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THE EXISTENCE OF BIPARTITE ALMOST SELF-COMPLEMENTARY 3-UNIFORM HYPERGRAPHS

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Abstract. An almost self-complementary 3-uniform hypergraph on n vertices exists if and only if n is congruent to 3 modulo 4. A hypergraph H with vertex set V and edge set E is called bipartite if V can be partitioned into two subsets V_1 and V_2 such that $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$ for any $e \in E$. A bipartite self-complementary 3-uniform hypergraph H with partition (V_1, V_2) of the vertex set V such that $|V_1| = m$ and $|V_2| = n$ exists if and only if either (i) $m = n$ or (ii) $m \neq n$ and either m or n is congruent to 0 modulo 4 or (iii) $m \neq n$ and both m and n are congruent to 1 or 2 modulo 4. In this paper we define a bipartite almost self-complementary 3-uniform hypergraph H with partition (V_1, V_2) of a vertex set V such that $|V_1| = m$ and $|V_2| = n$ and find the conditions on m and n for a bipartite 3-uniform hypergraph H to be almost self-complementary. We also prove the existence of bi-regular bipartite almost self-complementary 3-uniform hypergraphs.

Keywords: almost self-complementary 3-uniform hypergraph, bipartite hypergraph, bipartite self-complementary 3-uniform hypergraph, bipartite almost self-complementary 3-uniform hypergraph.

Mathematics Subject Classification: 05C65.

1. INTRODUCTION

Let V be a finite set with n vertices. By $\binom{V}{k}$ we denote the set of all k-subsets of V. A k-uniform hypergraph is a pair $H = (V; E)$, where $E \subset \binom{V}{k}$. V is called a vertex set, and E an edge set of H. Two k-uniform hypergraphs $H = (V; E)$ and $H' = (V'; E')$ are isomorphic if there is a bijection $\sigma : V \to V'$ such that σ induces a bijection of E onto E'. If $H = (V; E)$ is isomorphic to $H' = (V; \binom{V}{k} - E)$, then H is called a self-complementary k-uniform hypergraph. Every permutation $\pi : V \to V$ which induces a bijection $\pi' : E \to \binom{V}{k} - E$ is called a self-complementing permutation.

A. Symański, A.P. Wojda ([9, 10, 12]) and S. Gosselin [3], independently characterized n and k for which there exist k-uniform self-complementary hypergraphs of order n and gave the structure of corresponding complementing permutations.

A k-uniform hypergraph $H = (V;E)$ is called almost self-complementary if it is isomorphic with $H' = (V; \binom{V}{k} - E - \{e\})$, where e is an element of the set $\binom{V}{k}$. Almost self-complementary k-uniform hypergraph of order n may be called self-complementary in $K_n^k - e$. The almost self-complementary 2-uniform hypergraphs, i.e. almost self-complementary graphs are introduced by Clapham in [1]. In [5], almost self-complementary 3-uniform hypergraphs are considered. In [11], Wojda generalized corresponding results of [1] for $k = 2$ and of [5] for $k = 3$ for any $k \geq 2$. A.P. Wojda proved that an almost self-complementary k-uniform hypergraph of order n exists if and only if $\binom{n}{k}$ is odd.

T. Gangopadhyay and S.P. Rao Hebbare [2] studied bipartite self-complementary graphs. In [6] a bipartite self-complementary 3-uniform hypergraph H with partition (V_1, V_2) of a vertex set V such that $|V_1| = m$ and $|V_2| = n$ is defined and a necessary and sufficient conditions on m and n for its existence is proved.

In this paper, we extend the concept of almost self-complementary 3-uniform hypergraphs to bipartite self-complementary 3-uniform hypergraphs. In Section 3, we define a bipartite almost self-complementary 3-uniform hypergraph and prove a necessary and sufficient condition for its existence. Further in Section 4, we prove existence of bi-regular bipartite almost self-complementary 3-uniform hypergraphs.

2. PRELIMINARY DEFINITIONS AND RESULTS

Definition 2.1. A hypergraph H with vertex set V and edge set E is called *bipartite* if V can be partitioned into two subsets V_1 and V_2 such that $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$ for any $e \in E$.

Furthermore if $|e| = k$ for every $e \in E$ then we call H, a bipartite k-uniform hypergraph, and denote it as $H^k(V_1, V_2)$. If $|V_1| = m$ and $|V_2| = n$ then $H^k(V_1, V_2) = H^k_{(m,n)}$.

