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DETERMINISTIC PROPERTIES OF SERIALLY CONNECTED DISTRIBUTED LAG MODELS

Distributed lag models are an important tool in modeling dynamic systems in economics. In the analysis of composite forms of such models, the component models are ordered in parallel (with the same independent variable) and/or in series (where the independent variable is also the dependent variable in the preceding model). This paper presents an analysis of certain deterministic properties of composite distributed lag models composed of component distributed lag models arranged in sequence, and their asymptotic properties in particular. The models considered are in discrete form. Even though the paper focuses on deterministic properties of distributed lag models, the derivations are based on analytical tools commonly used in probability theory such as probability distributions and the central limit theorem.

Keywords: *distributed lags, limit properties, characteristics of distributed lags*

1. Introduction

Distributed lag models are an important tool used to describe many dynamic systems. Their aim is to explain the reaction, distributed over time, to a disturbance in the input, without using a feedback loop connecting the input to the output. One can often come across composite forms of such structures, consisting of elements arranged in a series, for example in logistic systems, where deliveries flow from the producer to the final customer via a chain of intermediary stocks/warehouses.

Distributed lag models were developed in order to enable the separation of factors influencing the value of a variable called the dependent variable, which represents a certain phenomenon. These factors are represented by: current and past values of the

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independent variable, which represents a phenomenon causing changes in the dependent variable; the lag coefficients, which measure the impact of past values of the independent variable on the dependent variable, as well as the random term representing measurement errors, as well as the impact of factors not accounted for. The latter is called the random component of the distributed lag model, while the former is the deterministic one. Using a classical approach, for example see [7, 8], the lag coefficients are assumed to be constant. However, in the literature, see for example [10, 9, 2, 11, 6], the assumptions concerning the variability of the lag coefficients are less strict.

In the first part of this paper, we present basic concepts related to distributed lag models. We show the results of a simulation experiment, which suggest convergence of the dependent random variable. The second part examines conditions ensuring such convergence. One should pay attention to the fact that the lag distribution is not a probability distribution. It is worth noting that the notation used in the first and second section changes in accordance with the conventions used in economics and probability theory, respectively.

A discrete time distributed lag model is written in the form of the following expression [7, 3, 8]:

$$y_t = \sum_{i=0}^{\infty} v_i x_{t-i} + \varepsilon_t \quad (1)$$

where: x_t – independent variable in period t , y_t – dependent variable in period t , v_i – lag coefficients of the lag structure fulfilling the conditions: $v_i \geq 0$, $i = 0, 1, 2, \dots$, ε_t – random variable with zero expected value and finite variance.

The dependent and independent variables should be in a causality relationship. Therefore, the value of the dependent variable depends on the value of the independent variable from the same and preceding periods. However, in some cases it is reasonable to also consider negative indices by assuming that for all $i < 0$, $v_i = 0$.

We have considered the distributed lag models obtained by composing several component distributed lag models. By employing some simplifying assumptions, we studied two ways of composing distributed lag models: summation (also called parallel composition) and superposition (also called serial composition).

Consider a finite number n , $1 < n$, of distributed lag models defined as in (1):

$$y_t^{(j)} = \sum_{i=0}^{\infty} v_i^{(j)} x_{t-i} + \varepsilon_t^{(j)}, \quad j = -1, \dots, n \quad (2)$$

where: x_t – independent variable in period t , $y_t^{(j)}$ – dependent variable of the j -th distributed lag model in period t , $v_i^{(j)}$ – coefficients of the j -th distributed lag model:

$v_i^{(j)} \geq 0, j = 1, \dots, n, i = 0, 1, 2, \dots, \varepsilon_t^{(j)}$ – random variable with zero expected value and finite variance.

Definition 1. For each $j, j = 1, \dots, n$, the sequence $V^{(j)}, V^{(j)} = \left(v_i^{(j)}\right)_{i=0}^{\infty}$, of the coefficients of a distributed lag model is called the j -th lag structure.

