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STABILITY AND BIFURCATION ANALYSIS FOR AN AIRFOIL MODEL WITH A HIGH-ORDER NONLINEAR SPRING

Shuqun Li, Liangqiang Zhou
Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, China, and
Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing, China
Corresponding author: Liangqiang Zhou, zlqrex@sina.com

In this paper, the stability and bifurcation of an airfoil model with a high-order nonlinear spring are investigated both analytically and numerically. Two possible types of bifurcation at the equilibrium point are studied. It is proved that the zero characteristic root can only be a single zero. With the help of the center manifold theory and the normal form theory, the expressions of critical bifurcation curves leading to initial bifurcation and secondary bifurcation are obtained. Numerical simulations confirm the theoretical results.

Keywords: nonlinear airfoil model, bifurcation, stability, normal form

1. Introduction

The stability properties and bifurcation research of aircraft, railroad wheelsets and other nonlinear systems have important practical significance (Carroll and Mehra, 1982; Knudsen *et al.*, 1994; Schy and Hannah, 1977; Thompson, 1983). Especially, the Hopf bifurcation (Malhotra and Namachchivaya, 1997; Namachchivaya and Van Roessel, 1986; 1990) has been paid considerable attention. A dynamical model of an airfoil system with cubic nonlinearity was proposed for the pitching stiffness (Zhao and Yang, 1990; Lee, 1986). Golubev and Tharayil discovered limit cycles and chaotic oscillations through high-precision viscosity analysis (Golubev *et al.*, 2009; Tharayil and Alleyne, 2004). Hence, controlling such unwanted and persistent oscillations has attracted interest among researchers. The first study of non-linear aeroelastic problems of an aircraft wing were perhaps the works of Woolston *et al.* (1957) and Shen (1959). Recently, using a precise integration method, the nonlinear effect on the airfoil system was also simulated by many scholars (Gordon *et al.*, 2008). In (Hao and Wu, 2020), the stochastic-aeroelastic nonlinear response of a three-degree-of-freedom structural nonlinear airfoil with a control flap was presented. Taking into account potential effects of the longitudinal and vertical turbulent flow, Hao *et al.* (2021) discussed stochastic airfoil flutter in an unsteady flow by the stochastic P-bifurcation method. With analytical and numerical methods, Zhou *et al.* (2013) studied chaotic motion of a two-dimensional airfoil system with cubic nonlinearity in supersonic flows. It was shown that the system was always in chaotic motion when the nonlinear stiffness coefficient crossed its critical value. Through the Gegenbauer polynomial approximation, Wu *et al.* (2007) investigated the effects of parameter uncertainty on flutter characteristics of a two-dimensional airfoil in an incompressible flow. The results showed the presence of intricate behavior of the system.

In this paper, the stability and bifurcation of an airfoil model with a high-order nonlinear spring are investigated with analytical and numerical methods. Possible bifurcation solutions and their stability conditions are obtained analytically. Numerical simulations confirm the analytical results.

2. Bifurcation analysis

Consider an airfoil model with a higher-order nonlinear spring with cubic pitching stiffness and viscous damping as shown in Fig. 1. The equation of motion for this model is (Rajagopal *et al.*, 2019)

$$\begin{aligned} &\ddot{h} + \frac{1}{4}\ddot{\alpha} + \frac{1}{10}\dot{h} + \frac{1}{5}h + \frac{1}{10}\beta\alpha + f(h) = 0 \\ &\frac{1}{4}\ddot{h} + \frac{1}{2}\ddot{\alpha} + \frac{1}{10}\alpha + k\alpha - d\beta\alpha + f(\alpha) = 0 \end{aligned} \tag{2.1}$$

where $f(\alpha)$ is the pitching stiffness and $f(h)$ is the plunging stiffness, respectively, the state variable h represents the plunging displacement, and α represents the pitching angle.

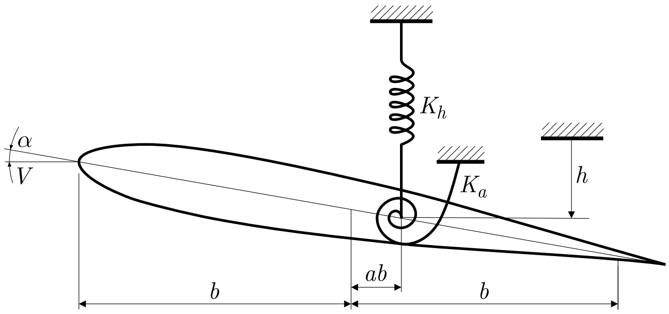

Fig. 1. Two-degree-of-freedom airfoil model

Substituting $x = \alpha$, $y = \dot{\alpha}$, $z = h$, $w = \dot{z}$ into Eq. (2.1), the dynamic equations of the new four-dimensional autonomous system are as follows

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \frac{1}{1.75}[(4d\beta + 0.1\beta - 2)x + 0.1w + 0.2z + f(z) - 4f(x) - 0.4y] \\ \frac{dz}{dt} &= w \\ \frac{dw}{dt} &= -\frac{1}{1.75}[(0.2d\beta + d\beta - 0.5)x + 0.2w + 0.4z + 2f(z) - f(x) - 0.1y] \end{aligned} \tag{2.2}$$

where $f(z) = 5z^2 + 10z^3 + 40z^5$, $f(x) = 5x^2 + 20x^3 + 40x^5$, d and β are system parameters. In addition $\beta = (V/b\omega_\alpha)^2$, V is the airspeed and ω_α is the eigenfrequency.

