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# New estimate for the curvature of an order-convex set and related questions*

by

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**Abstract:** It is well known that in discrete optimization problems, gradient (local) algorithms do not always guarantee an optimal solution. Therefore, the problem arises of finding the accuracy of the gradient algorithm. This is a fairly well-known problem and numerous publications have been devoted to it. In establishing accuracy, various approaches are used. One of these approaches is to obtain guaranteed estimates of the accuracy of the gradient algorithm in terms of the curvature of the admissible domain. With this approach, it is required to find the curvatures of the admissible region. Since finding the exact value of curvature is a difficult problem to solve, curvature estimates in terms of more or less simply calculated parameters of the problem are relevant. A new improved bound for the curvature of an order-convex set is found and is presented in this paper in terms of the steepness and parameters of strict convexity of the function.

**Keywords:** gradient, estimates, curvature, convexity, algorithm

## 1. Introduction

It is well known that in discrete optimization (DO) problems, gradient (local) algorithms do not always guarantee an optimal solution (see, for example, Emelichev, Kovalev and Ramazanov, 1992; Kovalev 2003; Ramazanov, 2008, 2018). Therefore, the problem arises of finding the accuracy of the gradient algorithm (Emelichev, Kovalev and Ramazanov, 1992; Emelichev and Ramazanov, 2016; Kovalev, 2003; Hausman and Korte, 1978; Ramazanov, 2008, 2011, 2018). This is a fairly well-known problem and numerous publications have been devoted to solving it (examples being provided by the references to this paper). In establishing accuracy, various approaches are used. One of these approaches is to obtain guaranteed estimates of the accuracy of the gradient algorithm in terms of the curvature of the admissible domain (see, for example, Kovalev, 2003; Hausman and Korte, 1978; Ramazanov 2008).

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In addition, in terms of curvature, one can characterize the set of feasible solutions, as well as analyze the stability of the gradient algorithm for small perturbations of the curvature of the feasible domain. For example, in Kovalev (2003) and Ramazanov (2008), guaranteed estimates of the accuracy of the gradient algorithm in terms of the curvature of an admissible region were obtained. With this approach, it is required to find the curvatures of the admissible region. Since finding the exact value of curvature is a difficult problem to solve, curvature estimates in terms of more or less simply calculated parameters of the problem are relevant (Kovalev, 2003 and Ramazanov, 2008). In particular, the present author found in Ramazanov (2008) a lower bound for the curvature of an order-convex set.

According to the results from Kovalev (2003) and Ramazanov (2008), the more accurate the curvature estimate, the more accurate will be the guaranteed estimates. Therefore, improving the estimate of curvature remains an urgent task. A new improved bound for the curvature of an order-convex set is found in terms of the steepness and parameters of the strict convexity of the function, from the constraint conditions. As a result, in terms of the curvature of the admissible domain, an improved estimate of the accuracy of the gradient algorithm is established. Examples are considered. The newly obtained results extend, supplement and refine the previously obtained results.

## 2. Definitions and notations

Let $\mathbb{Z}_+^n$ ($\mathbb{R}_+^n$) be a set of $n-$dimensional non-negative integer-valued (real) vectors. For $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{Z}_+^n$ we write $x = (x_1, ..., x_n) \leq y = (y_1, ..., y_n)$ if $x_i \leq y_i,\ i = 1, 2, ..., n.$ If among the last inequalities at least one represents strict inequality, then we write $x < y$. Let $P \subseteq \mathbb{Z}_+^n$. In what follows we assume that the set $P$ possesses the following properties:

(i) $|P| < +\infty$;
(ii) $0 \in P$;
(iii) $[0, x] = \{z \in \mathbb{Z}_+^n : 0 \leq z \leq x\} \subseteq P$ for any $x \in P$.

A set $P$ possessing properties (i)-(iii) is referred to as a finite ordinal-convex set with zero (see, for example, Emelichev and Ramazanov, 2016; Kovalev, 2003 and Ramazanov, 2008, 2011).

