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A GENERALIZATION OF MATTHEWS PARTIAL METRIC SPACE AND FIXED POINT THEOREMS

Abstract. In the present paper, we introduce the notion of a generalized partial metric space which is an extension of the partial metric space due to S. G. Matthews (*Partial metric topology, Papers on general topology and applications, Ann. New York Acad. Sci.,728* (*1994*), *183-197*). We investigate some basic properties of the generalized partial metric spaces and establish some new fixed point theorems for linear and non-linear contraction on such spaces.

Key words: fixed point, contraction mapping, metric space, partial metric space.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The notion of a partial metric space was introduced by S.G. Matthews [6] in 1994. The partial metric space is a generalization of the usual metric space in which the self distance is no longer necessarily zero. Matthews hit the idea of partial metric spaces while studying some problems related to computer applications. So the classical theory of a fixed point can be applied to solve some challenging problems occurring in computer sciences. This notion is so fascinating that till now it has been one of the active research areas to work on. Moreover, partial metric spaces have a wide area of application in the field of fixed point theory and have been used extensively for several generalizations of the Banach contraction principle (see [1, 2, 3, 4, 5, 8, 9, 10, 11] and references therein).

Our main motive is to further extend the partial metric spaces to their more generalized form. In this paper, we introduce the notion of generalized partial metric and generalized partial metric space (see Definition 2) which significantly extend the suitable notions of Matthews [6]. Motivated by the suitable results of ([6], [8] and [10]), we prove some new fixed point theorems for linear and nonlinear generalized contractions. Moreover, we also investigate some basic properties of generalized partial metric space and give some examples to illustrate our main results.

2. A generalization of Matthews partial metric space

We begin with definition of partial metric space due to Matthews [6] .

Definition 1. (*S.G. Matthews 1994*). *A function* $p : X \times X \to [0, \infty)$ *is called a partial metric if, for all* $x, y, z \in X$,
(*a*). $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
(*b*). $p(x, x) \leq p(x, y)$,
(*c*). $p(x, y) = p(y, x)$,
(*d*). $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.
The pair (X, p) *is called a Matthews partial metric space. This notion has turned out to be useful in the fixed point theory* (*see for instance [1, 2, 3, 4, 5, 7, 9, 10, 11]*).
Now we propose the following definition:

Definition 2. *Let* X *be a non-empty set and* $P : X \times X \times X \to \mathbb{R}$ *be a mapping which satisfy the following conditions for all* $x, y, z, a \in X$,
(*i*). $x = y \iff P(x, x, y) = 0$ *and* $P(y, y, x) = 0$,
(*ii*). $P(x, x, y) \geq 0$,
(*iii*). $P(x, y, z) = P(z, y, x)$,
(*iv*). $P(x, y, z) \leq P(x, y, a) + P(a, a, z)$.
In the sequel P *and the ordered pair* (X, P) *are referred to, respectively, as a generalized partial metric and a generalized partial metric space.*

If we define $P : X \times X \times X \to \mathbb{R}$ by

$$P(x, y, z) := p(x, z) - p(y, y), \tag{1}$$

then P satisfies all the conditions of Definition 1. It is easy to verify that conditions $(i) - (iii)$ of Definition 2 imply the conditions $(a) - (c)$ of Definition 1, respectively. If we take $a = y$ and $P(x, y, z) := p(x, z) - p(y, y)$ in condition (iv) of Definition 2 then we get condition (d) of Definition 1. Thus generalized partial metric and generalized partial metric space contain the respective Matthews notions as special cases.

