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APPROXIMATION OF THE OFFSET CURVES IN THE FORMATION OF TURBO ROUNDABOUTS

Anna BOROWSKA

Bialystok University of Technology, Faculty of Computer Science
ul. Wiejska 45A, 15-351 Bialystok, POLAND
e-mail: a.borowska@pb.edu.pl

Abstract. This article describes the results of numerical analysis of the following problem: the approximation of the offset curve $off(c(t); s)$ of the Archimedean spiral $c(t)$ at distance s by the other Archimedean spiral $c_1(t)$. We focus on the spirals which can be used to shape the turbo roundabouts. The paper contains mathematical facts which are helpful in designing the turbo roundabouts formed by means of the Archimedean spiral.

Keywords: Archimedean spiral, offset curve, turbo roundabout

1 Introduction

This article is a continuation of research from [1], [3] and [4]. In [4] it was suggested to analyze the properties of Archimedean spiral (see Figure 2) which can be used to shape the turbo roundabouts. A large safety coefficient is the advantage of such roundabouts, because the tracks of vehicles on the turbo roundabouts do not intersect. Statistics show a much higher capacity of such roundabouts in comparison to regular intersections (up to 20%) (cf. [3]). Additionally, if one traffic direction is dominant in terms of the intensity, specialists recommend the turbo roundabout with the elliptical central island (Figure 1).

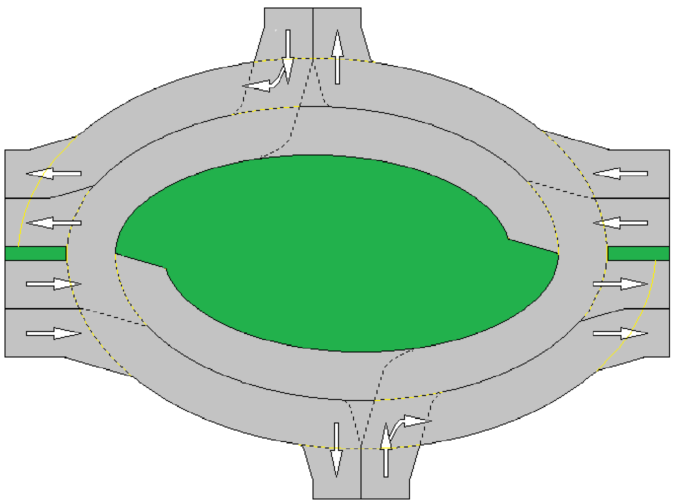

Figure 1: The typical elliptical turbo roundabout (cf. [1])

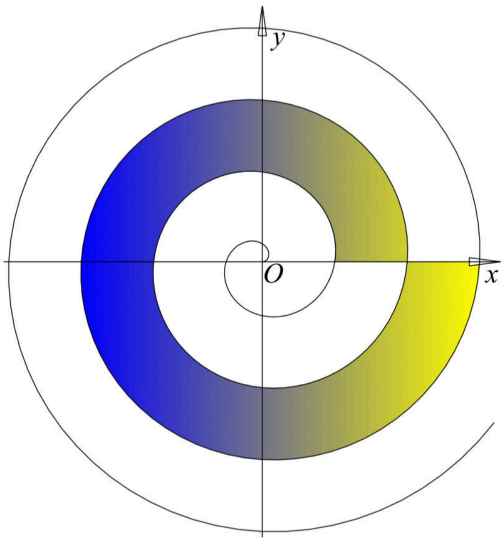

Figure 2: Archimedean spiral with one shaded coil

Section 2 contains mathematical facts necessary to describe research results. Section 3 contains results of the numerical analysis of Archimedean spiral properties. We considered this curve as a tool that can be used to shape the turbo roundabouts. The starting point $P_{t_s}(x_{t_s}, y_{t_s})$ (the angle t_s) of the spiral from which we can start delineating the central island was determined. It has been tested for which parameter values s (the width of the lane) and t_s

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(the angle corresponding to the starting point P_{t_s}) the approximation of the offset curve *off*(*c*(*t*); s) of the Archimedean spiral *c*(*t*) by the spiral $c_1(t)=c(t+2\pi)$ is satisfactory, i.e. for which parameter values *s* and t_s the maximum deviation between the two curves *off*(*c*(*t*); *s*) and $c_1(t)$ does not exceed the accuracy of the delineating the curves in the terrain. Section 4 includes the algorithm which assesses the possibility of approximating an ellipse offset curve by another ellipse.