The Jacobian matrix of system (2.2) at $(0, 0, 0, 0)$ is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{4}{7}\Big(4d\beta - 4k + \frac{1}{10}\beta\Big) & -\frac{8}{35} & \frac{4}{35} & \frac{2}{35} \\ \frac{4}{7}\Big(k - \frac{1}{5}\beta - d\beta\Big) & \frac{2}{35} & -\frac{8}{35} & -\frac{4}{35} \end{bmatrix} \tag{2.3}$$

The characteristic equation of the matrix $\mathbf{A}$ is

$$\lambda^4 + R_1\lambda^3 + R_2\lambda^2 + R_3\lambda + R_4 = 0 \tag{2.4}$$

where

$$\begin{aligned} &R_1 = \frac{12}{35} \qquad R_2 = \frac{44}{175} - \frac{16}{7}d\beta + \frac{16}{7}k - \frac{2}{35}\beta \\ &R_3 = \frac{8}{175} + \frac{8}{35}k - \frac{8}{35}d\beta \qquad R_4 = \frac{16}{35}k - \frac{16}{35}d\beta \end{aligned} \tag{2.5}$$

According to the Routh-Hurwitz criterion (Hu, 2000), the equilibrium point $O(0,0,0,0)$ is stable if the following conditions are satisfied

$$R_1 > 0 \qquad R_1R_2 - R_3 > 0 \qquad R_3(R_1R_2 - R_3) - R_1^2R_4 > 0 \qquad R_4 > 0 \tag{2.6}$$

The above conditions indicate that all eigenvalues of the Jacobian matrix $\mathbf{A}$ have negative real parts. When the above conditions are not satisfied, the initial equilibrium solution may become unstable and bifurcation will occur. First, we prove that zero can only be a simple eigenvalue of the matrix $\mathbf{A}$.

Theorem 1. *If zero is an eigenvalue of the matrix* $\mathbf{A}$*, then it can only be simple.*

Proof 1. *If the matrix* $\mathbf{A}$ *has double zero eigenvalues, then* $f(\lambda)$ *can be written in the following form*

$$\begin{aligned} f(\lambda) &= \lambda^4 + R_1\lambda^3 + R_2\lambda^2 + R_3\lambda + R_4 \equiv \lambda^2 f_1(\lambda) \\ &\equiv \lambda^2 f_1(\lambda^2 + R_1\lambda + R_2) \equiv \lambda^4 f_1(\lambda) + R_1\lambda^3 + r_2\lambda^2 \end{aligned} \tag{2.7}$$

Obviously, $R_3 = R_4 = 0$*, and in the above characteristic polynomial, when* $R_4 = (16/35)k - (16/35)d\beta = 0$*, we have* $R_3 = (8/175) + (8/35)k - (8/35)d\beta = (8/175) \neq 0$*, so zero can only be a single root of the characteristic polynomial.*

3. A pair of purely imaginary eigenvalues and a pair of complex eigenvalues with negative real parts

Consider an autonomous system described by

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + f(\mathbf{x}) \qquad \mathbf{x} \in R^n \qquad f:\ R^n \to R^n \tag{3.1}$$

where we assumed the non-linear function f be analytical and $f(0) = 0$. Further, system (3.1) is assumed to have a pair of purely imaginary eigenvalues $\pm iw_c$ at the equilibrium 0. Without loss of generality, it is assumed that $w_c = 1$ (otherwise one can use transformation $t' = w_c t$ to change frequency $w_c = 1$). The Jacobian matrix of system (3.1) at $\mathbf{0}$ is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix} \qquad \mathbf{A} \in R^{(n-2)\times(n-2)} \tag{3.2}$$

where it is assumed that all eigenvalues of $\mathbf{A}$ have negative real parts. Next, we introduce new independent variables

$$T_k = \varepsilon^k t, k = 0, 1, 2, \ldots \tag{3.3}$$

So the derivatives with respect to t now become expansions in terms of the partial derivatives with respect to T_k, given by

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \frac{dT_2}{dt}\frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots \tag{3.4}$$

where $D_k = \partial/\partial T_k$ is the differentiation operator.

Next, it is supposed that the solution to Eq. (3.1) is represented by a power expansion in the neighborhood of $X = 0$

$$x_i(t;\varepsilon) = \varepsilon x_{i1}(T_0, T_1, \ldots) + \varepsilon^2 x_{i2}(T_0, T_1, \ldots) + \cdots \qquad i = 1, 2, \ldots, n \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.1) with the aid of Eq. (3.4), and balancing the like powers of ε results in the ordered perturbation equations, we get

$$\begin{aligned}\varepsilon^1: \quad & D_0x_{11} = x_{21} \qquad D_0x_{21} = -x_{11} \qquad D_0x_{p1} = -\alpha_p x_{p1} \\ & \qquad p = 3,4,\dots,m_1+2 \\ & D_0x_{q1} = -\alpha_p x_{q1} + w_q x_{(q+1)1} \qquad q = m_2+3, m_2+5,\dots,n-1\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\varepsilon^2: \quad & D_0x_{12} = x_{22} - D_1x_{11} + f_{12}(x_{11},x_{21},\dots,x_{n1}) \\ & D_0x_{22} = -x_{12} - D_1x_{21} + f_{22}(x_{11},x_{21},\dots,x_{n1})\end{aligned} \tag{3.7}$$

etc., where $\alpha_p > 0\alpha_p > 0$, $w_q > 0$ and $m1 + 2m2 + 2 = n$.