If $H^3(V_1, V_2)$ is a bipartite 3-uniform hypergraph then every edge of $H^3(V_1, V_2)$ contains one vertex from one part and two vertices from the other part of the partition V_1 and V_2 of V. Thus any triple of vertices $\{x, y, z\}$ such that x, y, z belong to a single part of the partition of V is not an edge of $H^3(V_1, V_2)$.

Definition 2.2. A 3-uniform hypergraph H with the vertex set $V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$ and the edge set $E = \{e : e \subset V, |e| = 3 \text{ and } e \cap V_i \neq \emptyset, \text{ for } i = 1, 2\}$ is called the *complete bipartite* 3-*uniform hypergraph.* It is denoted as $K^3(V_1, V_2)$ or $K^3_{(m,n)}$.

Clearly, the total number of edges in $K^3_{(m,n)}$ is $m\binom{n}{2} + n\binom{m}{2} = \frac{mn(m+n-2)}{2}$.

Definition 2.3. Given a bipartite 3-uniform hypergraph $H = H^3(V_1, V_2)$, we define its *bipartite complement* to be the 3-uniform hypergraph $\bar{H} = \bar{H}^3(V_1, V_2)$, where $V(\bar{H}) = V(H)$ and $E(\bar{H}) = E(K^3(V_1, V_2)) - E(H)$.

Definition 2.4. A bipartite 3-uniform hypergraph $H = H^3(V_1, V_2)$ is said to be *self-complementary* if it is isomorphic to its bipartite complement $\bar{H} = \bar{H}^3(V_1, V_2)$, that is there exists a bijection $\sigma : V \to V$ such that e is an edge in H if and only if $\sigma(e)$ is an edge in $\bar{H}$.

That is, there exists a bijection $\sigma : V \to V$ such that $e = \{x, y, z\}$ is an edge in H if and only if $\sigma(e) = \{\sigma(x), \sigma(y), \sigma(z)\}$ is an edge in $\bar{H}$. Such a σ is called a complementing permutation.

In [5], the cycle structure of complementing permutation of almost self-complementary 3-uniform hypergraphs is analyzed. In [6], the cycle structure of complementing permutation of bipartite self-complementary 3-uniform hypergraphs is analyzed.

Definition 2.5. The *degree* of a vertex v in a hypergraph H is the number of edges containing the vertex v and is denoted as $d_H(v)$.

Definition 2.6. A hypergraph H is said to be *regular* if all vertices have the same degree.

Definition 2.7. A hypergraph H is said to be *bi-regular* if there exist two distinct positive integers d_1 and d_2 such that the degree of each vertex is either d_1 or d_2.

Definition 2.8. A hypergraph H is said to be *quasi-regular* if the degree of each vertex is either r or $r-1$ for some positive integer r.

It is clear that every quasi-regular hypergraph is bi-regular but not conversely.

Following theorem gives necessary and sufficient condition on the order of bipartite 3-uniform hypergraph $H^3_{(m,n)}$ to be self-complementary which is proved in [6].

Theorem 2.9. *There exists a bipartite self-complementary* 3*-uniform hypergraph* $H^3_{(m,n)}$ *if and only if either*

(i) $m = n$, *or*
(ii) $m \neq n$ *and either* m *or* n *is congruent to* 0 *modulo* 4*, or*
(iii) $m \neq n$ *and both* m *and* n *are congruent to* 1 *or* 2 *modulo* 4.

3. EXISTENCE OF BIPARTITE ALMOST SELF-COMPLEMENTARY 3-UNIFORM HYPERGRAPHS

In this section, we define a bipartite almost self-complementary 3-uniform hypergraph and prove its existence.

Definition 3.1. The hypergraph $\tilde{K}^3_{(m,n)} = K^3_{(m,n)} - e$ is called an *almost complete bipartite* 3*-uniform hypergraph.*

The hypergraph $\tilde{K}^3_{(m,n)}$ is a bipartite 3-uniform hypergraph obtained by deleting any edge e from $K^3_{(m,n)}$. We always denote by e the edge deleted from $K^3_{(m,n)}$, call it as the missing edge and the corresponding vertices of e the special vertices.