Let us note the following obvious

Remark 1. Condition (iii) is equivalent to the following

$$P(x, y, z) \leq P(z, y, x), \text{ for all } x, y, z \in X.$$

An immediate example of generalized partial metric space is given as below:

Example 1. Let $X = \mathbb{R}$. We consider the function $P : X \times X \times X \to \mathbb{R}$ given as follows

$$P(x, y, z) = |x + z - 2y| + |x - z|$$

for all $x, y, z \in X$. One can easily verify the properties $(i) - (iii)$ of generalized partial metric space (X, P). For property (iv), we have

$$\begin{aligned} P(x, y, z) &= |x + z - 2y| + |x - z| \\ &= |(x + a - 2y) + (z - a)| + |x - a + a - z| \\ &\leq |x + a - 2y| + |x - a| + |a + z - 2a| + |a - z| \\ &= P(x, y, a) + P(a, a, z). \end{aligned}$$

Hence the ordered pair (X, P) forms a generalized partial metric space.

Some other examples of a generalized partial metric space are given as follows:

(1). Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X. Then $P(x, y, z) = \|x + z - 2y\| + \|x - z\|$ is a generalized partial metric on X.
(2). Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X. Then $P(x, y, z) = \|x - y\| + \|y - z\|$ is a generalized partial metric on X.
(3). Let X be a non-empty set, and let d be an ordinary metric on X. Then $P(x, y, z) = d(x, y) + d(y, z)$ is a generalized partial metric on X.
(4). Let $X = [0, \infty)$. Then $P(x, y, z) = y - \min\{x, y, z\}$ is a generalized partial metric space on X.
(5). Let $X = [0, \infty)$. Then $P(x, y, z) = \max\{x, y, z\} - \min\{x, y, z\}$ is a generalized partial metric space on X.

Definition 3. *Let (X, P) be a generalized partial metric space. Then for $x \in X, r > 0$, we define the ball $B_P(x, r)$ and the closed ball $B_P[x, r]$ with center x and radius r as follows respectively:*

$$B_P(x, r) = \{y \in X : P(x, x, y) < r \textit{ and } P(y, y, x) < r\}$$

and

$$B_P[x, r] = \{y \in X : P(x, x, y) \leq r \textit{ and } P(y, y, x) \leq r\}.$$

Example 2. Let $X = \mathbb{R}$ and $P(x,y,z) = |x-y|+|y-z|$ be a generalized partial metric space on $\mathbb{R}$. Then the open ball of radius $1/2$, centered at $x = 1$ is the open interval $(1/2, 3/2)$, as we have

$$\begin{aligned} B_P(1,1/2) &= \{y \in \mathbb{R} : P(y,y,1) < 1/2 \text{ and } P(1,1,y) < 1/2\} \\ &= \{y \in \mathbb{R} : |y-y| + |y-1| < 1/2 \text{ and } |1-1| + |1-y| < 1/2\} \\ &= \{y \in \mathbb{R} : |1-y| < 1/2\} \\ &= \{y \in \mathbb{R} : |y-1| < 1/2\} \\ &= \{y \in \mathbb{R} : 1 - 1/2 < y < 1 + 1/2\} \\ &= (1/2, 3/2). \end{aligned}$$

Lemma 1. *If (X,P) is the generalized partial metric space, then $P(x,x,z) = P(z,x,x)$, for all $x, z \in X$.*

Proof. This lemma follows directly from property (iii) of generalized partial metric by taking $x = y$. ■

Definition 4. *Let (X,P) be a generalized partial metric space. A sequence (x_n) in X is called a Cauchy sequence if $P(x_n, x_n, x_m) \to 0$ and $P(x_m, x_m, x_n) \to 0$ as $n, m \to \infty$, that is for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $P(x_n, x_n, x_m) < \epsilon$ and $P(x_m, x_m, x_n) < \epsilon$, for each $n, m \geq n_0$.*

Definition 5. *Let (X,P) be a generalized partial metric space. A sequence (x_n) in X converges to x if $P(x_n, x_n, x) \to 0$ and $P(x, x, x_n) \to 0$ as $n \to \infty$, that is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $P(x_n, x_n, x) < \epsilon$ and $P(x, x, x_n) < \epsilon$, for each $n, m \geq n_0$.*