According to Eq. (3.6), we obtain

$$D_0^2x_{11} + x_{11} = 0 \tag{3.8}$$

It can be written in a general form

$$x_{11} = r(T_1,T_2,\dots)\cos[T_0 + \phi(T_1,T_2,\dots)] = r\cos(T_0+\phi) \equiv r\cos\theta \tag{3.9}$$

where ϕ, r represent the phase of motion and amplitude. $\theta = w_cT_0 + \phi = T_0 + \phi$ implies that

$$D_0r = 0 \qquad D_0\phi = 0 \tag{3.10}$$

The asymptotic ε^1 order solutions of the second group are given by

$$x_{i1} = 0 \qquad i = 3,4,\dots,n \tag{3.11}$$

Next, the procedure described above can be applied again to solve the ε^2 order perturbation Eq. (3.7). We obtain

$$D_0^2x_{12} + x_{12} = -D_1D_0x_{11} - D_1x_{21} + D_0f_{12} + f_{22} \tag{3.12}$$

by substituting the solutions x_{11}, x_{21} into Eq. (3.7) expressed in terms of trigonometric functions $\cos[k(T_0+\phi)]$, $\sin[k(T_0+\phi)]$, $k = 0,1,2$. In order to eliminate possible secular terms in x_{12}, it is required that $\cos[k(T_0+\phi)] = 0$, $\sin[k(T_0+\phi)] = 0$. And explicit expressions for D_1r, $D_1\phi$ can also be obtained. So we can know that $x_{22} = D_0x_{12} + D_1x_{11} - f_{12}(x_{11},x_{21},\dots,x_{n1})$. This procedure can be carried out to any high order perturbation equations. Finally, the normal forms which are given in polar co-ordinates can be written as

$$\begin{aligned}\frac{dr}{dt} &= \frac{\partial r}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial r}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial r}{\partial T_2}\frac{\partial T_2}{\partial t} + \dots = D_0r + \varepsilon D_1r + \varepsilon^2D_2r + \cdots \\ \frac{d\theta}{dt} &= w_c + \frac{\partial \phi}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial \phi}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial \phi}{\partial T_2}\frac{\partial T_2}{\partial t} + \dots = 1 + D_0\phi + \varepsilon D_1\phi + \varepsilon^2D_2\phi + \cdots\end{aligned} \tag{3.13}$$

Now, we consider an example. Equation (2.4) has a pair of purely imaginary characteristic roots and a pair of complex characteristic roots with real part at the initial equilibrium point if and only if the following conditions are satisfied

$$\frac{R_3}{R_1}\Big(\frac{R_3}{R_1} - R_2\Big) + R_4 = 0 \qquad \frac{R_3}{R_1} > 0 \qquad R_1 > 0 \qquad R_4 > 0 \tag{3.14}$$

Substituting $\beta = 18/19$, $d = 1/35$, $k = 1/7$ into Eq. (2.5), we have $R_1 = 12/35$, $R_2 = 1536/3325$, $R_3 = 48/665$, $R_4 = 176/3325$.

The eigenvalues of characteristic equation (2.4) in this case are

$$\lambda_{1,2} = \pm\frac{2\sqrt{19}}{19}I \qquad \lambda_{3,4} = -\frac{6}{36} \pm \frac{4\sqrt{17}}{35}I \tag{3.15}$$

Taking k as the disturbance parameter, substituting $k = k_0 + \delta$, and using the following transformation

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -\frac{19}{8} & -\frac{1}{8}\sqrt{19} & \frac{4085}{1144} & \frac{285}{1144}\sqrt{17} \\ \frac{1}{4} & -\frac{\sqrt{19}}{4} & -\frac{57}{52} & -\frac{19}{52}\sqrt{17} \\ 0 & -\frac{\sqrt{19}}{2} & -\frac{15}{22} & -\frac{5}{11}\sqrt{17} \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3.16}$$

system (2.2) can be rewritten as follows

$$\begin{aligned} x_1 &= -\frac{1615}{4564}\delta x_1 + \Big(\frac{2}{19}\sqrt{19} - \frac{85}{4564}\sqrt{19}\delta\Big)x_2 + \frac{347225}{652652}\delta x_3 + \frac{24225}{652652}\sqrt{17}\delta x_4 + Na_1 \\ x_2 &= \Big(-\frac{2}{19}\sqrt{19} + \frac{1425}{4564}\sqrt{19}\delta\Big)x_1 - \frac{1425}{4564}\delta x_2 + \frac{306375}{652652}\sqrt{19}\delta x_3 + \frac{21375}{652652}\sqrt{323}\delta x_4 + Na_2 \\ x_3 &= -\frac{4579}{4564}x_1 - \frac{241}{4564}\sqrt{19}\delta x_2 + \Big(-\frac{6}{35} + \frac{984485}{652652}\delta\Big)x_3 \\ &\quad + \Big(\frac{4}{35}\sqrt{17} + \frac{68685}{652652}\sqrt{17}\delta\Big)x_4 + Na_3 \\ x_4 &= \frac{36651}{77588}\sqrt{17}\delta x_1 + \frac{1929}{77588}\sqrt{323}\delta x_2 + \Big(-\frac{4}{15}\sqrt{17} - \frac{7879965}{11095084}\sqrt{17}\delta_2\Big)x_3 \\ &\quad + \Big(-\frac{6}{35} + \frac{549765}{652652}\delta_2\Big)x_4 + Na_4 \end{aligned} \tag{3.17}$$

where the nonlinear terms Na_j $(j = 1, 2, 3, 4)$ are ommited.

At the critical value $\delta_c = 0$, the standard form of the Jacobian matrix for the initial equilibrium solution of the equation is

$$\mathbf{J} = \begin{bmatrix} 0 & \frac{2}{19}\sqrt{19} & 0 & 0 \\ -\frac{2}{19}\sqrt{19} & 0 & 0 & 0 \\ 0 & 0 & -\frac{6}{35} & \frac{4}{35}\sqrt{17} \\ 0 & 0 & -\frac{4}{35}\sqrt{17} & -\frac{6}{35} \end{bmatrix}. \tag{3.18}$$

Using the time scale transformation $\tau' = \tau/w_c = \tau/(2\sqrt{19}/19)$, the method of multiple scales and computer algebra (Yu, 1998), the canonical form of Eq. (3.17) in the polar coordinates can be obtained as follows

$$\frac{dr}{d\tau'} = a\delta r + br^3 \qquad \frac{d\theta}{d\tau'} = 1 + cr^2 + d\delta \tag{3.19}$$

where $a = 0.608127$, $b = -0.429761$, $c = 0.016749$, $d = 0.026493$, $ab < 0$. Taking $dr/d\tau' = 0$, the initial solution $r_1 = 0$ and Hopf bifurcation solution $r_2^2 = 1.4150035\delta$ are obtained. The stability of the equilibrium solution depends on

$$\frac{d}{dr}\Big(\frac{dr}{d\tau'}\Big) = a\delta + 3r^2 b \tag{3.20}$$

Substituting the initial solution $r_1 = 0$ into Eq. (3.20), it can be obtained that $d/dr(dr/d\tau') = 0.608127\delta$, so the initial solution is stable when $\delta < 0$ and unstable when $\delta > 0$. In the same way, substituting the Hopf bifurcation solution into Eq. (3.20), we get $d/dr(dr/d\tau') = -1.216258\delta$, so the initial solution is stable when $\delta > 0$ and unstable when $\delta < 0$.