We define the *maximal height* of set $P \subseteq \mathbb{Z}_+^n$ as
$h = h(P) = \max\{h(x) :\ x \in P\ \}$, where
$h(x) = h(0, x)$ is the height of its element,
$h(x, y) = \sum\limits_{i \in N(x,y)} h(x_i, y_i)$,
$h(x_i, y_i) = |\{z_i : x_i \leq z_i \leq y_i\}| - 1,\ 1 \leq i \leq n$,
$N(x, y) = \{i : x = (x_1, ..., x_n) \leq (y_1, ..., y_n) = y,\ x_i < y_i,\ \ 1 \leq i \leq n\}$.

The curvature of $P \subseteq \mathbb{Z}_+^n$ is defined as follows (see Kovalev, 2003; Ramazanov, 2008):

$$\theta(P) = \min\left\{ \quad \frac{l(P\bigcap[0,x])}{h(P\bigcap[0,x])} : x \in \mathbb{Z}_+^n, x \neq 0 \quad \right\},$$

where $l(P) = \min\{h(x) : x \in P^{\max}\}$, $P^{\max}$ is the set of all maximal elements of the partially ordered set $(P, \leq)$, which can be defined as $P^{\max} = \{x \in P : \pi_i(x) \notin P\}$, $\pi_i(x) = (x_1, ..., x_{i-1}, x_i^+ + 1, x_{i+1}, ..., x_n)$. Obviously, $0 < \theta(P) \leq 1$. If $\theta(P) = 1$, then the set $P$ is called a *supermatroid* (Kovalev, 2003; Ramazanov, 2008).

Following the works of Emelichev, Kovalev and Ramazanov (1992), Emelichev and Ramazanov (2016), Kovalev (2003), and Ramazanov (2011), for a function $f : \mathbb{Z}_+^n \to \mathbb{R}$ ($\mathbb{R}$ being the set of real numbers) we introduce the notion of the $i$-gradient

$$\Delta_i f(x) = f(\pi_i(x)) - f(x),$$

and the $(i, j)$-gradient

$$\Delta_{ij} f(x) = \Delta_j f(\pi_i(x)) - \Delta_j f(x).$$

Let $\rho = (\rho_1, ..., \rho_n) \in \mathbb{R}_+^n$ and $\Re_\rho(\mathbb{Z}_+^n)$ be the class of $\rho$-coordinate-convex functions on $\mathbb{Z}_+^n$ (see Emelichev, Kovalev and Ramazanov, 1992; Emelichev and Ramazanov, 2016; Ramazanov, 2011), that is, of functions $f : \mathbb{Z}_+^n \to \mathbb{R}$ such that for any $x \in \mathbb{Z}_+^n$

$$\begin{array}{l} \Delta_{ij} f(x) \leq 0 \ \ , \ i \neq j, \quad 1 \leq i, j \leq n, \\ \Delta_{ii} f(x) \leq -\rho_i \ , \ \ 1 \leq i \leq n. \end{array}$$

If $f(x) \in \Re_0(\mathbb{Z}_+^n)$ (that is, if $\rho = (0, ..., 0)$), then the function $f(x)$ is called *coordinate-convex* (Kovalev, 2003).

Let $fes(x, P) = \{1 \leq i \leq n : \pi_i(x) \in P\}$, $x \in P$. A function $f : \mathbb{Z}_+^n \to \mathbb{R}$ is said to be non-decreasing on the set $P \subseteq \mathbb{Z}_+^n$ for any $i \in fes(x, P)$ and $x \in P$ if the inequality $\Delta_i f(x) \geq 0$ is true (that is, the function $f(x)$ is non-decreasing along the coordinate chains).

## 3. Statement of the problem

We consider the following convex discrete optimization problem A: find

$$\max\{f(x) : x = (x_1, ..., x_n) \in P_g\},$$

where $f(x) \in \Re_\rho(\mathbb{Z}_+^n)$, $f(x)$ is a non-decreasing function on the set $P$, $P \subseteq \mathbb{Z}_+^n$−order-convex set,

$$P_g = \{x = (x_1, ..., x_n) \in P \subseteq \mathbb{Z}_+^n : g(x) \leq 0,\ g(0) \leq 0,\ -g(x) \in \Re_q(\mathbb{Z}_+^n),$$
$$q = (q_1, ..., q_n) \in \mathbb{R}_+^n\}.$$

Let $x^* = (x_1^*, ..., x_n^*)$ be an optimal solution of the problem A, that is, $f(x^*) = \max\{f(x) : x \in P_g\}$.