Whenever the sum of the coefficients of a non-trivial lag structure is finite, the j -th model, (2), can be described in the following form:

$$y_t^{(j)} = a^{(j)} \sum_{i=0}^{\infty} w_i^{(j)} x_{t-i} + \varepsilon_t^{(j)}, \quad j = 1, \dots, n \quad (3)$$

where: $a^{(j)}$ – coefficient called the long-term multiplier

$$a^{(j)} = \sum_{i=0}^{\infty} v_i^{(j)}$$

$w_i^{(j)}$ – standardized coefficient in the distributed lag model (3):

$$w_i^{(j)} = \frac{v_i^{(j)}}{\sum_{i=0}^{\infty} v_i^{(j)}} = \frac{v_i^{(j)}}{a^{(j)}}, \quad j = 1, \dots, n \quad (4)$$

The coefficients $w_i^{(j)}$ are non-negative and moreover, for all $j, j = 1, \dots, n$, $w_i^{(j)} \geq 0$ and $\sum_{i=0}^{\infty} w_i^{(j)} = 1$.

Definition 2. For each $j, j = 1, \dots, n$, the sequence $W^{(j)} = \left(w_i^{(j)}\right)_{i=0}^{\infty}$ of the standardized coefficients of the distributed lag is called the j -th lag distribution.

In further considerations we assume that all the distributed lags have lag distributions with finite expected values and variances defined by the following expressions*:

$$M(W^{(j)}) = \sum_{i=0}^{\infty} i w_i^{(j)}, \quad j = 1, \dots, n \quad (5)$$

*The notions of the expected value and variance of a distributed lag are widely used, for example in [3].

$$D^2(W^{(j)}) = \sum_{i=0}^{\infty} [i - M(W^{(j)})]^2 w_i^{(j)} = \sum_{i=0}^{\infty} i^2 w_i^{(j)} - M^2(W^{(j)}), \quad j = 1, \dots, n \quad (6)$$

A parallel connection (sum) of n distributed lag models occurs when a dependent variable $y_t^{[n]}$ can be expressed as the sum of the dependent variables of the constituent distributed lag models*:

$$y_t^{[n]} = \sum_{j=0}^n y_t^{(j)} = \sum_{j=1}^n \left(\sum_{i=0}^{\infty} v_i^{(j)} x_{t-i} + \varepsilon_t^{(j)} \right) = \sum_{i=0}^{\infty} v_i^{[n]} x_{t-i} + \varepsilon_t^{[n]} = a^{[n]} \sum_{i=0}^{\infty} w_i^{[n]} x_{t-i} + \varepsilon_t^{[n]} \quad (7)$$

where:

$$v_i^{[n]} = \sum_{j=1}^n v_i^{(j)}$$

and the random variable $\varepsilon_t^{[n]}$

$$\varepsilon_t^{[n]} = \sum_{j=1}^n \varepsilon_t^{(j)}$$

is also a random variable with finite expected value and variance.

The long-term multiplier $a^{[n]}$ of model (7) equals the sum of the long-term multipliers of the component distributed lag models:

$$a^{[n]} = \sum_{j=1}^n a^{(j)} = \sum_{j=1}^n \sum_{i=0}^{\infty} v_i^{(j)} = \sum_{i=0}^{\infty} \sum_{j=1}^n v_i^{(j)} \quad (8)$$

and the coefficients of the lag distribution $W^{[n]}$ are equal to:

$$w_i^{[n]} = \sum_{j=1}^n \frac{a^{(j)}}{a^{[n]}} w_i^{(j)} \quad (9)$$

If the j -th distributed lag model has lag distribution $W^{(j)}$ with finite mean value $M(W^{(j)})$, $j = 1, 2, \dots, n$, then the mean value $M(W^{[n]})$ of the lag distribution $W^{[n]}$ can be expressed by the following equation:

*Properties of the sum of distributed lag models have been presented in [6].

$$M(W^{[n]}) = \frac{a^{(1)}}{a^{[n]}} M(W^{(1)}) + \frac{a^{(2)}}{a^{[n]}} M(W^{(2)}) + \dots + \frac{a^{(n)}}{a^{[n]}} M(W^{(n)}) \quad (10)$$

which shows that the mean value $M(W^{[n]})$ of the lag distribution $W^{[n]}$ is a weighted average of the mean values of the component lag distributions with the weight coefficients being the ratio of the j -th long term multiplier $a^{(j)}$ to the value of the long term multiplier $a^{[n]}$ of the lag distribution $W^{[n]}$.