Definition 3.2. A 3-uniform bipartite hypergraph $H(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is *almost self-complementary* if it is isomorphic with its complement $\bar{H}(V_1, V_2)$ with respect to $\tilde{K}^3_{(m,n)} = K^3_{(m,n)} - e$.

This means that a 3-uniform hypergraph $H(V_1, V_2)$ is almost self-complementary if $\tilde{K}^3_{(m,n)}$ can be decomposed into two isomorphic factors with $H(V_1, V_2)$ as one of the factors.

We use the shortform "bipasc" for bipartite almost self-complementary 3-uniform hypergraph.

Let $H(V_1, V_2)$ be a bipartite almost self-complementary 3-uniform hypergraph such that $|V_1| = m$ and $|V_2| = n$. Let the edges of $H(V_1, V_2)$ be coloured red and the remaining edges of $\tilde{K}^3_{(m,n)} = K^3_{(m,n)} - e$ be coloured green. Since the 2 factors are isomorphic, there is a permutation σ of the vertices of $\tilde{K}^3_{(m,n)}$ that induces a mapping of the red edges onto the green edges. We consider σ as a permutation of the vertices of $K^3_{(m,n)}$, and denote by σ' the corresponding mapping induced on the set of edges of $K^3_{(m,n)}$. Thus σ' maps each red edge onto a green edge. However, the mapping σ' need not necessarily map each green edge onto a red edge. This would be so if σ' mapped e onto itself, but it may happen that σ' maps e onto a red edge and some green edge onto e. Such a σ (which, for definiteness, we shall always assume that it induces a mapping from red to green) will (as for s.c. 3-uniform hypergraphs) be called a *complementing permutation*. Note that a cycle of σ' that does not include e must be of even length, consisting of edges alternately red and green. The cycle of σ' that includes e has odd length, consisting of e followed by red and green edges alternately. Further this length equals 1 when σ' maps e onto itself.

Since $\tilde{K}^3_{(m,n)}$ has $\frac{mn(m+n-2)}{2} - 1$ edges, its decomposition into two isomorphic factors is possible only if this number is divisible by 2. This means that $K^3_{(m,n)}$ should have an odd number of edges. We know that $K^3_{(m,n)}$ has an even number of edges if $m = n$ and for $m \neq n$, either (i) either m or n is congruent to 0 modulo 4 or (ii) both m and n are congruent to 1 or 2 or 3 modulo 4. That is, $K^3_{(m,n)}$ has an odd number of edges if $m \neq n$ and, either (i) one is congruent to 1 modulo 4 and the other is congruent to 2 or 3 modulo 4 or (ii) one is congruent to 2 modulo 4 and the other is congruent to 1 or 3 modulo 4.

In the following theorem we prove that the above conditions on m and n for existence of bipasc are not only necessary but are also sufficient.

Theorem 3.3. *There exists a bipasc 3-uniform hypergraph $H^3_{(m,n)}$ with partition (V_1, V_2) of vertex set V, where $|V_1| = m$ and $|V_2| = n$ if and only if $m \neq n$ and either*

(i) *one is congruent to* 1 *modulo* 4 *and the other is congruent to* 2 *or* 3 *modulo* 4*, or*
(ii) *one is congruent to* 2 *modulo* 4 *and the other is congruent to* 1 *or* 3 *modulo* 4*.*

Proof. Necessity follows from the above discussions. To prove the sufficiency, we need to construct a bipasc $H^3_{(m,n)}$ with partition (V_1, V_2) of vertex set V, where $|V_1| = m$ and $|V_2| = n$, for all possible values of m and n.

Case (1). Suppose m is congruent to 1 modulo 4 and n is congruent to 2 modulo 4. Let $m = 4k + 1$ and $n = 4l + 2$ for some positive integers k and l.

Let

$$V_1 = \{u_1, u_2, \ldots, u_{4k}, x\} \quad \text{and} \quad V_2 = \{v_1, v_2, \ldots, v_{4l}, y_1, y_2\}.$$

Let G_1 be a self-complementary graph on $m = 4k+1$ vertices with vertex set V_1 and G_2 be a self-complementary graph on $4l$ vertices with vertex set $V_{G_2} = \{v_1, v_2, \dots, v_{4l}\}$. Let σ_1 and σ_2 be complementing permutations of G_1 and G_2, respectively.