Different values of the parameters can be taken to verify the results of previous analysis. Numerical simulation can be performed on system (2.2). When $\delta = -0.1$, $(x_1, x_2, x_3, x_4) = (0.02, 0.02, 0, -0.1)$, the numerical solution converges to the origin as shown in Fig. 2. At this time, the initial equilibrium solution is stable. If we select $\delta = 0.1$, $(x_1, x_2, x_3, x_4) = (-0.1, 0.05, 0.03, -0.03)$, a limit cycle is obtained from the starting point as shown in Fig. 3. Obviously, the numerical results are consistent with the analytical results.

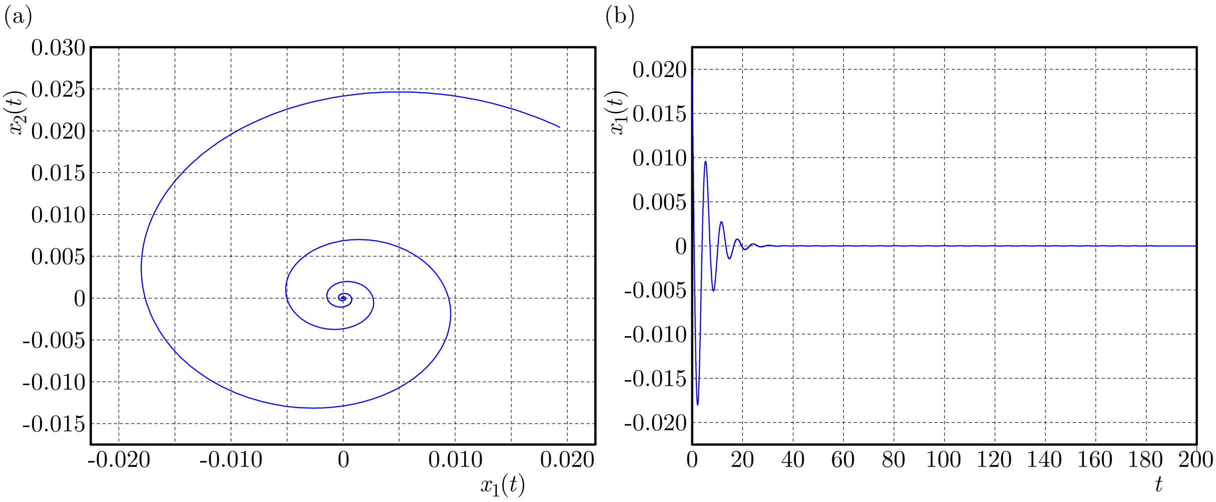

Fig. 2. The phase diagram projection that converges to the initial equilibrium solution and the time history diagram of $x_1(t)$

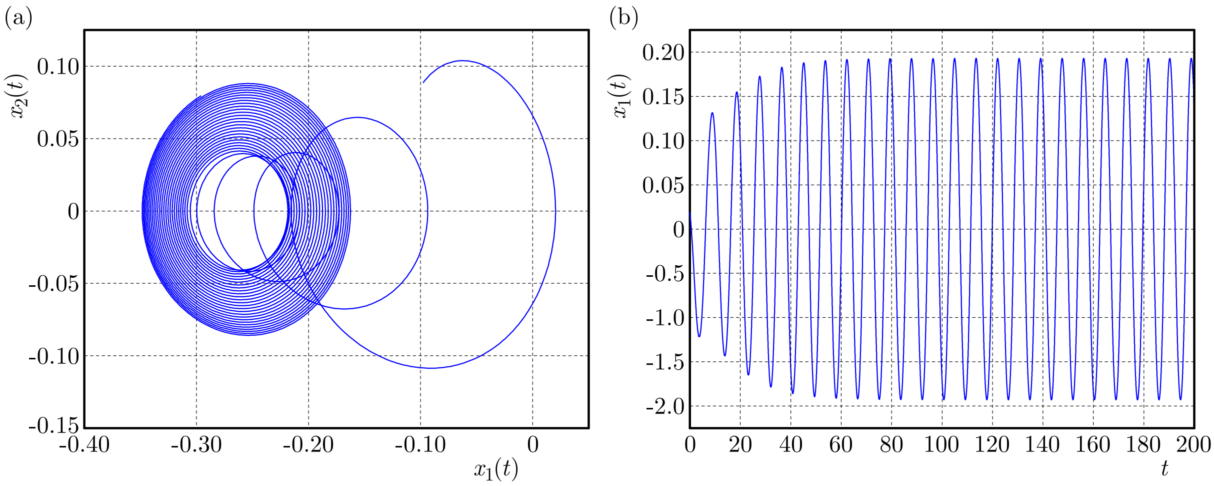

Fig. 3. The phase diagram projection that converges to the Hopf bifurcation solution and the time history diagram of $x_1(t)$

4. The case of a single zero and a pair of purely imaginary eigenvalues

Consider system (3.1), but where the Jacobian $\mathbf{J}$ is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \qquad A \in R^{(n-3)\times(n-3)} \tag{4.1}$$

where it is assumed that all eigenvalues of A have negative real parts.

