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MATRIX METHODS IN EVALUATION OF INTEGRALS

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Abstract. The method of evaluating the integrals through use of the matrix inversion, presented here, was introduced by J.W. Rogers and then generalized by Matlak, Słota and Wituła. This method is still developed and one of its other possible applications is presented in this paper. This application concerns a new way of evaluating the integral $\int \sec^{2n+1} x\,dx$ on the basis of the discussed method. Additionally, many other applications of the obtained original recursive formula for this type of integral are given here. Some of them are used to generate the interesting identities for inverses of the central binomial coefficients and the trigonometric limits. The historical view is also presented as well as the connections between the received and previously known identities.

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1. Introduction

In this note a new formula for $I_n = \int \sec^{2n+1} x dx$ is obtained. Our formula for I_n, in comparison with the one obtained by W.V. Parker [1] by using the integration by parts, leads to some interesting trigonometric identities for inverses of the central binomial coefficients (see also [2–5] which are probably the basic papers considering this subject).

Moreover, referring to [1], some interesting limits are presented in this paper. We note that the method of evaluating I_n, presented in the current paper, is different than the previously discussed methods of analytical-combinatoric nature [1, 6, 7]. The method of evaluating the integrals, presented here, was introduced by Rogers in paper [8] and generalized by Matlak, Słota and Wituła in [9]. The method was also discussed by Meemark and Sriwongsa in the paper [10].

The binomial sums (and the special binomial sums) are of interest to physicists who use them, for example, in performing calculations of higher order corrections of scattering processes in the particle physics [11,12] and also in many relevant topics of quantum mechanics (especially in q-quantum calculus). The central binomial sums have become extremely popular in connection to the Apery's proof of irrationality of $\zeta(3)$ as well as in reference to many well-known conjectures on the series for powers of π and other important constants.

In our paper, the following definition of the double factorial (for all integers n) is used

$$n!! = \begin{cases} \prod_{k=0}^{\frac{n-1}{2}} (n-2k), & \text{if } n \geqslant 1 \text{ is odd} \\ \prod_{k=0}^{\frac{n-2}{2}} (n-2k), & \text{if } n \geqslant 2 \text{ is even} \\ 1, & \text{if } n \leqslant 0 \end{cases} =$$

$$= \begin{cases} n(n-2)\cdots 1, & \text{if } n \geqslant 1 \text{ is odd;} \\ n(n-2)\cdots 2, & \text{if } n \geqslant 2 \text{ is even;} \\ 1, & \text{if } n \leqslant 0. \end{cases} \tag{1}$$

2. Main result

We note that all formulae, given below, hold over the set $\mathbb{R} \setminus \frac{\pi}{2}\mathbb{Z}$.

First let us observe that

$$\frac{d}{dx}(\sec^n x) = n \sec^n x \tan x \tag{2}$$

which implies

$$\cot x \frac{d}{dx}(\sec^n x) = n \sec^n x, \tag{3}$$

and then, by using (3), we have

$$\frac{d}{dx}(\cot x \sec^n x) = \cot x \frac{d}{dx}(\sec^n x) + \sec^n x \frac{d}{dx}(\cot x) = n \sec^n x - \frac{1}{\sin^2 x} \sec^n x. \tag{4}$$

Application of the identity

$$\frac{1}{\sin^2 x} = 1 + \frac{\cos^2 x}{\sin^2 x} \tag{5}$$

to (4) yields

$$\frac{d}{dx}\big(\cot x \sec^n x\big) = (n-1) \sec^n x - \frac{1}{\sin^2 x} \sec^{n-2} x. \tag{6}$$

Since we are interested in integrating the odd powers of $\sec(x)$, we restrict n to be an odd integer. Now we use (5) repeatedly in (6) which gives us the following identity

$$\frac{d}{dx}\big(\cot x \sec^n x\big) = (n-1)\sec^n x - \sec^{n-2} x - \sec^{n-4} x - \cdots - \sec x - \frac{\cos x}{\sin^2 x}. \tag{7}$$

Since $\dfrac{d}{dx}(-\csc x) = \dfrac{\cos x}{\sin^2 x}$, we can then rewrite equation (7) as follows

$$\frac{d}{dx}\big(-\csc x + \cot x \sec^n x\big) = (n-1)\sec^n x - \sec^{n-2} x - \sec^{n-4} x - \cdots - \sec x \tag{8}$$

for each odd positive integer n.