2 Mathematical formulas

Let $c(t)=(x(t),y(t))$ $(t\in[\alpha,\beta])$ be a parametric representation of a planar curve (functions $x(t)$, $y(t)$ will be described as x_t, y_t). The *curvature of the curve* *c*(*t*) at the point $P(x_t,y_t)$ is determined by the formula (cf. [8], p. 225)

$$\kappa=\frac{|x_t' y_t'' - y_t' x_t''|}{\sqrt{(x_t'^2+y_t'^2)^3}} \tag{1}$$

The *arc length of the smooth curve* *c*(*t*) (where functions $x(t)$, $y(t)$ have continuous derivatives on $[\alpha,\beta]$) is expressed by the formula (cf. [8], p. 317)

$$L=\int_\alpha^\beta \sqrt{x_t'^2+y_t'^2}\,dt \tag{2}$$

Let us assume that the curve $c(t)=(x(t),y(t))$, $t\in[\alpha,\beta]$ fulfills the conditions 1-2 (1. functions $x(t)$, $y(t)$ are continuous on $[\alpha,\beta]$; 2. the function $x(t)$ is increasing and has a continuous derivative on $[\alpha,\beta]$). The area bounded by the arc of the curve *c*(*t*), the *x*-axis, and two lines $x_1=x(\alpha)$, $x_2=x(\beta)$ is given by the formula (3a) (cf. [2], p. 179). If above conditions are fulfilled and the function $x(t)$ is decreasing on $[\alpha,\beta]$ then the defined area is given by the formula (3b).

$$\text{(a) } P=\int_\alpha^\beta |y_t|\,x_t'\,dt \quad \text{(b) } P=-\int_\alpha^\beta |y_t|\,x_t'\,dt \tag{3}$$

The *normal vector* to the curve *c*(*t*) at the point $P(x_t,y_t)$ is as follows $\mathrm{n}=[-y_t',x_t']$ (cf. [5]). The *unit normal vector* at the point $P(x_t,y_t)$ is defined as (cf. [9], p. 335, [7])

$$\mathrm{n}_{\mathrm{ver}}=\frac{[-y_t',x_t']}{\sqrt{(x_t')^2+(y_t')^2}}.$$

Definition 1 (an *offset curve* c_d *at distance d*) (cf. [9], p. 335, [6]):

For a smooth planar curve *c*, we define an *offset curve* c_d *at distance d* in the following way. On each curve normal, we mark the two points that are at distance *d* from the curve *c*. The set of all of these points forms the offset $c_d=(c_d'\cup c_d'')$.

The offset $c_d(t)$ at distance *d* to the curve *c*(*t*) is obtained as $c_d(t)=c(t)\pm d\mathrm{n}_{\mathrm{ver}}(t)$.

The curve *c* and its offset curves c_d' and c_d'' are seldom of the same type. The offset curves of the circle are circles and the ellipse offset curves are not ellipses. The interesting offset surfaces (offset curves) are described in [6], [7] and [3].

Lemma 1:

$$\int t\sin(2t)dt=-(1/2)t\cos(2t)+(1/4)\sin(2t)+C, \tag{4}$$

$$\int t^2\sin^2(t)dt=(t^3/6)-(t/4)\cos(2t)+((1-2t^2)/8)\sin(2t)+C. \tag{5}$$

Justification. For checking, it is enough to determine the derivative of the result.

3 Archimedean spiral

Presented below facts can be helpful in designing the turbo roundabouts formed by using the Archimedean spiral. The *Archimedean spiral* is a curve outlined by point P moving with the constant speed v along the half-line, which has the origin in point O (the origin of the coordinate system), and turns around it at a constant angular velocity ω (cf. [4]). The parametric equations of the Archimedean spiral are of the form (cf. [8], p. 229)

$$x = at\cos(t)\,,\ \ y = at\sin(t)\,,\ \text{where } a>0,\ t \ge 0\,. \tag{6}$$

The distance $r = |\overline{OP}|$ is proportional to the angle of rotation t ($r=at$), a=const.

The Archimedean spiral can be used to shape the turbo roundabout because the length of the segment lying on any radius between the two nearest branches of the spiral is fixed and equals $2\pi a$. The existence of a simple method for delineating the spiral in the terrain is also important (cf. [4]).