Let $x^g = (x_1^g, ..., x_n^g)$ be the gradient solution (the gradient maximum of the function $f(x)$ on the set $P_g$) of the problem A, that is, the point obtained by the following gradient algorithm of coordinate–wise lifting (Emelichev, Kovalev and Ramazanov, 1992; Emelichev and Ramazanov, 2016; Kovalev, 2003; Ramazanov, 2011):

$$x^{t+1} = \pi_{i(t)}(x^t),\ t = 0, 1, ...,\ \ x^0 = 0 = (0, ..., 0),$$
$$i(t) = \arg\max_i \{\Delta_i f(x^t) : i \in fes(x^t, P_g)\}.$$

The algorithm stops at step $\tau$ if either $\Delta_{i(\tau)} f(x^\tau) \leq 0$ or $fes(x^\tau, P_g) = \emptyset$.

The guaranteed (relative) estimate for the error of the gradient algorithm solving the problem A is, as usual, a number $\varepsilon \geq 0$ such that

$$\frac{f(x^*) - f(x^g)}{f(x^*) - f(0)} \leq \varepsilon.$$

The primary objective of this paper is to find new guaranteed errors of the gradient coordinate-wise lifting algorithm for the problem A in the parameters of the problem.

## 4. The main result

We start with formulating one result known earlier.

Theorem 4.1 *(Ramazanov, 2008) If $-g(x) \in \Re_q(\mathbb{Z}_+^n)$ is a non-decreasing function, then the set $P_g$ is order-convex.*

Definition 4.2 *(Emelichev and Ramazanov, 2016; Ramazanov, 2011, 2018) Let $-g(x) \in \Re_q(\mathbb{Z}_+^n)$ be a non-decreasing function on a set $P \subseteq \mathbb{Z}_+^n$. The steepness of the function $g(x)$ on the set $P \subseteq \mathbb{Z}_+^n$ is*

$$c(g) = \min_i \left\{ \frac{q_i}{\Delta_i g(\pi_i(x)} : \Delta_i g(\pi_i(x)) > 0,\ i \in fes(\pi_i(x)) \right\}.$$

Theorem 4.3 *Suppose that $-g(x) \in \Re_q(\mathbb{Z}_+^n)$ is a non-decreasing function on a set $P \subseteq \mathbb{Z}_+^n$ and $\Omega(q) = \sum_{i=1}^n q_i > 0$. Then*

$$\theta(P_g) \geq Q = \frac{c(g)}{q^* - c(g)q_*},$$

*where* $q_* = \min_i \{q_i : 1 \le i \le n\}$, $q^* = \max_i \{q_i : 1 \le i \le n\}$.

PROOF From the inclusion

$$-g(x) \in \Re_q(\mathbb{Z}^n_+)$$

it is easy to see that

$$\Delta_i g(\pi_i(x)) - \Delta_i g(x) \ge q_i,\ i \in fes(x, P).$$

Then

$$c(g) \le \frac{q_i}{\Delta_i g(\pi_i(x))} \le \frac{q_i}{\Delta_i g(x) + q_i},$$

$$\Delta_i g(x) \le \frac{q_i - c(g)q_i}{c(g)} \le \frac{q^* - c(g)q_*}{c(g)}, \forall i \in fes(x, P_g).$$

Furthermore, we use the statement of Theorem 5.2 from Kovalev (2003), which in our notation, can be reformulated as

$$\theta(P_g) \ge \frac{1}{\Delta_i g(x)}, \forall x \in P_g, i \in fes(x, P_g).$$

Combining this with the previous inequality, we complete the proof of Theorem 4.2. □

THEOREM 4.4 *The gradient algorithm for Problem A has the guaranteed error estimate*

$$\frac{f(x^*) - f(x^g)}{f(x^*) - f(0)} \le B(c, q) = B,$$

*where*

$$B(c, q) = \frac{Q}{1 + Q},$$

$$Q = \begin{cases} Q^0 \ \mathit{if}\, Q^0 > 1, \\ \frac{1}{Q^0} \ \mathit{if}\, 0 < Q^0 \le 1, \end{cases}$$

$$Q^0 = \frac{c(g)}{q^* - c(g)q_*}.$$

In order to prove the theorem, we need a lemma.