Definition 6. *A generalized partial metric space (X,P) is said to be complete if every Cauchy sequence is convergent.*

Lemma 2. *Let (X,P) be a generalized partial metric space. If the sequence (x_n) in X converges to x, then x is unique.*

Proof. Let (x_n) converge to x and y. Then for each $\epsilon > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \Longrightarrow P(x_n, x_n, x) < \frac{\epsilon}{2} \quad \wedge \quad P(x, x, x_n) < \frac{\epsilon}{2}$$

and

$$n \geq n_2 \Longrightarrow P(x_n, x_n, y) < \frac{\epsilon}{2} \quad \wedge \quad P(y, y, x_n) < \frac{\epsilon}{2}.$$

If $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ and from property (iv) of a generalized partial metric we get

$$P(x, x, y) \leq P(x, x, x_n) + P(x_n, x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $P(x,x,y)=0$ and similarly $P(y,y,x)=0$ so $x=y$. ■

Lemma 3. *In a generalized partial metric space every convergent sequence is a Cauchy sequence.*

Proof. Assume that (X,P) is a generalized partial metric space and (x_n) is a sequence from X which converges to x. Then for every $\epsilon>0$ there exist $k_1,k_2\in\mathbb{N}$ such that

$$n\geq k_1 \implies P(x_n,x_n,x)<\frac{\epsilon}{2} \quad\wedge\quad P(x,x,x_n)<\frac{\epsilon}{2}$$

and

$$m\geq k_2 \implies P(x_m,x_m,x)<\frac{\epsilon}{2} \quad\wedge\quad P(x,x,x_m)<\frac{\epsilon}{2}.$$

Setting $k_0=\max\{k_1,k_2\}$, by property (iv) of generalized partial metric, we get

$$P(x_n,x_n,x_m)\leq P(x_n,x_n,x)+P(x,x,x_m)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Similarly

$$P(x_m,x_m,x_n)\leq P(x_m,x_m,x)+P(x,x,x_n)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

■

Lemma 4. *Let (X,P) be a generalized partial metric space. If there exist sequences (x_n) and (y_n) such that $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$, then $\lim_{n\to\infty}P(x_n,x_n,y_n)=P(x,x,y)$ and $\lim_{n\to\infty}P(y_n,y_n,x_n)=P(y,y,x)$.*

Proof. Since (x_n) converges to x and (y_n) converges to y, then for each $\epsilon>0$ there exist $k_1,k_2\in\mathbb{N}$ such that

$$n\geq k_1 \implies P(x_n,x_n,x)<\frac{\epsilon}{2} \quad\wedge\quad P(x,x,x_n)<\frac{\epsilon}{2}$$

and

$$n\geq k_2 \implies P(y_n,y_n,y)<\frac{\epsilon}{2} \quad\wedge\quad P(y,y,y_n)<\frac{\epsilon}{2}.$$

Let $k_0=\max\{k_1,k_2\}$. Then, by property (iv) of generalized partial metric,

$$\begin{aligned}P(x_n,x_n,y_n)&\leq P(x_n,x_n,x)+P(x,x,y_n)\\&\leq P(x_n,x_n,x)+P(x,x,y)+P(y,y,y_n)\\&<\frac{\epsilon}{2}+\frac{\epsilon}{2}+P(x,x,y)=\epsilon+P(x,x,y).\end{aligned}$$

Hence we obtain

$$P(x_n, x_n, y_n) - P(x, x, y) < \epsilon. \tag{2}$$

On the other hand, we have

$$\begin{aligned} P(x, x, y) &\leq P(x, x, x_n) + P(x_n, x_n, y) \\ &\leq P(x, x, x_n) + P(x_n, x_n, y_n) + P(y_n, y_n, y) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} + P(x_n, x_n, y_n), \end{aligned}$$

that is

$$P(x, x, y) - P(x_n, x_n, y_n) < \epsilon. \tag{3}$$

From (2) and (3), we have $|P(x_n, x_n, y_n) - P(x, x, y)| < \epsilon$, that is

$$\lim_{n\to\infty} P(x_n, x_n, y_n) = P(x, x, y).$$

Similarly, we can prove that $\lim_{n\to\infty} P(y_n, y_n, x_n) = P(y, y, x)$, which completes our proof. ■

3. A fixed point result for linear generalized contractions on generalized partial metric space

Now we begin with a fixed point theorem for linear generalized contractions on generalized partial metric space.