The unit normal vector to the Archimedean spiral $c(t)$ at the point $P(x_t, y_t)$ is of the form

$$\mathrm{n}_{\mathrm{ver}} = \frac{[-(\sin(t)+t\cos(t)),(\cos(t)-t\sin(t))]}{\sqrt{1+t^2}}.$$

The equation of the offset curves of the Archimedean spiral is as follows

$$off\,(c(t);s):[X,Y]=[x_t, y_t]\pm\frac{s[-(\sin(t)+t\cos(t)),(\cos(t)-t\sin(t))]}{\sqrt{1+t^2}}. \tag{7}$$

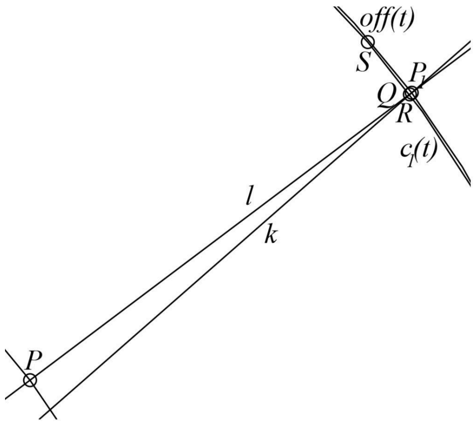
Assumption 1. We assume in the whole section 3 that the distance between the branches of the spiral is s (the width of the lane), hence $a = s/(2\pi)$. All calculations were made for the following widths of the lane s=3.5, 4, 4.5, 5, 5.5 (for roundabouts approved in [10]).

3.1 Approximation of the Archimedean spiral offset curve

Offset curves are used to shape the turbo roundabouts. We checked the possibility of approximating the Archimedean spiral offset curve *off*(c(t); s) by the other Archimedean spiral $c_1(t)$ defined as follows

$$x = a(t+2\pi)\cos(t)\,,\ \ y = a(t+2\pi)\sin(t)\,,\ \ a = s/2\pi \tag{8}$$

Let us assume that P is any point of the spiral $c(t)$ (i.e. for any fixed angle t) and l is the normal line to $c(t)$ at the point P. Point P_1 lies on the normal line l at distance s from P. Q is the intersection point of the normal l with the curve $c_1(t)$. S is the point of the spiral $c_1(t)$ for the angle t (i.e. the point of the spiral $c(t)$ for the angle $t+2\pi$). (see Figure 3)



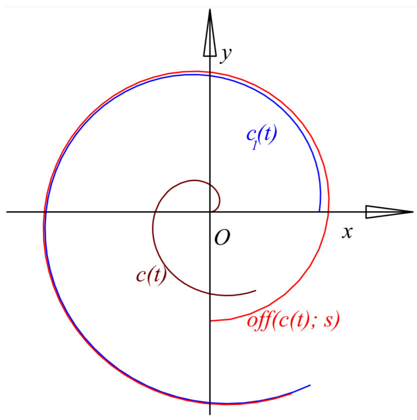


Figure 3: The arrangement of points P, P_1, S, Q, R and lines l and k

Figure 4: The Archimedean spiral $c(t)$ for $t\in[0, 5\pi/3]$, the offset *off*(c(t), s), the Archimedean spiral $c_1(t)$ for $t\in[0, 5\pi/3]$

Fact 1 (cf. [3]). The approximation of the offset curve *off*(c(t); s) by the curve $c_1(t)$ is satisfactory if for any point P of the curve $c(t)$ there is $d_{P_1Q} \le k$ for k=0.01, i.e. when the deviation d_{P_1Q} does not exceed the accuracy of the delineating the curves in the terrain.

Let us draw a straight line k passing through the origin of the coordinate system O and the point P_1. Let R be the intersection point of the line k and the spiral $c_1(t)$. Let the angle φ correspond to the point R of the curve $c_1(t)$. $r = |\overline{OP_1}|$. α is the angle between the x-axis and the line k. $E(x)$ represents the floor function.

$$r = \sqrt{x_{P_1}^2 + y_{P_1}^2},\ \alpha = \begin{cases} \arccos(x_{P_1}/r) & for\ y_{P_1} \ge 0 \\ -\arccos(x_{P_1}/r) & for\ y_{P_1} < 0 \end{cases},\ w = E((t-\alpha)/180^\circ),\ \varphi = (w+2)\pi + \alpha$$

Table 1 contains the following distances between the point P_1 and the curve $c_1(t)$: $d_1 = |\overline{P_1S}|$, $d_2 = |\overline{P_1R}|$, d_3 – the smallest distance between the point P_1 and the curve $c_1(t)$. λ denotes the angle corresponding to the point F (of the curve $c_1(t)$) such that $d_3 = |\overline{P_1F}|$. Calculations were made for the following lane widths s=3.5, 4, 4.5, 5, 5.5.