We let $e = \{x, y_1, y_2\}$ to be the missing edge.

Consider following subsets of the edge set of $\tilde{K}^3_{(m,n)}$.

$$\begin{aligned}
E_1 &= \{e_1 \cup \{v\} \mid v \in V_2 \text{ and } e_1 \text{ is an edge in } G_1\},\\
\bar{E}_1 &= \{e_1' \cup \{v\} \mid v \in V_2 \text{ and } e_1' \text{ is an edge in of the complement of } G_1\},\\
E_2 &= \{e_2 \cup \{u\} \mid u \in V_1 \text{ and } e_2 \text{ is an edge in } G_2\},\\
\bar{E}_2 &= \{e_2' \cup \{u\} \mid u \in V_1 \text{ and } e_2' \text{ is an edge in of the complement of } G_2\},\\
E_{y_1} &= \{\{y_1, u, v\} \mid u \in V_1, v \in V_{G_2}\},\\
\bar{E}_{y_1} &= \emptyset,\\
E_{y_2} &= \emptyset,\\
\bar{E}_{y_2} &= \{\{y_2, u, v\} \mid u \in V_1, v \in V_{G_2}\}.
\end{aligned}$$

Since σ_1 is a complementing permutation of the self-complementary graph G_1, $\sigma_1 = C_1C_2 \ \cdots \ C_p \ (x)$, where every cycle C_i is of length being a multiple of 4 for each $i = 1, 2, \dots, p$. Then each cycle $C_i,\ i = 1, 2, \dots, p$ is of even length greater than 2.

Let

$$C_i = (u_{i_1} u_{i_2} \dots u_{i_{2s_i}}) \quad \text{for } i = 1, 2, \dots, p,$$

and

$$\begin{aligned}
E^i_{(y_1,y_2)} &= \{\{y_1, y_2, u_{i_k}\} \mid k = 1, 3, \dots, 2s_i - 1\},\\
\bar{E}^i_{(y_1 y_2)} &= \{\{y_1, y_2, u_{i_k}\} \mid k = 2, 4, \dots, 2s_i\}.
\end{aligned}$$

All the above subsets form a partition of $\tilde{K}^3_{(m,n)}$.

Let H be a 3-uniform hypergraph whose edge set is

$$E = E_1 \cup E_2 \cup E_{y_1} \cup E_{y_2} \cup \left(\bigcup_{i=1}^{p} E^i_{(y_1,y_2)} \right).$$

H is bipartite 3-uniform hypergraph with partition (V_1, V_2) of vertex set

$$V = \{u_1, u_2, \dots, u_{4k}, x, v_1, v_2, \dots, v_{4l}, y_1, y_2\}.$$

Define a bijection $\sigma : V(H) \to V(H)$ as $\sigma = \sigma_1\sigma_2(y_1\ y_2)$.

It can be easily checked that H is almost self-complementary with σ as its complementing permutation.

Case (2). Suppose m is congruent to 1 modulo 4 and n is congruent to 3 modulo 4. Let $m = 4k+1$ and $n = 4l+3$ for some positive integers k and l. Let

$$V_1 = \{u_1, u_2, \dots, u_{4k}, x\} \quad \text{and} \quad V_2 = \{v_1, v_2, \dots, v_{4l}, y, y_1, y_2\}.$$

Let G_1 be a self-complementary graph on $m = 4k + 1$ vertices with vertex set V_1 and G_2 be a self-complementary graph on $4l + 1$ vertices with vertex set $V_{G_2} = \{v_1, v_2, \ldots, v_{4l}, y\}$. Let σ_1 and σ_2 be complementing permutations of G_1 and G_2, respectively. Let $e = \{x, y_1, y_2\}$ be the missing edge.

We construct a bipasc exactly the same way as in Case (1).

Case (3). Suppose m is congruent to 3 modulo 4 and n is congruent to 2 modulo 4. Let $m = 4k + 3$ and $n = 4l + 2$ for some positive integers k and l.

Let

$$V_1 = \{u_1, u_2, \ldots, u_{4k}, x, x_1, x_2\} \quad \text{and} \quad V_2 = \{v_1, v_2, \ldots, v_{4l}, y_1, y_2\}.$$

Let $e = \{x, y_1, y_2\}$ be the missing edge.