Lemma 4.5 *(Kovalev, 2003; Ramazanov, 2008) The gradient algorithm for Problem A has the guaranteed error estimate*

$$\frac{f(x^*)-f(x^g)}{f(x^*)-f(0)} \leq \frac{1}{1+\theta(P_g)} = B_1.$$

Proof of Theorem 4.3. From Theorem 4.2 and Lemma 4.1 it follows that

$$\frac{f(x^*)-f(x^g)}{f(x^*)-f(0)} \leq \frac{1}{1+\theta(P)} \leq \frac{1}{1+1/Q} = B(c,g).$$

Theorem 4.3 has been proven. □

## 5. Corollaries and examples

Corollary 5.1 *Under the conditions of Theorem 4.2, if $Q = 1$, then the set $P_g$ is a supermatroid.*

Existence of problems with $Q = 1$ is shown in Example 5.1.

In Ramazanov (2008) (Theorem 4) it is proven that

$$\theta(P_g) \geq \frac{1}{\Omega^*(q,h)} = L$$

where

$$\Omega^*(q,h) = \begin{cases} \Omega^0(q,h) \text{ if } \Omega^0(q,h) > 1, \\ \frac{1}{\Omega^0(q,h)} \text{ if } 0 < \Omega^0(q,h) \leq 1, \end{cases}$$

$$\Omega^0 = \Omega^0(q,h) = (h(P)+\frac{1}{2})\Omega(q\ ),\ h = h(P), \quad \Omega(q\ ) = \sum_{i=1}^{n} q_i.$$

Corollary 5.2 *If $c(g) \geq \frac{2}{2h+1}$, then the estimate of the curvature of an order-convex set is improvable.*

For this we show the correctness of the inequality

$$Q = \frac{c(g)}{\rho^* - c(g)\rho_*} \geq \frac{2}{(2h+1)\Omega(\rho)} = L.$$

From the chain of inequalities we get

$$\rho^* - c(g)\rho_* \leq \rho^* \leq \Omega(\rho), \quad Q = \frac{c(g)}{\rho^* - c(g)\rho_*} \geq \frac{2}{(2h+1)\Omega(\rho)} = L.$$

Corollary 5.3 *The guaranteed estimates of the accuracy of the gradient algorithm, given in Theorem 4 from Ramazanov (2008) are improvable. In order to illustrate this we construct the following example.*

Example 5.4 *Let*

$$g(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_1 + \frac{1}{2}x_2^2 + \frac{1}{2}x_2 - 4,$$

*then*

$$P_g = \{(0,0),(1,0),(2,0),(0,1),(0,2)\}$$

*and the gradients are defined as follows:*

$$\begin{aligned}
&\Delta_1 g(x) = x_1 + 1,\ \Delta_2 g(x) = x_2 + 1, q = (1,1),\\
&\Delta_1 g(\pi_1(x)) = x_1 + 2,\\
&\Delta_2 g(\pi_2(x)) = x_2 + 2, h = 2,\\
&c(g) = \min\{1/2,\ 1/2\} = 1/2,\ Q = 1, L = 1/5.
\end{aligned}$$

*From here we have* $B = 1/2,\ B_1 = 5/6,\ B < B_1$.

## 6. Conclusion

The class of discrete optimization problems, where the local (gradient) extremum coincides with the global one is rather narrow. Due to high computational complexity of discrete optimization problems, in many real-life cases only approximate algorithms can be used efficiently. Therefore, finding the accuracy of gradient-type algorithms is one of the intensively investigated and current directions of modern discrete optimization theory. One option for analyzing the accuracy is to obtain an estimate in terms of the curvature of the feasible region. In this work we derived an easily calculated estimate for the curvature, which allows for finding new improved estimates for the accuracy of gradient algorithms. The results of this paper refine and improve the previously known results.

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