Table 1: Distances between the point P_1 of the offset curve *off*($c(t)$; s) and the curve $c_1(t)$. $d_1 = |\overline{P_1S}|$, $d_2 = |\overline{P_1R}|$, d_3 – the smallest distance between the point P_1 and the curve $c_1(t)$

	t=0° t_1=360° φ=270° λ=273°	t=90° t_1=450° φ=423°50'42'' λ=424°5'42''	t=180° t_1=540° φ=528°12'29'' $\lambda\approx\varphi$	t=270° t_1=630° φ=623°9'4'' $\lambda\approx\varphi$	t=360° t_1=720° φ=715°28'41'' $\lambda\approx\varphi$	t=450° t_1=810° φ=806°46'34'' $\lambda\approx\varphi$	t=540° t_1=900° φ=897°34'41'' $\lambda\approx\varphi$	t=630° t_1=990° φ=988°6'40'' $\lambda\approx\varphi$	t=720° t_1=1080° φ=1078°29'2'' $\lambda\approx\varphi$
s=3.5	d_1=4.949747 d_2=0.870 d_3=0.860753	d_1=1.957726 d_2=0.143371 d_3=0.142123	d_1=1.074329 d_2=0.059393 $d_3\approx d_2$	d_1=0.730534 d_2=0.033828 $d_3\approx d_2$	d_1=0.551836 d_2=0.022177 $d_3\approx d_2$	d_1=0.442953 d_2=0.015761 $d_3\approx d_2$	d_1=0.369805 d_2=0.011818 $d_3\approx d_2$	d_1=0.317328 d_2=0.009208 $d_3\approx d_2$	d_1=0.277862 d_2=0.007386 $d_3\approx d_2$
s=4	d_1=5.656854 d_2=1 d_3=0.983718	d_1=2.237401 d_2=0.163853 d_3=0.162427	d_1=1.227804 d_2=0.067878 $d_3\approx d_2$	d_1=0.834895 d_2=0.038661 $d_3\approx d_2$	d_1=0.630670 d_2=0.025341 $d_3\approx d_2$	d_1=0.506232 d_2=0.018012 $d_3\approx d_2$	d_1=0.422634 d_2=0.013507 $d_3\approx d_2$	d_1=0.362660 d_2=0.010524 $d_3\approx d_2$	d_1=0.317557 d_2=0.008441 $d_3\approx d_2$
s=4.5	d_1=6.363961 d_2=1.125 d_3=1.106683	d_1=2.517076 d_2=0.184335 d_3=0.182730	d_1=1.381280 d_2=0.076363 $d_3\approx d_2$	d_1=0.939257 d_2=0.043493 $d_3\approx d_2$	d_1=0.709503 d_2=0.028509 $d_3\approx d_2$	d_1=0.569511 d_2=0.020264 $d_3\approx d_2$	d_1=0.475464 d_2=0.015195 $d_3\approx d_2$	d_1=0.407993 d_2=0.011839 $d_3\approx d_2$	d_1=0.357252 d_2=0.009496 $d_3\approx d_2$
s=5	d_1=7.071068 d_2=1.25 d_3=1.229647	d_1=2.796752 d_2=0.204816 d_3=0.203034	d_1=1.534755 d_2=0.084847 $d_3\approx d_2$	d_1=1.043619 d_2=0.048326 $d_3\approx d_2$	d_1=0.788337 d_2=0.031677 $d_3\approx d_2$	d_1=0.632790 d_2=0.022515 $d_3\approx d_2$	d_1=0.528293 d_2=0.016883 $d_3\approx d_2$	d_1=0.453325 d_2=0.013155 $d_3\approx d_2$	d_1=0.396946 d_2=0.010551 $d_3\approx d_2$
s=5.5	d_1=7.778174 d_2=1.378976 d_3=1.352612	d_1=3.076427 d_2=0.225298 d_3=0.223337	d_1=1.688231 d_2=0.093332 $d_3\approx d_2$	d_1=1.147981 d_2=0.053158 $d_3\approx d_2$	d_1=0.867171 d_2=0.034844 $d_3\approx d_2$	d_1=0.696068 d_2=0.024767 $d_3\approx d_2$	d_1=0.581122 d_2=0.018572 $d_3\approx d_2$	d_1=0.498658 d_2=0.014470 $d_3\approx d_2$	d_1=0.436641 d_2=0.011606 $d_3\approx d_2$

For the established angle t (for all analyzed values s) we have the same values t_1 (=$t+2\pi$), φ and λ.

Starting from t=134°30' we can assume that the smallest distance between the point P_1 and the curve $c_1(t)$ is $d_2 = |\overline{P_1R}| \approx d_3$. This property was checked for s=0.5, …, 6000. Therefore, the problem whether the approximation of the offset curve *off*($c(t)$; s) by the curve $c_1(t)$ is satisfactory can be reduced to checking the condition $d_2 = d_{P_1R} \le 0.01$.