Theorem 1. *Let (X, P) be a complete generalized partial metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$P(Tx, Tx, Ty) \leq kP(x, x, y) \tag{4}$$

for all $x, y \in X$ and some $k \in [0, 1)$. Then T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. If $x_0 = Tx_0$ then the proof completes. So we assume $x_0 \neq Tx_0$ and $x_1 = Tx_0$. Continuing this process we construct a sequence (x_n) such that

$$x_{n+1} = Tx_n \quad \text{and} \quad x_{n+1} \neq x_n, \text{ for all } n \in \mathbb{N}_0.$$

Now taking $x = y = x_n$, and $z = x_{n+1}$ in (4), we get

$$\begin{aligned} P(x_n, x_n, x_{n+1}) &= P(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq kP(x_{n-1}, x_{n-1}, x_n) \\ &= kP(Tx_{n-2}, Tx_{n-2}, Tx_{n-1}) \\ &\leq k^2 P(x_{n-2}, x_{n-2}, x_{n-1}) \\ &\quad \vdots \\ &\leq k^n P(x_0, x_0, x_1). \end{aligned} \tag{5}$$

Similarly if we take $x = y = x_{n+1}$, and $z = x_n$ in (4), we get

$$\begin{aligned} P(x_{n+1}, x_{n+1}, x_n) &= P(Tx_n, Tx_n, Tx_{n-1}) \\ &\leq kP(x_n, x_n, x_{n-1}) \\ &= kP(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) \\ &\leq k^2 P(x_{n-1}, x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq k^n P(x_1, x_1, x_0). \end{aligned} \tag{6}$$

Let $n, m \in \mathbb{N}$ and $m > n$. Then in view of inequality (5) and from property (iv) of generalized partial metric, we have

$$\begin{aligned} P(x_n, x_n, x_m) &\leq P(x_n, x_n, x_{n+1}) + P(x_{n+1}, x_{n+1}, x_m) \\ &\leq P(x_n, x_n, x_{n+1}) + P(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\quad + \cdots + P(x_{m-1}, x_{m-1}, x_m) \\ &\leq k^n P(x_0, x_0, x_1) + k^{n+1} P(x_0, x_0, x_1) \\ &\quad + \cdots + k^{m-1} P(x_0, x_0, x_1) \\ &= k^n (1 + k + k^2 + \cdots + k^{m-n-1}) P(x_0, x_0, x_1) \\ &= k^n \left[\frac{1 - k^{m-n}}{1 - k} \right] P(x_0, x_0, x_1) \\ &< \frac{k^n}{1 - k} P(x_0, x_0, x_1). \end{aligned} \tag{7}$$

Making $n, m \to \infty$, we get $P(x_n, x_n, x_m) \to 0$. Similarly, in view of (6) and from property (iv) of generalized partial metric, we have

$$\begin{aligned} P(x_m, x_m, x_n) &\leq P(x_m, x_m, x_{m-1}) + P(x_{m-1}, x_{m-1}, x_n) \\ &\leq P(x_m, x_m, x_{m-1}) + P(x_{m-1}, x_{m-1}, x_{m-2}) \\ &\quad + \cdots + P(x_{n+1}, x_{n+1}, x_n) \\ &\leq k^{m-1} P(x_1, x_1, x_0) + k^{m-2} P(x_1, x_1, x_0) \\ &\quad + \cdots + k^n P(x_1, x_1, x_0) \\ &= k^n (1 + k + k^2 + \cdots + k^{m-n-1}) P(x_0, x_0, x_1) \\ &= k^n \left[\frac{1 - k^{m-n}}{1 - k} \right] P(x_1, x_1, x_0) \\ &< \frac{k^n}{1 - k} P(x_1, x_1, x_0). \end{aligned} \tag{8}$$