Construct H_1 as in Case (1) on

$$V_1' = \{u_1, u_2, \ldots, u_{4k}, x\} \cup \{v_1, v_2, \ldots, v_{4l}, y_1, y_2\}$$

vertices with complementing permutation $\sigma_1\sigma_2(y_1\ y_2)$ such that

$$\sigma_1 = (x)\prod_{i=1}^{p} C_i \quad \text{and} \quad \sigma_2 = \prod_{j=1}^{q} C_j',$$

where each C_i and C_j' is a cycle of length a multiple of 4.

Let

$$C_i = (u_{i_1} u_{i_2} \ldots u_{i_{2s_i}}) \quad \text{for } i = 1, 2, \ldots, p$$

and let

$$C_i' = (v_{i_1} v_{i_2} \ldots v_{i_{2t_i}}) \quad \text{for } i = 1, 2, \ldots, q.$$

$$\begin{aligned}
E_{x_1} &= \{\{x_1\} \cup e \mid e \text{ is an edge in } G_2\} \cup \{\{x_1, u, v\} \mid u \in V_1', v \in V_2\},\\
\bar{E}_{x_1} &= \{\{x_1\} \cup e \mid e \text{ is not an edge in } G_2\},\\
E_{x_2} &= \{\{x_2\} \cup e \mid e \text{ is an edge in } G_2\},\\
\bar{E}_{x_2} &= \{\{x_2\} \cup e \mid e \text{ is not an edge in } G_2\} \cup \{\{x_2, u, v\} \mid u \in V_1',\ v \in V_2\},\\
E^j_{(x_1,x_2)} &= \{\{x_1, x_2, v_{j_k}\} \mid k = 1, 3, \ldots, 2t_j - 1\},\\
\bar{E}^j_{(x_1,x_2)} &= \{\{x_1, x_2, v_{j_k}\} \mid k = 2, 4, \ldots, 2t_j\}.
\end{aligned}$$

Consider H on $V_1 \cup V_2$ with edges containing the edge set of H_1 and $\{x_1, x_2, y_1\} \cup E_{x_1} \cup E_{x_2} \cup (\bigcup_j (E^j_{(x_1,x_2)})$. It can be easily checked that H is bipasc with $\sigma = \sigma_1\sigma_2(x_1\ x_2)(y_1\ y_2)$. □

Bipartite almost self-complementary 3-uniform hypergraphs in a sense fill the gap where bipartite self-complementary 3-uniform hypergraphs do not exist.

4. EXISTENCE OF BI-REGULAR BIPARTITE ALMOST SELF-COMPLEMENTARY 3-UNIFORM HYPERGRAPHS

It is known that a regular self-complementary 3-uniform hypergraph exists if and only if n is congruent to 1 or 2 modulo 4 [8]. In [5], it is proved that there does not exist a regular almost self-complementary 3-uniform hypergraph on n vertices, where n is congruent to 3 modulo 4. In [7], it is proved that, there exists a regular bipartite self-complementary 3-uniform hypergraph $H(V_1, V_2)$ with $|V_1| = m, |V_2| = n, m+n > 3$ if and only if $m = n$ and n is congruent to 0 or 1 modulo 4.

We have every edge of $\tilde{K}^3(V_1, V_2) = \tilde{K}^3_{(m,n)}$ contains one vertex from one part and two vertices from the other. Let $u \in V_1, v \in V_2$. Suppose u and v are not special vertices. If $e_1 = \{u, x, y\}$ is an edge containing u then either (i) $x \in V_1$ and $y \in V_2$, or (ii) $x, y \in V_2$. Considering both the possibilities we get that there are $n(m-1) + \binom{n}{2}$ number of edges containing u. Hence the degree of u in $\tilde{K}^3(V_1, V_2) = \tilde{K}^3_{(m,n)}$ is $n(m-1) + \binom{n}{2}$. Similarly, the degree of v in $\tilde{K}^3(V_1, V_2) = \tilde{K}^3_{(m,n)}$ is $m(n-1) + \binom{m}{2}$.

If u and v are special vertices, that is u and v are in the deleted edge e then the degree of u in $\tilde{K}^3(V_1, V_2) = \tilde{K}^3_{(m,n)}$ is $n(m-1) + \binom{n}{2} - 1$ and the degree of v in $\tilde{K}^3(V_1, V_2) = \tilde{K}^3_{(m,n)}$ is $m(n-1) + \binom{m}{2} - 1$.
This shows that there does not exist a regular bipartite almost self-complementary 3-uniform hypergraph.