Table 2: Data for the Archimedean spiral $c(t)$ (for $a=s/2\pi$, where s=3.5, 4, 4.5, 5, 5.5): the smallest angle φ such that for $\varphi_1 \ge \varphi$ $d_2(\varphi_1) \le 0.01$, the angle t (for the point $P(x_t, y_t)$) corresponding to φ and the distance d_2

	s=3.5	s=4	s=4.5	s=5	s=5.5
t	599°	650°	698°	744°	787°
$t_1=t+2\pi$	959°	1010°	1058°	1104°	1147°
φ	956°	1008°12'18''	1056°24'13''	1102°33'52''	1145°41'36''
d_2	0.009998306	0.009998113	0.009999058	0.009988815	0.009979469

3.2 The size of the central island

The diameter of the smallest central island of the circular roundabout in the build-up area (approved in [10]) equals 17m.

Let us assume that P_t is any point of the spiral $c(t)$ (for the established angle t), $r_t = |\overline{OP_t}|$ is the radius, $p_t = |\overline{P_t P_{t+\pi}}|$ is the segment passing through the origin of the coordinate system O ($p_t = r_t + r_{t+\pi}$), γ is the smallest angle such that the length of the segment $p_\gamma = |\overline{P_\gamma P_{\gamma+\pi}}| \geq 17$, γ_1 - the angle γ which is rounded to full π.

Example 1. We need to determine the segment $p_{\gamma_1} = |\overline{P_{\gamma_1} P_{\gamma_1+\pi}}|$ for the Archimedean spiral $c(t)$ (for $a=s/2\pi$, where $s=3.5$). We have $r_{4\pi} = a4\pi = 7$, $r_{5\pi} = a5\pi = 8.75$, $r_{6\pi} = a6\pi = 10.5$. $p_{4\pi} = r_{4\pi} + r_{5\pi} = 7 + 8.75 = 15.75$, $p_{5\pi} = r_{5\pi} + r_{6\pi} = 8.75 + 10.5 = 19.25$, $\gamma_1 = 5\pi$.

Table 3: Data for the Archimedean spiral $c(t)$ (for $a=s/2\pi$, where $s=3.5$, 4, 4.5, 5, 5.5): the angle γ, the length of the segment p_γ, the angle γ_1 (the angle γ which is rounded to full π)

	$s=3.5$	$s=4$	$s=4.5$	$s=5$	$s=5.5$
γ	785°=4π+65°	675°=3π+135°	590°=3π+50°	522°=2π+162°	467°=2π+107°
p_γ	17.01388888	17.0	17.0	17.0	17.01944444
γ_1	5π	4π	4π	3π	3π

For the Archimedean spiral $c(t)$ for $t > \gamma$ each segment $p_t = |\overline{P_t P_{t+\pi}}| \geq 17$.

3.3 Curvature of the curve *c*(*t*)

The curvature of the Archimedean spiral $c(t)$ at the point $P(x_t, y_t)$ is defined as (cf. (1), [8])

$$\kappa = \frac{2+t^2}{|a|\sqrt{(1+t^2)^3}}.$$

The curvature of the Archimedean spiral $c(t)$ (for $t \in [\gamma_1, \gamma_1 + 2\pi]$ (cf. Table 3)) for the width of the lane $s=3.5$, 4, 4.5, 5, 5.5 ($a=s/2\pi$) was determined. The points of the spiral were taken (every $\pi/4$).

Table 4: The curvature of the edge of the smallest central island of the turbo roundabout