The general form of solution to Eq. (3.1) can be written as

$$x_{12} = x_{12}^h + x_{12}^p \tag{4.2}$$

where x_{12}^h is the solution to equation $D_0^2 x_{12} + x_{12} = 0$, whereas x_{12}^p represent a particular solution to Eq. (3.12). And the solution of x_{12}^h can be written as

$$x_{12}^h = r[A_{211} z \cos(T_0) + B_{211} z \sin(T_0)] \tag{4.3}$$

In addition,we have

$$x_{32}^h = C_{220} r^2 + C_{202} z^2 \tag{4.4}$$

where four arbitrary coefficients A_{211}, B_{211}, C_{220}, C_{202} can be used in the third-order perturbation equations. Thus, it is easy to find general formulas for the homogeneous solution for an even number n

$$\begin{aligned}
x_{1n}^h &= (A_{nn-11} r^{n-1} z + A_{nn-33} r^{n-2} z^3 + \cdots + A_{n1n-1} r z^{n-1}) \cos(T_0) + (B_{nn-11} r^{n-1} z \\
&\quad + B_{nn-33} r^{n-2} z^3 + B_{n1n-1} r z^{n-1}) \sin(T_0) \\
x_{3n}^h &= C_{nn0} r^n + C_{nn-22} r^{n-2} z^2 + \cdots + C_{n0n} z^n
\end{aligned} \tag{4.5}$$

For an even n, we get

$$\begin{aligned}
\dot{r} &= r(b_{101} y + b_{120} r^2 + b_{140} r^4) \\
\dot{\theta} &= 1 + b_{201} y + b_{220} r^2 + \sum_{i=2}^{m_3} b_{20\,2i} y^{2i} \\
\dot{y} &= b_{302} y^2 + b_{303} y^3 + \sum_{i=1}^{m_3} b_{3\,2i\,0} y^{2i} + \sum_{i=1}^{m_1} b_{3\,2i\,1} r^{2i} y
\end{aligned} \tag{4.6}$$

therefore, from the pattern of SNF described by Eq. (4.6), we may find that the procedure is similar to that for the Hopf bifurcation while solving the coefficients from the ordered perturbation equations.

Now, we consider an example. Equation (2.4) has a single zero characteristic root and a pair of purely imaginary characteristic roots at the initial equilibrium point if and only if the following conditions are satisfied

$$R_1 > 0 \qquad R_2 > 0 \qquad R_3 > 0 \qquad R_4 = 0 \qquad R_1 R_2 - R_3 = 0 \tag{4.7}$$

Substituting $\beta = 31/15$, $d = 1/25$, $k = 31/375$ into Eq. (2.5), we have $R_1 = 12/35$, $R_2 = 26/525$, $R_3 = 8/175$, $R_4 = 0$.

The eigenvalues of characteristic Eq. (2.4) in this case are: $\lambda_1 = 0$, $\lambda_2 = -12/35$, $\lambda_{3,4} = \pm(\sqrt{30}/15)I$, where $I = \sqrt{-1}$.

Taking β, k as the disturbance parameter, letting $\beta = 31/15 + \delta_1$, $k = 31/375 + \delta_2$, and using the following transformation

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & -\frac{30}{31} & -\frac{15}{26} & \frac{5}{26}\sqrt{30} \\ -\frac{5}{6} & 0 & -\frac{5}{13} & -\frac{1}{26}\sqrt{30} \\ -\frac{35}{12} & 1 & 0 & -\frac{1}{2}\sqrt{30} \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{4.8}$$

system (2.2) can be transformed into the following new system

$$
\begin{aligned}
x_1 &= \Big(-\frac{12}{35} - \frac{874}{2305}\delta_1 + \frac{3470}{461}\delta_2\Big)x_1 + \Big(\frac{10488}{100037}\delta_1 - \frac{208200}{100037}\delta_2\Big)x_2 \\
&\quad + \Big(\frac{2622}{41951}\delta_1 - \frac{52050}{41951}\delta_2\Big)x_3 + \Big(\frac{874\sqrt{30}}{41951}\delta_1 + \frac{17350\sqrt{30}}{41951}\delta_2\Big)x_4 + Nb_1 \\
x_2 &= \Big(-\frac{217}{150}\delta_1 + \frac{217}{6}\delta_2\Big)x_1 + \Big(\frac{2}{5}\delta_1 - 10\delta_2\Big)x_2 + \Big(\frac{31}{130}\delta_1 - \frac{155}{26}\delta_2\Big)x_3 \\
&\quad + \Big(\frac{31\sqrt{30}}{390}\delta_1 + \frac{155\sqrt{30}}{78}\delta_2\Big)x_4 + Nb_2 \\
x_3 &= \Big(-\frac{1162}{11525}\delta_1 - \frac{2548}{461}\delta_2\Big)x_1 + \Big(\frac{1992}{71455}\delta_1 - \frac{21840}{14291}\delta_2\Big)x_2 + \Big(\frac{498}{29965}\delta_1 - \frac{420}{461}\delta_2\Big)x_3 \\
&\quad + \Big(\frac{\sqrt{30}}{15} + \frac{166\sqrt{30}}{29965}\delta_1 - \frac{140\sqrt{30}}{461}\Big)x_4 + Nb_3 \\
x_4 &= \Big(-\frac{1309}{57625}\delta_1 + \frac{2184\sqrt{30}}{2305}\delta_2\Big)x_1 + \Big(\frac{2244\sqrt{30}}{357275}\delta_1 - \frac{3744\sqrt{30}}{14291}\delta_2\Big)x_2 \\
&\quad + \Big(-\frac{\sqrt{30}}{15} + \frac{561\sqrt{30}}{149825}\delta_1 - \frac{72}{461}\delta_2\Big)x_3 + \Big(\frac{1122}{29965}\delta_1 + \frac{720}{461}\delta_2\Big)x_4 + Nb_4
\end{aligned}
\tag{4.9}
$$

where the nonlinear terms Nb_i $(i = 1, 2, 3, 4)$ are ommited.

