *Measurement Automation Monitoring*, **62**(4), (2016), 132–135.

[21] KACZOREK T. and OSTALCZYK P.: Positivity and stability of fractional discrete-time systems. The model without a time shift in the difference, *Control and Cybernetics*, **46**(1), (2017).

[22] KACZOREK T. and ROGOWSKI K.: Fractional Linear Systems and Electrical Circuits, *Studies in Systems, Decision and Control*, **13**, Springer, 2015.

[23] VIRNIK E.: Stability analysis of positive descriptor systems, *Linear Algebra and its Applications*, **429**, 2640–2659.