	The angle t (point P_t) of the spiral and the curvature of the spiral at the point P_t								
$s=3.5$	$t=5\pi$ $\kappa=0.114515$	$t=5\pi+\pi/4$ $\kappa=0.109042$	$t=5\pi+\pi/2$ $\kappa=0.104069$	$t=5\pi+3\pi/4$ $\kappa=0.099530$	$t=6\pi$ $\kappa=0.095371$	$t=6\pi+\pi/4$ $\kappa=0.091546$	$t=6\pi+\pi/2$ $\kappa=0.088017$	$t=6\pi+3\pi/4$ $\kappa=0.084750$	$t=7\pi$ $\kappa=0.081717$
$s=4$	$t=4\pi$ $\kappa=0.125390$	$t=4\pi+\pi/4$ $\kappa=0.117973$	$t=4\pi+\pi/2$ $\kappa=0.111386$	$t=4\pi+3\pi/4$ $\kappa=0.105497$	$t=5\pi$ $\kappa=0.100201$	$t=5\pi+\pi/4$ $\kappa=0.095412$	$t=5\pi+\pi/2$ $\kappa=0.091060$	$t=5\pi+3\pi/4$ $\kappa=0.087089$	$t=6\pi$ $\kappa=0.083450$
$s=4.5$	$t=4\pi$ $\kappa=0.111458$	$t=4\pi+\pi/4$ $\kappa=0.104865$	$t=4\pi+\pi/2$ $\kappa=0.099010$	$t=4\pi+3\pi/4$ $\kappa=0.093775$	$t=5\pi$ $\kappa=0.089067$	$t=5\pi+\pi/4$ $\kappa=0.084810$	$t=5\pi+\pi/2$ $\kappa=0.080942$	$t=5\pi+3\pi/4$ $\kappa=0.077412$	$t=6\pi$ $\kappa=0.074178$
$s=5$	$t=3\pi$ $\kappa=0.134065$	$t=3\pi+\pi/4$ $\kappa=0.123655$	$t=3\pi+\pi/2$ $\kappa=0.114750$	$t=3\pi+3\pi/4$ $\kappa=0.107045$	$t=4\pi$ $\kappa=0.100312$	$t=4\pi+\pi/4$ $\kappa=0.094378$	$t=4\pi+\pi/2$ $\kappa=0.089109$	$t=4\pi+3\pi/4$ $\kappa=0.084398$	$t=5\pi$ $\kappa=0.080161$
$s=5.5$	$t=3\pi$ $\kappa=0.121877$	$t=3\pi+\pi/4$ $\kappa=0.112413$	$t=3\pi+\pi/2$ $\kappa=0.104318$	$t=3\pi+3\pi/4$ $\kappa=0.097313$	$t=4\pi$ $\kappa=0.091193$	$t=4\pi+\pi/4$ $\kappa=0.085798$	$t=4\pi+\pi/2$ $\kappa=0.081008$	$t=4\pi+3\pi/4$ $\kappa=0.076725$	$t=5\pi$ $\kappa=0.072873$

The curvature of the edge of the smallest central island (with diameter d=17m) of the circular roundabout in the build-up area (approved in [10]) at any point is equal κ_c=0.117647058.

In [4] it was assumed that *the ellipse forming the interior traffic lane of a turbo roundabout cannot, at any point, be shaped by a radius smaller than the minimal admissible radius destined for a circular roundabout*. Therefore, it should be assumed that the curvature of a fragment of a spiral that shapes the central island of the roundabout, at no point, can be greater than κ_c.

Based on Tables 2, 3 and 4, we can determine (suggest) the starting point P_{t_s} (the angle t_s) such that the fragment of the spiral forming the central island (which has the smallest size approved in [10]) can be plotted starting from this point.

Table 5: The suggested angle t_s (for the starting point P_{t_s}) for the smallest central island (approved in [10])

	s=3.5	s=4	s=4.5	s=5	s=5.5
t_s	5π	5π	4π	4π	4π

Example 2. The starting point P_{t_s} (the angle t_s) for the smallest central island (approved in [10]) shaped by the Archimedean spiral (for $a=s/2\pi$, where s=3.5) will be determined.
(a) The smallest angle γ_1 such that the segment $p_{\gamma_1} =|\overline{P_{\gamma_1} P_{\gamma_1+\pi}}|\geq 17$ equals $\gamma_1=5\pi$ (Table 3). Therefore, we can assume that $t_s=5\pi$. (b) The curvature of the spiral $c(t)$ at the point $P(x_t, y_t)$ for $t=5\pi$ equals κ=0.114515<κ_c (Table 4). (c) The approximation of the curve $off(c(t); s)$ by the spiral $c_1(t)$ is satisfactory starting from the angle t=599°=3*180°+59° (Table 2). (d) Therefore, for s=3.5 we can establish $t_s=5\pi$.

3.4 The arc length of the Archimedean spiral

The arc length of the Archimedean spiral $c(t)$ is expressed by the formula (cf. (2))

$$L = a\int_{\alpha}^{\beta}\sqrt{(\cos(t)-t\sin(t))^2+(\sin(t)+t\cos(t))^2}\,dt = a\int_{\alpha}^{\beta}\sqrt{t^2+1}\,dt =$$

$$=\frac{1}{2}a\left[t\sqrt{t^2+1}+\ln(t+\sqrt{t^2+1})\right]_{\alpha}^{\beta}$$

In order to determine the length of the fragment of the spiral $c(t)$ shaping the roundabout (the edge of the central island, the axis of the road and the outer edge of the roundabout) we must sum up the lengths of 3 coils of the spiral $c(t)$ starting from the starting point P_{t_s} (the angle t_s) (cf. Table 5). The knowledge of the length of the edge of the roundabout is necessary, among other things, to order a suitable amount of building materials (e.g. curb).