At the critical value $\delta_{1c} = \delta_{2c} = 0$, the standard form of the Jacobian matrix for the initial equilibrium solution of the equation is

$$
\mathbf{J} = \begin{bmatrix} -\frac{12}{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{15}\sqrt{30} \\ 0 & 0 & -\frac{1}{15}\sqrt{30} & 0 \end{bmatrix}
\tag{4.10}
$$

The local dynamic behavior of the system in the domain of the degenerate equilibrium point can be described by x_1, x_2, x_3. According to the reference (Yu and Huseyin, 1988), using the approximate identity nonlinear transformation $z_i = y_i + g_i(y_i)$ and polar coordinate transformation $x_1 = y$, $x_2 = r\cos\theta$, $x_3 = \sin\theta$, $x_4 = x_4$, the canonical type of system (4.9) in the polar coordinates can be obtained as

$$
\begin{aligned}
y &= \Big(-\frac{74\sqrt{30}}{235}\delta_1 + \frac{40\sqrt{30}}{461}\delta_2\Big)y - \frac{360}{3227}\sqrt{30}y^3 + \Big(\frac{75\sqrt{30}}{461} + \frac{124}{25}\Big)yr^2 \\
r &= -\frac{17\sqrt{30}}{50}\delta_1 r + \frac{2}{41951}(31\sqrt{30} - 262)r^3 + \Big(\frac{253}{15} - \frac{4\sqrt{30}}{5}\Big)y^2 r \\
\theta &= \frac{\sqrt{30}}{15} + \frac{14\sqrt{30}}{9}r^2 + \frac{360\sqrt{30} - 41}{3277}y^2
\end{aligned}
\tag{4.11}
$$

The bifurcation solution of the system is as follows:

➢ The initial equilibrium solution (E.S.)

$$
y = r = 0 \tag{4.12}
$$

Its Jacobian matrix is

$$
\mathbf{J} = \begin{bmatrix} -\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 & 0 \\ 0 & -\frac{177}{50}\sqrt{30}\delta_1 \end{bmatrix}
\tag{4.13}
$$

The stability condition is

$$
\delta_1 > 0 \qquad -\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 < 0 \tag{4.14}
$$

Therefore, the stable boundary of E.S. is obtained.
The critical curves are

$$\begin{aligned} &L_1: \quad \delta_1 = 0 \qquad \Big(-\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 < 0\Big) \\ &L_2: \quad -\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 = 0 \qquad (\delta_1 > 0) \end{aligned} \tag{4.15}$$

➢ The static bifurcation solution (S.B.) is

$$y^2 = \frac{3277\sqrt{30}}{10800}\Big(-\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2\Big) \qquad r = 0 \tag{4.16}$$

The initial equilibrium solution is unstable along L_2. A static bifurcation solution occurs. Estimating the Jacobian matrix at the static bifurcation solution yields

$$\mathbf{J} = \begin{bmatrix} -2\Big(\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2\Big) & 0 \\ 0 & -\frac{17\sqrt{30}}{50}\delta_1 + \frac{253-12\sqrt{30}}{15}y^2 \end{bmatrix} \tag{4.17}$$

The stability condition for obtaining the static bifurcation solution is

$$\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 > 0 \qquad -\frac{17\sqrt{30}}{50}\delta_1 + \frac{253 - 12\sqrt{30}}{15}y^2 < 0 \tag{4.18}$$

Therefore, a static bifurcation boundary of S.B. is obtained.
The critial curves are

$$\begin{aligned} &L_2: \quad -\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 = 0 \qquad (\delta_1 > 0) \\ &L_3: \quad -\frac{17\sqrt{30}}{50}\delta_1 + \frac{253 - 12\sqrt{30}}{15}y_2 = 0 \qquad \Big(-\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 > 0\Big) \end{aligned} \tag{4.19}$$

➢ The initial Hopf bifurcation solution (H.B.(I)) is

$$y = 0 \qquad r^2 = \frac{713167\sqrt{30}\delta_1}{100(31\sqrt{30} - 2622)} \tag{4.20}$$

Obviously, when $\delta_1 < 0$, it has initial Hopf bifurcation solution.
The Jacobian matrix evaluated at initial Hopf bifurcation solution is

$$\mathbf{J} = \begin{bmatrix} -\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 + \Big(\frac{75}{461}\sqrt{30} + \frac{124}{25}\Big)r^2 & 0 \\ 0 & -\frac{17\sqrt{30}}{50}\delta_1 + \frac{6(31\sqrt{30}-2622)}{41951}r^2 \end{bmatrix} \tag{4.21}$$

It can be obtained that the stability condition of the Hopf bifurcation solution

$$-\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 + \Big(\frac{75}{461}\sqrt{30} + \frac{124}{25}\Big)r^2 > 0 \qquad \delta_1 < 0 \tag{4.22}$$

Therefore, a Hopf bifurcation solution H.B. is obtained.
The critical curve is

$$L_4: \quad -\frac{74}{235}\delta_1 + \frac{40}{461}\delta_2 + \Big(\frac{75}{461}\sqrt{30} + \frac{124}{25}\Big)r^2 = 0 \qquad (\delta_1 < 0) \tag{4.23}$$

So when $\delta_1 > 0$, Hopf bifurcation solution is unstable.