The variance $D^2(W^{[n]})$ of the lag distribution $W^{[n]}$ is also related to the weighted average of the variances $D^2(W^{(j)})$ of the lag distributions of the component distributed lag models:

$$D^2(W^{[n]}) \geq \sum_{j=1}^n \frac{a^{(j)}}{a^{[n]}} D^2(W^{(j)}) \quad (11)$$

The superposition of lag distribution models occurs when a dependent variable $y_t^{[n]}$ is described by a distributed lag model with independent variable $y_t^{[n-1]}$, which in turn is a dependent variable of another distributed lag model with independent variable $y_t^{[n-2]}$, etc.

$$\begin{aligned} y_t^{[n]} &= \sum_{i=0}^{\infty} v_i^{(n)} y_{t-i}^{[n-1]} + \varepsilon_t^{(n)} \\ y_t^{[n-1]} &= \sum_{i=0}^{\infty} v_i^{(n-1)} y_{t-i}^{[n-2]} + \varepsilon_t^{(n-1)} \\ &\dots \\ y_t^{[1]} &= \sum_{i=0}^{\infty} v_i^{(1)} y_{t-i}^{[0]} + \varepsilon_t^{(1)} = \sum_{i=0}^{\infty} v_i^{(1)} x_{t-i} + \varepsilon_t^{(1)} \\ y_t^{[0]} &= x_t \end{aligned} \quad (12)$$

In further considerations of Eq. (12), the concept of the shift operator L , $Lx_t = x_{t-1}$, will be helpful.

Using the shift operator, the impact of the independent variable $y_t^{(0)} = x_t$ on the dependent variable $y_t^{(n)}$ can be written in the following form:

$$y_t^{[n]} = \prod_{i=1}^n V^{(i)}(L)x_t + \sum_{j=0}^{n-2} \prod_{i=j+2}^n V^{(i)}(L) \varepsilon_t^{(j+1)} + \varepsilon_t^{(n)} \quad (13)$$

where $V^{(i)}(L)$, $i = 1, \dots, n$, is the polynomial operator:

$$\sum_{i=0}^{\infty} v_i^{(j)} L^i, \quad j = 1, \dots, n \quad (14)$$

The study of the expected value of the dependent variable $y_t^{[n]}$ limits analysis of the distributed lag model to its deterministic part:

$$E(y_t^{[n]}) = \prod_{i=1}^n V^{(i)}(L)x_t \quad (15)$$

where the operator E is the expected value in the probability space where the random variables $\varepsilon_t^{(i)}$ are defined. In further considerations, we will focus on the deterministic part of the problem. For notational simplicity, the symbol $E(y_t^{[n]})$ will be substituted by $y_t^{[n]}$.

If the lag distributions $W^{(1)}$, $W^{(2)}$, ..., $W^{(n)}$ exist, Eq. (15) can be rewritten in the following form:

$$\begin{aligned} y_t^{[n]} &= aW(L)x_t = a^{(1)}W^{(n)}(L)a^{(n-1)}W^{(n-1)}(L) \\ &\times \dots \times a^{(1)}W^{(1)}(L)x_t = \prod_{i=1}^n a^{(i)} \prod_{i=1}^n W^{(i)}(L)x_t \end{aligned} \quad (16)$$

and

$$y_t^{[n]} = a^{[n]}W^{[n]}(L)x_t \quad (17)$$

where

$$W^{[n]}(L) = \prod_{i=1}^n W^{(i)}(L) \quad \text{and} \quad a^{[n]} = \prod_{i=1}^n a^{(i)}$$

In order to simplify the notation and facilitate transformations, the generating function of the j -th lag distribution, $W^{(j)}(\theta)$, will be employed, where

$$W^{(j)}(\theta) = \sum_{i=0}^{\infty} w_i^{(j)} \theta^i, \quad j = 1, \dots, n \quad (18)$$

has the following properties:

$$W^{(j)}(1) = 1, \quad j = 1, \dots, n$$

$$W^{[n]}(1) = \prod_{i=1}^n W^{(i)}(1) = 1$$

$$M(W^{(i)}) = \frac{dW^{(i)}(1)}{d\theta}, \quad j = 1, \dots, n$$

$$\begin{aligned} D^2(W^{(i)}) &= \frac{d^2W^{(i)}(1)}{d\theta^2} + \frac{dW^{(i)}(1)}{d\theta} - \left[\frac{dW^{(i)}(1)}{d\theta} \right]^2 \\ &= \frac{d^2W^{(i)}(1)}{d\theta^2} + M(W^{(i)}) - M^2(W^{(i)}), \quad j = 1, \dots, n \end{aligned}$$