Again making $n, m \to \infty$, we get $P(x_m, x_m, x_n) \to 0$. Since $P(x_n, x_n, x_m) \to 0$ and $P(x_m, x_m, x_n) \to 0$ as $m, n \to 0$, therefore the sequence (x_n) is a

Cauchy in X. Since X is a complete generalized partial metric space so there exists a point x in X such that $P(x_n, x_n, x) \to 0$ and $P(x, x, x_n) \to 0$ as $n \to \infty$. Now we will prove that $Tx = x$. In view of property (iv) of generalized partial metric and using (4), we have

$$\begin{aligned} P(Tx, Tx, x) &\leq P(Tx, Tx, Tx_n) + P(Tx_n, Tx_n, x) \\ &\leq kP(x, x, x_n) + P(x_{n+1}, x_{n+1}, x). \end{aligned}$$

Making $n \to \infty$, we get $P(Tx, Tx, x) = 0$, hence by property (i) of generalized partial metric $Tx = x$.

Now we will show the uniqueness of the fixed point of T. Suppose there exist $x, y \in X$ with $x = Tx$ and $y = Ty$. Then

$$P(x, x, y) = P(Tx, Tx, Ty) \leq kP(x, x, y)$$

and therefore $P(x, x, y) = 0$ implies $x = y$. ∎

Example 3. Let $X = \mathbb{R}$. Then $P(x, y, z) = |x - y| + |y - z|$ is a complete generalized partial metric space. Define a self-map T on X by $T(x) = \dfrac{\sin x + x}{3}$, for all $x \in X$. We have

$$\begin{aligned} P(Tx, Tx, Ty) &= |Tx - Tx| + |Ty - Ty| \\ &= \frac{1}{3}\left|\big(\sin x + x\big) - \big(\sin y + y\big)\right| \\ &\leq \frac{1}{3}|\sin x - \sin y| + \frac{1}{3}|x - y| \\ &\leq \frac{1}{3}|x - y| + \frac{1}{3}|x - y| = \frac{2}{3}|x - y| = \frac{2}{3}P(x, x, y). \end{aligned}$$

Thus all the assumptions of Theorem 1 hold and, consequently, $x = 0 \in X$ is the only fixed point of T.

4. A fixed point result for nonlinear generalized contractions on generalized partial metric space

Theorem 2. *Let (X, P) be a complete generalized partial metric space and $T : X \to X$ be a mapping. If there exists a nondecreasing function $\gamma : [0, \infty) \to [0, \infty)$ such that the sequence $(\gamma^n)_{n\in\mathbb{N}}$ of iterates of γ converges pointwise to 0 , and*

$$(9) \qquad P(Tx, Tx, Ty) \leq \gamma\big(P(x, x, y)\big), \quad \textit{for all } x, y \in X,$$

then T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary element in X. Define the sequence (x_n) in X by

$$x_{n+1} = Tx_n, \text{ for all } n \geq 1.$$

It follows from (9), by induction, that for all $n \in \mathbb{N} \cup \{0\}$,

$P(x_{n+1}, x_{n+1}, x_n) \leq \gamma^n\big(P(x_1, x_1, x_0)\big)$ and $P(x_n, x_n, x_{n+1}) \leq \gamma^n\big(P(x_0, x_0, x_1)\big)$.