➢ The Hopf bifurcation solution of the second kind (H.B.(II)) is

$$\begin{aligned} y^2 &= \frac{\sqrt{30}[(742\sqrt{30}+24361)\delta_1-(251\sqrt{30}+87642)\delta_2]}{273(461\sqrt{30}+217642)} \\ r^2 &= \frac{\sqrt{30}[(860+271\sqrt{30})\delta_1-(741+204\sqrt{30})\delta_2]}{273(503\sqrt{30}+419624)} \end{aligned} \tag{4.24}$$

Its Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4.25}$$

where

$$\begin{aligned} a_{11} &= -\frac{74\sqrt{30}}{235}\delta_1 + \frac{40\sqrt{30}}{461}\delta_2 - \frac{1008}{3227}\sqrt{30}y^2 + \Big(\frac{75\sqrt{30}}{461}+\frac{124}{25}\Big)r^2 \\ a_{12} &= 2\Big(\frac{75\sqrt{30}}{461}+\frac{124}{25}\Big)yr \\ a_{21} &= 2\Big(\frac{253}{15}-\frac{4\sqrt{30}}{5}\Big)yr \\ a_{22} &= -\frac{17\sqrt{30}}{50}\delta_1 + \frac{6}{41951}(31\sqrt{30}-262)r + \Big(\frac{253}{15}-\frac{4\sqrt{30}}{5}\Big)y^2 \end{aligned}$$

The stability condition can be obtained from the trace and determinant of the above matrix as follows

$$T_r = a_{11}+a_{22} < 0 \qquad \det = a_{11}a_{22}-a_{12}a_{21} > 0 \tag{4.26}$$

So when

$$\begin{aligned} &(742\sqrt{30}+24361)\delta_1-(251\sqrt{30}+87642)\delta_2 > 0 \\ &\sqrt{30}[(860+271\sqrt{30})\delta_1+(741+204\sqrt{30})\delta_2] > 0 \end{aligned} \tag{4.27}$$

the second type of the Hopf bifurcation solution exists.

The critical curve is

$$\begin{aligned} L_5: \quad &\Big(\frac{35420}{24689}-\frac{6804}{102678}\sqrt{30}\Big)\delta_1 + \Big(\frac{3241}{102678}\sqrt{30}-\frac{245}{42631}\Big)\delta_2 = 0 \\ &\Big(-\frac{17\sqrt{30}}{50}\delta_1 + \frac{253-12\sqrt{30}}{15}y^2 < 0\Big) \end{aligned} \tag{4.28}$$

From the above analysis, it can be obtained that initial equilibrium solution (4.12) bifurcates along the critical curve L_2 of static bifurcation solution (4.16). When the parameters pass through the critical curve L_3, static bifurcation solution (4.16) loses stability and becomes a Hopf bifurcation solution of the second kind along the L_3.

The critical curve is shown in Fig. 4.

The values of parameters (δ_1,δ_2) can be selected from different areas of Fig. 1 to verify the results of the previous analysis. Firstly, the parameters $(\delta_1,\delta_2) = (0.1, 0.2)$ are selected in the stable region of the initial equilibrium solution, then the trajectory starting from the initial value $(x_1,x_2,x_3,x_4) = (-0.02, 0.0175, -0.1, 0.1)$ will eventually converge to the origin. The phase diagram projection that converges to the initial equilibrium solution and the time history diagram of $x_1(t)$ are shown in Fig. 5.

Secondly, the parameters $(\delta_1,\delta_2) = (0.03, 0.01)$ are selected in the region where the static equilibrium solution is stable, then the trajectory starting from the initial value $(x_1,x_2,x_3,x_4) =$

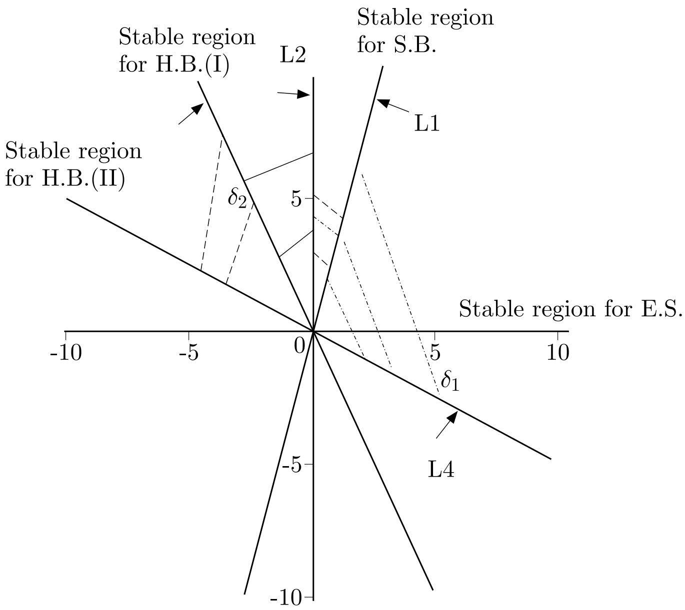

Fig. 4. The transition curve of tuning parameters in the case of a single zero and a pair of purely imaginary eigenvalues

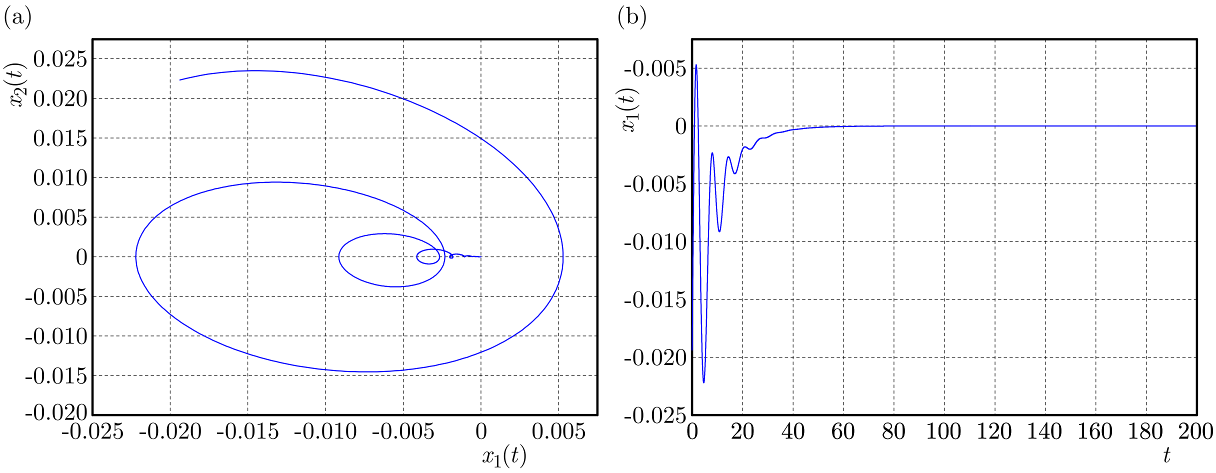

Fig. 5. The phase diagram projection that converges to the initial equilibrium solution and the time history diagram of $x_1(t)$

$(0.02, 0.02, 0, 0.03)$ will converge to the static equilibrium solution. The phase diagram projection that converges to the static bifurcation solution and the time history diagram of $x_1(t)$ are shown in Fig. 6.