The mean value $M(W^{[n]})$ of the result of the superposition of n distributed lag models is the sum of the mean values $M(W^{(i)})$, $i = 1, 2, \dots$, of the component distributions:

$$M(W^{[n]}) = M(W^{(1)}) + M(W^{(2)}) + \dots + M(W^{(n)}) \quad (19)$$

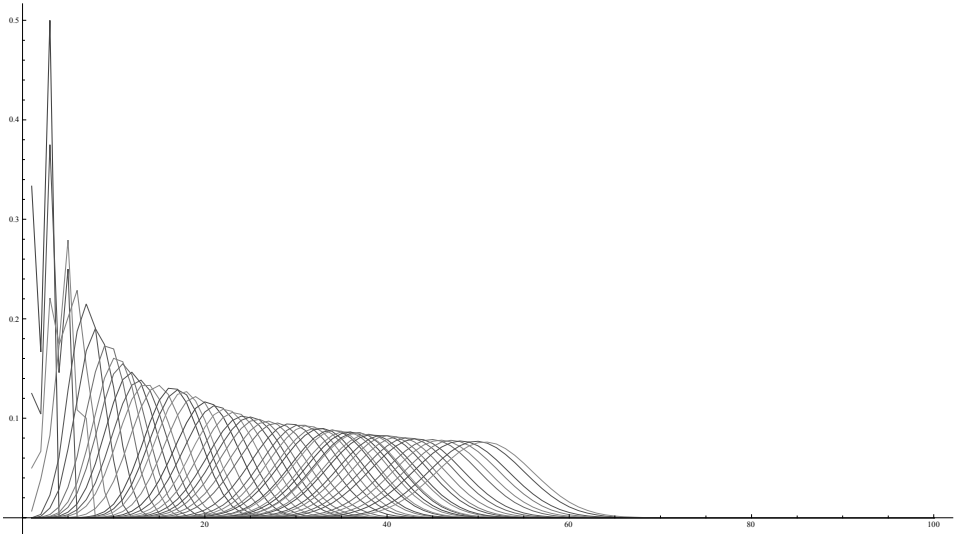


Fig. 1. Lag distributions for an increasing number of serially connected distributed lag models. Source: authors' own computations

The variance $D^2(W^{[n]})$ of the lag distribution obtained from the superposition of n distributed lag models is the sum of variances $D^2(W^{(i)})$, $i = 1, 2, \dots$, of the component models:

$$D^2(W^{[n]}) = D^2(W^{(1)}) + D^2(W^{(2)}) + \dots + D^2(W^{(n)}) \quad (20)$$

Simulations aimed at investigating the properties of distributed lag models composed of serially connected distributed lag models have shown that the resulting lag distribution resembles a normal distribution, see Fig. 1. The component lag distributions had supports consisting of five points with randomly generated coefficients from the uniform distribution over the interval $[0, 5]$.

The result presented above was the motivation for an investigation of whether the resemblance of the lag distribution to the normal distribution is accidental or has a sound explanation. The aim of the analysis below is to find a sufficient condition for the convergence of the lag distributions.

2. Asymptotic properties of the superposition of distributed lag models

For $1 \leq k \leq n$, suppose $(w_i^{(k)})_{i=0}^{\infty}$ are sequences of nonnegative coefficients such that $\sum_{i=0}^{\infty} w_i^{(k)} = 1$ (compare with Definition 2).

Let $\tau^{(k)}$ be random variables with the distribution given by*

$$P(\tau^{(k)} = i) = w_i^{(k)}, \quad i = 0, 1, 2, \dots, \quad k = 1, 2, \dots, n \quad (21)$$

It follows from probability theory that the sequence of random variables $\tau^{(1)}, \tau^{(2)}, \dots, \tau^{(n)}$ can be assumed to be independent.

Let $(w_i^{[n]})_{i=1}^{\infty}$ be a sequence of coefficients defined recursively by:

$$\begin{aligned} (w_i^{[1]})_{i=0}^{\infty} &= (w_i^{(1)})_{i=0}^{\infty} \\ w_i^{[n]} &= \sum_{j=0}^i w_j^{[n-1]} w_{i-j}^{(n)}, \quad i = 0, 1, 2, \dots, \text{ for } n > 1 \end{aligned} \quad (22)$$

*A similar approach can be found in [3], p. 4.

The expected values and variances of the random variables used in this section will be denoted by E and Var . It is worth noticing that $M(W^{(j)}) = E\tau^{(j)}$ and $D^2W^{(j)} = \text{Var}\tau^{(j)}$. Formula (22) follows from (16).