Consequently, letting $n \to \infty$, we get

$$\lim_{n\to\infty} P(x_{n+1}, x_{n+1}, x_n) = 0 = \lim_{n\to\infty} P(x_n, x_n, x_{n+1}).$$

Thus, for $\epsilon > 0$, we can choose $n \in \mathbb{N}$ such that

$$\max\{P(x_n, x_n, x_{n+1}), P(x_{n+1}, x_{n+1}, x_n)\} \leq \epsilon - \gamma(\epsilon).$$

Now we define

$$M := \{x \in X : \max\{P(x, x, x_n), P(x_n, x_n, x)\} \leq \epsilon\}.$$

By (9) and by virtue of γ, we have for any $y \in M$,

$$\begin{aligned} P(Ty, Ty, x_n) &\leq P(Ty, Ty, Tx_n) + P(Tx_n, Tx_n, x_n) \\ &\leq P(Ty, Ty, Tx_n) + P(x_{n+1}, x_{n+1}, x_n) \\ &\leq \gamma(\epsilon) + \big(\epsilon - \gamma(\epsilon)\big) = \epsilon. \end{aligned}$$

Similarly

$$\begin{aligned} P(x_n, x_n, Ty) &\leq P(x_n, x_n, Tx_n) + P(Tx_n, Tx_n, Ty) \\ &= P(x_n, x_n, x_{n+1}) + P(Tx_n, Tx_n, Ty) \\ &\leq \epsilon - \gamma(\epsilon) + \gamma(\epsilon) = \epsilon. \end{aligned}$$

Thus, $Ty \in M$, that is $T(M) \subseteq M$, which implies that for all $k, m > n$,

$$P(x_k, x_k, x_m) \leq 2\epsilon \text{ and } P(x_m, x_m, x_k) \leq 2\epsilon.$$

This shows that (x_n) is a Cauchy sequence in generalized partial metric space X and in view of completeness of X, there exists a point x in X such that $P(x_n, x_n, x) \to 0$ and $P(x, x, x_n) \to 0$ as $n \to \infty$. Now we will prove that x is a fixed point of T. In view of (*iv*) and using (9), we have

$$\begin{aligned} P(Tx, Tx, x) &\leq P(Tx, Tx, Tx_n) + P(Tx_n, Tx_n, x) \\ &\leq \gamma\big(P(x, x, x_n)\big) + P(x_{n+1}, x_{n+1}, x) \\ &< P(x, x, x_n) + P(x_{n+1}, x_{n+1}, x). \end{aligned}$$

Making $n \to \infty$, we get $P(Tx, Tx, x) = 0$ implies $Tx = x$. Hence x is a fixed point of mapping T in X. The argument for the uniqueness of fixed point is similar to the one used in Theorem 1. ■

Example 4. Let $X = [0,1]$ then $P(x,y,z) = |x+z-2y| + |x-z|$ is a complete generalized partial metric space. Define a self-map T on X by $T(x) = \sin x - \dfrac{x^2}{3}$, for all $x \in X$ and $\gamma : [0,\infty) \to [0.\infty)$ such that

$$\gamma(t) = \begin{cases} t - \dfrac{t^2}{3}, & t \in [0,1], \\ 1, & t \in (1,\infty). \end{cases}$$

Then for $x, y \in X$, with $x - y = t > 0$, we have

$$\begin{aligned} P(Tx,Tx,Ty) &= 2|Tx - Ty| \\ &= 2\left|\left(\sin x - x^2/3\right) - \left(\sin y - y^2/3\right)\right| \\ &\leq 2|\sin x - \sin y| + \frac{2}{3}|x^2 - y^2| \\ &\leq 2|x-y| + \frac{2}{3}|x^2 - y^2| \\ &= 2|x-y|\left(1 - \frac{1}{3}|x+y|\right) \\ &\leq 2|x-y|\left(1 - \frac{1}{3}|x-y|\right) = 2t(1-t/3) \\ &= \gamma\big(P(x,x,y)\big). \end{aligned}$$

Thus all the assumptions of Theorem 2 hold and, consequently, $x = 0 \in X$ is the only fixed point of T.

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