Finally, the parameters $(\delta_1, \delta_2) = (-0.01, 0.041)$ are selected in the region where the second type of the Hopf bifurcation solution is stable. The trajectory when the original value is $(x_1, x_2, x_3, x_4) = (0.1, 0.1, -0.04, 0.04)$ finally converges to a stable limit cycle. The phase diagram projection that converges to the Hopf bifurcation solution and the time history diagram of $x_1(t)$ are shown in Fig. 7.

5. Conclusions

In this paper, two possible types of bifurcation at the equilibrium point of the system are discussed in detail. When the stable conditions of stationary solutions for the initial equilibrium

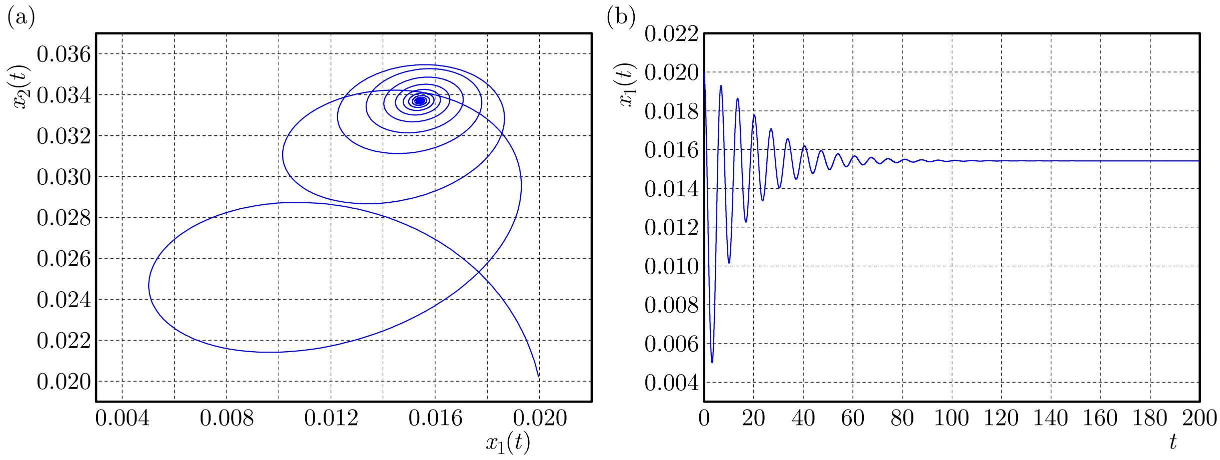

Fig. 6. The phase diagram projection that converges to the static bifurcation solution and the time history diagram of $x_1(t)$

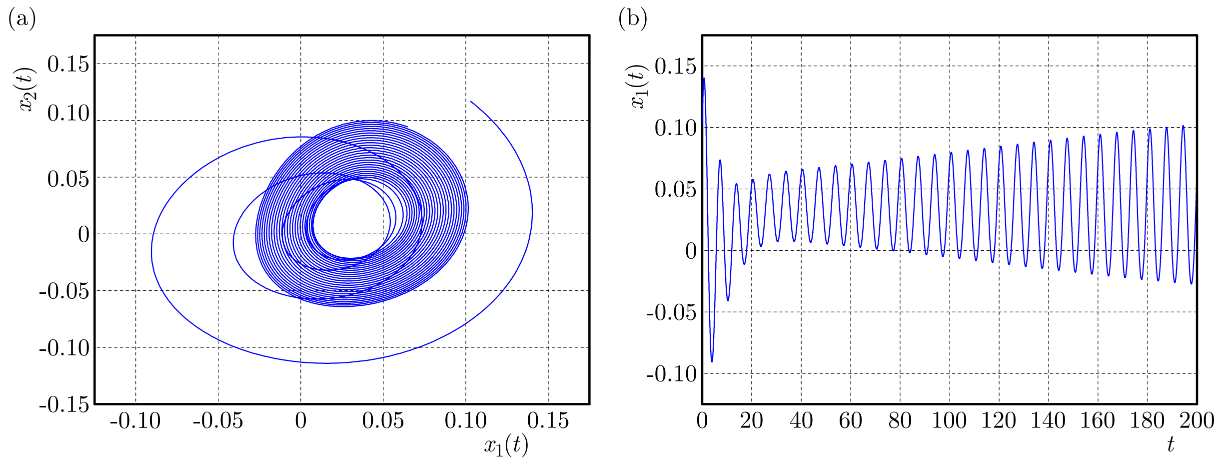

Fig. 7. The phase diagram projection that converges to the Hopf bifurcation solution and the time history diagram of $x_1(t)$

solution are not satisfied for the airfoil model with a high-order spring, bifurcations including Hopf bifurcation, 2D tori may occur. Stable regions and critical bifurcation curves for some equilibrium solutions are presented. Flutter in an aeroelastic structure such as an aircraft wing may cause structural unstability. The results provide some inspiration and guidance for the analysis and dynamics design of this class of systems. For example, while designing the structure of an airfoil, we should reasonably choose parameters to avoid the occurrence of persistent oscillations, etc.

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