Let $\tau^{[n]}$ be a random variable with the distribution defined by the sequence $(w_i^{[n]})_{i=0}^\infty$, i.e.

$$P(\tau^{[n]} = i) = w_i^{[n]}, \quad i = 0, 1, 2, \dots \quad (23)$$

as well as

$$W^{[n]}(\theta) = \sum_{i=0}^{\infty} w_i^{[n]} \theta^i$$

Remark 1. If we define $\tau^{[n]}$ as the sum $\tau^{[n]} = \tau^{(1)} + \dots + \tau^{(n)}$, where $\tau^{(1)}, \tau^{(2)}, \dots, \tau^{(n)}$ are independent, then (22) holds. Moreover, for the sequence of random variables $\tau^{(1)}, \tau^{(2)}, \dots, \tau^{(n)}$ and the random variable $\tau^{[n]}$ the following equalities hold:

$$e^{(k)} = E\tau^{(k)} = \frac{dW^{(k)}(1)}{d\theta}, \quad k = 1, 2, \dots, n \quad (24)$$

$$e^{[n]} = E\tau^{[n]} = \frac{dW^{[n]}(1)}{d\theta} = \sum_{k=1}^n \frac{dW^{(k)}(1)}{d\theta} = \sum_{k=1}^n e^{(k)} \quad (25)$$

$$(\sigma^{(k)})^2 = \text{Var}\tau^{(k)} = \frac{d^2W^{(k)}(1)}{d\theta^2} + \frac{dW^{(k)}(1)}{d\theta} - \left(\frac{dW^{(k)}(1)}{d\theta} \right)^2, \quad k = 1, 2, \dots, n \quad (26)$$

$$(\sigma^{[n]})^2 = \text{Var}\tau^{[n]} = \sum_{k=1}^n \frac{d^2W^{(k)}(1)}{d\theta^2} + \sum_{k=1}^n \frac{dW^{(k)}(1)}{d\theta} - \sum_{k=1}^n \left(\frac{dW^{(k)}(1)}{d\theta} \right)^2 = \sum_{k=1}^n (\sigma^{(k)})^2 \quad (27)$$

The symbols $e^{(k)}$, $e^{[n]}$, $(\sigma^{(k)})^2$ and $(\sigma^{[n]})^2$ in (24)–(27) were introduced to shorten the formulas in the reminder of this section. We additionally introduce the following notation.

Let Φ_n^S be the cumulative distribution function of the random variable $(\tau^{[n]} - e^{[n]})/\sigma^{[n]}$ and Φ_n be the cumulative distribution function of the random vari-

able $\tau^{[n]}$ for each $n = 1, 2, \dots$. Moreover, let $\Phi_{\mu, \sigma}$ be the cumulative distribution function of the normal distribution $N(\mu, \sigma)$. In particular, we denote the distribution function of the standard normal distribution by Φ , i.e. $\Phi = \Phi_{0,1}$.

In our further considerations, we will apply the version of the central limit theorem formulated in [1]. Throughout the remainder of this paper, standard statistical notation will be used.

Theorem 1 (Lindeberg–Feller). Assume that $\{X_k\}_{k=1}^{\infty}$ is a sequence of independent random variables such that $EX_k = \mu_k$ and $\text{Var}X_k = \sigma_k^2 < \infty$. Let $s_n = \sqrt{\sum_{k=1}^n \sigma_k^2}$. If for each $\eta > 0$

$$\frac{1}{s_n^2} \sum_{k=1}^n \int_{|x-\mu_k|>\eta s_n} (x-\mu_k)^2 dF_k(x) \xrightarrow{n \rightarrow \infty} 0 \quad (28)$$

where F_k is the cumulative distribution function of X_k , then

$$\frac{\sum_{k=1}^n (X_k - \mu_k)}{s_n} \xrightarrow{d} N(0, 1)$$

Remark 2. The integral $\int_{|x-\mu_k|>\eta s_n} (x-\mu_k)^2 dF_k(x)$ in (28) can be written in the form

$$E(X_k - \mu_k)^2 I_{|X_k - \mu_k| > \eta s_n}.$$

We attempt to find a sufficient condition for the asymptotic normality of the random variable $\tau^{[n]}$. The following theorem states the convergence in distribution of the sequence $(\tau^{[n]} - e^{[n]})/\sigma^{[n]}$, $n \in N$, to the standard normal distribution $N(0, 1)$.