3.5 Area of two-lane road of the roundabout

Let the Archimedean spiral $c(t)$ be defined by the parametric equations (6). By virtue of formula (3), we obtain the following formula for the area of the fragment of the spiral coil (a) for $y_t \geq 0$, (b) for $y_t < 0$.

$$|P|=\begin{cases} -\int_{\alpha}^{\beta}|y_t|\,x_t'dt = a^2\left(\int_{\alpha}^{\beta}t^2\sin^2(t)dt-(1/2)\int_{\alpha}^{\beta}t\sin(2t)dt\right) & for\ \alpha=k\pi, \beta=(k+1)\pi, k \bmod 2=0 \\ \int_{\alpha}^{\beta}|y_t|\,x_t'dt = a^2\left(\int_{\alpha}^{\beta}t^2\sin^2(t)dt-(1/2)\int_{\alpha}^{\beta}t\sin(2t)dt\right) & for\ \alpha=k\pi, \beta=(k+1)\pi, k \bmod 2=1 \end{cases}$$

$|P|=a^2\left[(t^3/6)-(t^2/4)\sin(2t)\right]_{\alpha}^{\beta}$ (cf. Lemma 1).

Example 3. $|P_{(0,\pi)}|=a^2\pi^3/6$, $|P_{(0,2\pi)}|=(2^3-1^3)\,a^2\pi^3/6+(1^3-0^3)\,a^2\pi^3/6=4\,a^2\pi^3/3$.

In order to determine the area of two-lane road of the turbo roundabout shaped by the spiral $c(t)$ we must sum up the areas of 2 coils of the spiral. The knowledge of the area of two-lane road of the roundabout is necessary, among other things, to order a suitable amount of building materials (e.g. asphalt).

4 Algorithm

The algorithm presented below checks the possibility of the approximation of the ellipse offset curve *off*(*el*(*t*); *s*) by the ellipse $el_1(t)$, when the acceptable deviation *k* between these two curves is given. The ellipses *el*(*t*) and $el_1(t)$ are defined as follows

$$el(t)\ \ x = a\cos(t)\,,\ \ y = b\sin(t)\,, t\in[0,2\pi]\,,$$

$$el_1(t)\ \ x = (a+s)\cos(t)\,,\ \ y = (b+s)\sin(t)\,,\ t\in[0,2\pi]\,.$$

Let us assume that *P* is any point of the ellipse *el*(*t*), *l* is the normal line to *el*(*t*) at the point *P*. Point P_1 lies on the normal line *l* at distance *s* from *P*. Q_1 is the intersection point of the normal *l* with the ellipse $el_1(t)$. Point D_1 lies on the line *l* at distance *k* from P_1. Point B_1 belongs to the ellipse $el_1(t)$ and has the same coordinate *x* as point D_1 (see Figures 5 and 6).
We shall focus on the ellipse fragment for $t\in[0,\pi/2]$. The following equations are true (see Figures 5 and 6)

$$\frac{s}{s-k}=\frac{x_{P_1}-x_P}{x_{D_1}-x_P}\ \text{ and }\ \frac{k}{s}=\frac{y_{P_1}-y_{D_1}}{y_{P_1}-y_P}\,,\ \text{where}\ \begin{cases}x_P=a\cos(t)\\ y_P=b\sin(t)\end{cases},$$

$$\begin{cases}x_{P_1}=x_P+\dfrac{s}{\sqrt{1+v}}\\ y_{P_1}=y_P+s\sqrt{\dfrac{v}{1+v}}\end{cases},\ v=\left(\frac{a}{b}\tan(t)\right)^2.\ \text{Hence}\ \begin{cases}x_{D_1}=\dfrac{s-k}{s}(x_{P_1}-x_P)+x_P\\ y_{D_1}=y_{P_1}-\dfrac{k}{s}(y_{P_1}-y_P)\end{cases}.$$

We shall determine such angle φ that $x_{D_1}=x_{B_1}=(a+s)\cos(\varphi)$ (see point B_1 Figure 5). From the above calculations and the assumption $t\in[0,\pi/2]$ we have $\varphi=\arccos(x_{D_1}/(a+s))$.

If $y_{B_1}=(b+s)\sin(\varphi)\geq y_{D_1}$ then $d_{P_1Q_1}=|\overline{P_1Q_1}|\leq k$, otherwise $d_{P_1Q_1}>k$ (Figures 5 and 6).