In [4], it is proved that a quasi-regular self-complementary 3-uniform hypergraph of order n exists if and only if $n \geq 4$ and n is congruent to 0 modulo 4. In [5] it is proved that there exist a quasi regular almost self-complementary 3-uniform hypergraph on n vertices, where n is congruent to 3 modulo 4.

In [7], authors proved that a bipsc $H(V_1, V_2)$ with $|V_1| = m, |V_2| = n$ is quasi-regular if and only if either $m = 3, n = 4$ or $m = n$ and n is congruent to 2 or 3 modulo 4. The following theorem gives the conditions for existence of a bi-regular bipasc $H(V_1, V_2)$ with $|V_1| = m$ and $|V_2| = n$.

Theorem 4.1. *There exists a bi-regular bipartite almost self-complementary 3-uniform hypergraph $H(V_1, V_2)$ with $|V_1| = m, |V_2| = n, m \neq n, m + n > 3$ if and only if either one of m and n is 3 and the other is 1 or 2.*

Proof. Suppose there exists a bi-regular bipasc $H(V_1, V_2)$ with $|V_1| = m$, $|V_2| = n$, $m \neq n$. Let d_1 and d_2 be the degrees of vertices of H. Let σ be a complementing permutation of $H(V_1, V_2)$.

For any vertex t in $H(V_1, V_2)$ we have $d_H(t) + d_H(\sigma(t))$ = degree of t in $\tilde{K}^3(V_1, V_2) = \tilde{K}^3_{(m,n)}$. Using this, we get following equations.

If $u \in V_1$ is not a special vertex, then

$$d_H(u) + d_H(\sigma(u)) = n(m-1) + \binom{n}{2}. \tag{4.1}$$

If $x \in V_1$ is a special vertex, then

$$d_H(x) + d_H(\sigma(x)) = n(m-1) + \binom{n}{2} - 1. \tag{4.2}$$

Similarly, if $v \in V_2$ is not a special vertex, then

$$d_H(v) + d_H(\sigma(v)) = m(n-1) + \binom{m}{2}. \tag{4.3}$$

If $y \in V_2$ is a special vertex, then

$$d_H(y) + d_H(\sigma(y)) = m(n-1) + \binom{m}{2} - 1. \tag{4.4}$$

From equations (4.1) and (4.2) we get that either

$$d_1 + d_2 = n(m-1) + \binom{n}{2} \tag{4.5}$$

and

$$2d_1 = n(m-1) + \binom{n}{2} - 1 \tag{4.6}$$

or

$$2d_1 = n(m-1) + \binom{n}{2} \tag{4.7}$$

and

$$d_1 + d_2 = n(m-1) + \binom{n}{2} - 1. \tag{4.8}$$

Similarly, from equations (4.3) and (4.4) we get that either

$$d_1 + d_2 = m(n-1) + \binom{m}{2} \tag{4.9}$$

and

$$2d_1 = m(n-1) + \binom{m}{2} - 1 \tag{4.10}$$

or

$$2d_1 = m(n-1) + \binom{m}{2} \tag{4.11}$$

and

$$d_1 + d_2 = m(n-1) + \binom{m}{2} - 1. \tag{4.12}$$

(Note that in equations (4.6), (4.7), (4.10) and (4.11) degree is either d_1 or d_2.)

Equating $d_1 + d_2$ from equations (4.5), (4.6), (4.9) and (4.10), we get that

$$m(n-1) + \binom{m}{2} = n(m-1) + \binom{n}{2}.$$

Solving this equation we get that $(m-n)(m+n-3)=0$. That is, $m=n$ or $m+n-3=0$, a contradiction.

Equating d_1+d_2 from equations (4.5), (4.6), (4.11) and (4.12) we get that

$$n(m-1)+\binom{n}{2}=m(n-1)+\binom{m}{2}-1.$$

Solving this equation we get that $m^2-n^2-3m+3n=2$. That is, $(m-n)(m+n-3)=2$. Suppose $m>n$. Then we get that either $m-n=2$, $m+n-3=1$ or $m-n=1$, $m+n-3=2$. If $m-n=2$, $m+n-3=1$, then we get that $m=3$, $n=1$. And if $m-n=1$, $m+n-3=2$, then we get that $m=3, n=2$.