Theorem 2. Assume that the random variables $\tau^{(1)}, \tau^{(2)}, \dots, \tau^{(n)}$ are independent, have finite second moments, i.e.

$$E\left(\tau^{(k)}\right)^2 = \frac{d^2 W^{(k)}(1)}{d\theta^2} + \frac{dW^{(k)}(1)}{d\theta} < \infty, \quad k = 1, 2, \dots, n \quad (29)$$

and satisfy the following condition:

$$\forall \eta > 0 \quad \frac{1}{(\sigma^{[n]})^2} \sum_{k=1}^n \left(\sum_{|i-e^{(k)}| > \eta \sigma^{[n]}} (i - e^{(k)})^2 w_i^{(k)} \right) \xrightarrow{n \rightarrow \infty} 0 \quad (30)$$

Then as $n \rightarrow \infty$,

$$\frac{\tau^{[n]} - e^{[n]}}{\sigma^{[n]}} = \sum_{k=1}^n \frac{\tau^{(k)} - e^{(k)}}{\sigma^{[n]}} \xrightarrow{d} N(0, 1) \quad (31)$$

In particular,

$$\forall t \in R \lim_{n \rightarrow \infty} \Phi_n^S(t) = \Phi(t) \quad (32)$$

Proof: Since for each k ($1 \leq k \leq n$)

$$E \left\{ \left(\tau^{(k)} - e^{(k)} \right)^2 I_{|\tau^{(k)} - e^{(k)}| > \eta \sigma^{[n]}} \right\} = \sum_{|i-e^{(k)}| > \eta \sigma^{[n]}} (i - e^{(k)})^2 w_i^{(k)}$$

condition (30) can be written in the form

$$\forall \eta > 0 \quad \frac{1}{(\sigma^{[n]})^2} \sum_{k=1}^n E \left\{ \left(\tau^{(k)} - e^{(k)} \right)^2 I_{|\tau^{(k)} - e^{(k)}| > \eta \sigma^{[n]}} \right\} \xrightarrow{n \rightarrow \infty} 0$$

Therefore, the sequence of random variables $\tau^{(1)}, \tau^{(2)}, \dots, \tau^{(n)}$ satisfies the assumptions of Theorem 1 where $X_k = \tau^{(k)}$, $\mu_k = e^{(k)}$, $k = 1, 2, \dots, n$ and $s_n = \sigma^{[n]}$. Thus as $n \rightarrow \infty$,

$$\sum_{k=1}^n \frac{\tau^{(k)} - e^{(k)}}{\sigma^{[n]}} \xrightarrow{d} N(0, 1)$$

From the convergence in distribution (31), we obtain the convergence of the cumulative distribution function (32). \square

Lemma 1. Under the above assumptions, for sufficiently large $n \in N$ and any $t \in R$,

$$\Phi_n(t) \approx \Phi_{e^{[n]}, \sigma^{[n]}}(t)$$

Proof: Let $t \in R$ and $s = \frac{t - e^{[n]}}{\sigma^{[n]}}$

$$P(\tau^{[n]} \leq t) = P\left(\frac{\tau^{[n]} - e^{[n]}}{\sigma^{[n]}} \leq s\right) = \Phi_n^S(s)$$

From (32) $\Phi_n^S(s) \approx \Phi(s)$ for sufficiently large $n \in N$. Thus for a random variable X with standard normal distribution

$$\Phi_n(t) \approx \Phi(s) = P(X \leq s) = P\left(X \leq \frac{t - e^{[n]}}{\sigma^{[n]}}\right) = P(e^{[n]} + \sigma^{[n]}X \leq t) = \Phi_{e^{[n]}, \sigma^{[n]}}(t)$$

since the random variable $e^{[n]} + \sigma^{[n]}X$ has distribution $N(e^{[n]}, \sigma^{[n]})$. □

3. Conclusions

In this paper we have considered the asymptotic properties of the deterministic part of serially connected distributed lag models. These models enable the separation of deterministic and random factors affecting a certain phenomenon.

We analyzed the convergence of the lag distribution of a composite distributed lag model composed of serially connected distributed lag models. Because of the strong resemblance of the lag distributions (of a deterministic nature) to probabilistic distributions, some analytical tools from probability theory were adapted. Applying the central limit theorem we found a sufficient condition for the convergence of the lag distribution of the composite model to a lag distribution which can be approximated by a normal probability distribution. This condition is expressed in terms of the characteristics of the component lag distributions.

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