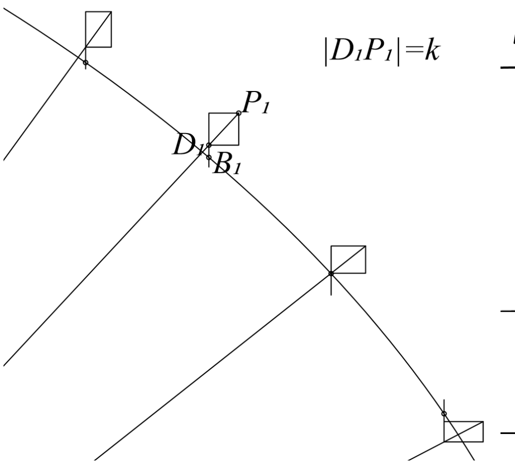


Figure 5: The arrangement of points and curves

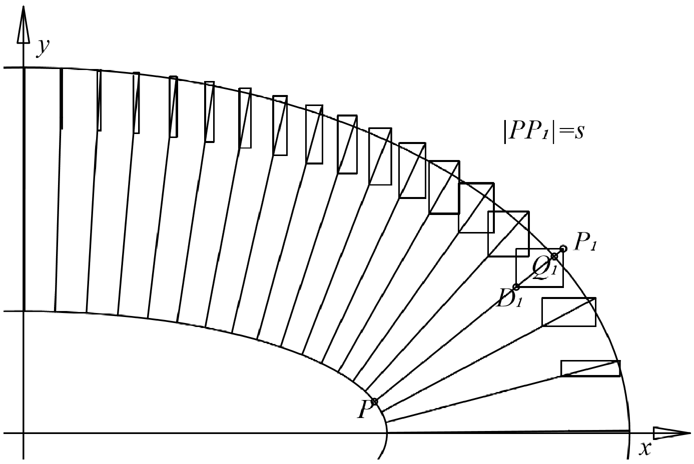


Figure 6: The arrangement of points and curves

The following algorithm checks the condition $d_{P_1Q_1}\leq k$ for consecutive points $P(x_t, y_t)$ (the angle *t* is taken (every *dt*)) using the method described above.

```
void ellipse::Algorithm(double dt, double s, double k){
//definitions of variables
t=dt;
while(t<π/2){
  v=((a/b)*tan(t))²;
  xP=a*cos(t);                                  yP=b*sin(t);
  xP1=xP+s/√(1+v);                              yP1=yP+s*√(v/(1+v));
  xD1=(s-k)*(xP1-xP)/s+xP;           yD1=yP1-k*(yP1-yP)/s;
  fi=arccos(xD1/(a+s));                    yB1=(b+s)*sin(fi);
  if(yB1>=yD1)cout<< t*180/π<< "+"; else cout<< t*180/π << "-");
```

```
  t+=dt;}
```

Results of the algorithm:

```
Data:        a=60, b=20, s=40, k=10, dt=p/36 (see Figure 6)
Result: 5+ 10+ 15+ 20+ 25+ 30+ 35+ 40+ 45+ 50+ 55+ 60+ 65+ 70+ 75+ 80+ 85+ 90+
Data:        a=20.5, b=17, s=3.5, k=0.01, dt=p/36 (see Figure 5)
Result: 5+ 10+ 15+ 20+ 25+ 30- 35- 40- 45- 50- 55- 60+ 65+ 70+ 75+ 80+ 85+ 90+
```

The method presented above (after a proper modification) can be used to analyze the approximation problem for other parametric curves.

5 Conclusions

The possibility of approximating the offset curve $off(c(t); s)$ of the Archimedean spiral $c(t)$ at distance s by the other Archimedean spiral $c_1(t)$ was numerically analyzed. The spirals that can be used to shape the turbo roundabouts were considered. This article focuses on presenting mathematical facts which are helpful in designing the turbo roundabouts formed by means of the Archimedean spirals.

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APROKSYMACJA KRZYWYCH OFFSETOWYCH W KSZTAŁTOWANIU ROND TURBINOWYCH

W pracy przedstawione są rezultaty analizy numerycznej dla problemu aproksymacji krzywej offsetowej $off(c(t); s)$ spirali Archimedesa $c(t)$ (o odległości s) przez inną spiralę Archimedesa $c_1(t)$. Szczególnie rozważane były te spirale, które mogą posłużyć do kształtowania rond turbinowych. Artykuł zawiera matematyczne fakty i wyniki obliczeń pomocne w projektowaniu rond turbinowych kształtowanych za pomocą spirali Archimedesa.