Equating d_1+d_2 from equations (4.7), (4.8), (4.9) and (4.10) we get that

$$n(m-1)+\binom{n}{2}-1=m(n-1)+\binom{m}{2}.$$

Solving this equation we get that $m^2-n^2-3m+3n=-2$. That is, $(m-n)(m+n-3)=-2$. Suppose $m>n$. Then we get that either $m-n=1$, $m+n-3=-2$ or $m-n=2$, $m+n-3=-1$. If $m-n=1$, $m+n-3=-2$ then we get that $m=1$, $n=0$, a contradiction. And if $m-n=2$, $m+n-3=-1$ then we get that $m=2$, $n=0$, a contradiction.

Equating d_1+d_2 from equations (4.7), (4.8), (4.11) and (4.12) we get that

$$m(n-1)+\binom{m}{2}=n(m-1)+\binom{n}{2}.$$

Solving this equation we get that $(m-n)(m+n-3)=0$. That is, $m=n$ or $m+n-3=0$, a contradiction.

Hence, if there exists a bi-regular bipartite almost self-complementary 3-uniform hypergraph $H(V_1,V_2)$ with $|V_1|=m, |V_2|=n, m\neq n, m+n>3$ then either one of m and n is 3 and the other is 1 or 2.

Conversely, the following Example 4.2 gives a bi-regular bipartite almost self-complementary 3-uniform hypergraph for $m=1$, $n=3$ and $m=2$, $n=3$. □

Example 4.2. (i) Consider $K^3_{(1,3)}$. Let $V_1=\{u_1\}$ and $V_2=\{v_1,v_2,v_3\}$ be a partition of the vertex set $V=\{u_1,v_1,v_2,v_3\}$. The edge set of $K^3_{(1,3)}$ is

$$E=\{\{u_1,v_1,v_2\},\{u_1,v_1,v_3\},\{u_1,v_2,v_3\}\}.$$

We delete the edge $e=\{u_1,v_2,v_3\}$ from $K^3_{(1,3)}$. Consider H with edge set $E_1=\{\{u_1,v_1,v_2\}\}$. Then $\bar{H}$ has edge set $E_2=\{\{u_1,v_1,v_3\}\}$. Clearly, H is isomorphic to $\bar{H}$ with complementing permutation $\sigma=(u_1)(v_1)(v_2\ v_3)$ or $\sigma=(u_1)(v_1\ v_3\ v_2)$ with the missing edge $\{u_1,v_2,v_3\}$.

(ii) Consider $K^3_{2,3}$. Let $V_1=\{u_1,u_2\}$ and $V_2=\{v_1,v_2,v_3\}$ be a partition of the vertex set $V=\{u_1,u_2,v_1,v_2,v_3\}$. The edge set of $K^3_{(2,3)}$ is

$$\begin{aligned}E=\{&\{u_1,v_1,v_2\},\{u_1,v_1,v_3\},\{u_1,v_2,v_3\},\{u_2,v_1,v_2\},\{u_2,v_1,v_3\},\\&\{u_2,v_2,v_3\},\{u_1,u_2,v_1\},\{u_1,u_2,v_2\},\{u_1,u_2,v_3\}\}.\end{aligned}$$

We delete $\{u_1, u_2, v_3\}$ from $K^3_{(2,3)}$. Consider H with edge set

$$E_1 = \{\{u_1, u_2, v_1\}, \{v_1, v_2, u_2\}, \{u_1, v_1, v_3\}, \{u_1, v_2, v_3\}\}.$$

Then $\bar{H}$ has the edge set

$$E_2 = \{\{u_1, u_2, v_2\}, \{u_1, v_1, v_2\}, \{u_2, v_2, v_3\}, \{u_2, v_1, v_3\}\}.$$

Clearly, H is isomorphic to $\bar{H}$ with complementing permutation $\sigma_1 = (u_1\ u_2)(v_1\ v_2)(v_3)$ with the missing edge $\{u_1, u_2, v_3\}$.

Remark 4.3. The bipasc given in Example 4.2 are in fact quasi-regular. Thus for bipasc 3-uniform hypergraphs we have that bi-regular if and only if quasi-regular.

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