Integrating this equation we obtain the identity

$$\begin{aligned} &C - \csc x + \cot x \sec^n x = \\ &\qquad = (n-1)\int \sec^n x\,dx - \int \sec^{n-2} x\,dx - \int \sec^{n-4} x\,dx - \ldots - \int \sec x\,dx. \end{aligned}$$

Hence, we deduce the system of equalities

$$\begin{aligned} &C_{2n+1} - \csc x + \cot x \sec^{2n+1} x = \\ &\quad = 2n\int \sec^{2n+1} x\,dx - \int \sec^{2n-1} x\,dx - \int \sec^{2n-3} x\,dx - \ldots - \int \sec x\,dx, \\ &C_{2n-1} - \csc x + \cot x \sec^{2n-1} x = \\ &\quad = 2(n-1)\int \sec^{2n-1} x\,dx - \int \sec^{2n-3} x\,dx - \int \sec^{2n-5} x\,dx - \ldots - \int \sec x\,dx, \\ &\quad \vdots \\ &C_3 - \csc x + \cot x \sec^3 x = 2\int \sec^3 x\,dx - \int \sec x\,dx, \\ &\int \sec x\,dx = \int \sec x\,dx, \end{aligned}$$

which can be written in a more compact form as the following matrix equation

$$\begin{bmatrix} 2n & -1 & -1 & \cdots & -1 & -1 \\ 0 & 2(n-1) & -1 & \cdots & -1 & -1 \\ 0 & 0 & 2(n-2) & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \int \sec^{2n+1} x\,dx \\ \int \sec^{2n-1} x\,dx \\ \int \sec^{2n-3} x\,dx \\ \vdots \\ \int \sec^{3} x\,dx \\ \int \sec x\,dx \end{bmatrix} =$$

$$= \begin{bmatrix} C_{2n+1} - \csc x + \cot x \sec^{2n+1} x \\ C_{2n-1} - \csc x + \cot x \sec^{2n-1} x \\ C_{2n-3} - \csc x + \cot x \sec^{2n-3} x \\ \vdots \\ C_3 - \csc x + \cot x \sec^3 x \\ \int \sec x\,dx \end{bmatrix}. \quad (9)$$

The above $(n+1)\times(n+1)$ matrix will be denoted by $\mathbf{A_n}$ for every $n \in \mathbb{N}$. We can deduce that

$$\mathbf{A_n}^{-\mathbf{1}} = \frac{\mathbf{1}}{(\mathbf{2n})!!} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} & a_{1(n+1)} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} & a_{2(n+1)} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} & a_{3(n+1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} & a_{n(n+1)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & (2n)!! \end{bmatrix}, \quad (10)$$

where

$$a_{ii} = \frac{(2n)!!}{2(n-i+1)}$$

for every $i = 1, \ldots, n$, and

$$a_{ij} = \frac{(2n)!!}{(2(n-i+1))!!} \frac{(2(n-i)+1)!!}{(2(n-j+1)+1)!!} (2(n-j))!!$$

for $i < j \leqslant n+1$. In particular, for $i = 1$ we obtain

$$a_{11} := \frac{(2n)!!}{2n} = (2(n-1))!!, \qquad a_{12} := \frac{(2n)!!}{4n(n-1)} = (2(n-2))!!, \tag{11}$$

$$\begin{aligned} a_{1k} &:= 2^{n-k}(n-k)!(2n-1)(2n-3)\cdots(2n-2k+5) \\ &= (2(n-k))!!\frac{(2n-1)!!}{(2(n-k)+3)!!} \end{aligned} \tag{12}$$

for every $k = 2,3,\ldots,n$ and at last we get

$$a_{1(n+1)} := (2n-1)!!. \tag{13}$$

For example, we have

$$\mathbf{A_1}^{-1} = \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}, \quad \mathbf{A_2}^{-1} = \frac{\mathbf{1}}{\mathbf{8}}\begin{bmatrix}2 & 1 & 3\\ 0 & 4 & 4\\ 0 & 0 & 8\end{bmatrix}, \quad \mathbf{A_3}^{-1} = \frac{\mathbf{1}}{\mathbf{48}}\begin{bmatrix}8 & 2 & 5 & 15\\ 0 & 12 & 6 & 18\\ 0 & 0 & 24 & 24\\ 0 & 0 & 0 & 48\end{bmatrix}.$$

It can be proven by induction that

$$\sum_{k=1}^{n} a_{1k} = (2n-1)!!. \tag{14}$$

In fact, the more general identity

$$\sum_{k=i}^{n} a_{ik} = a_{i(n+1)}, \qquad 1 \leqslant i \leqslant n,$$

holds (by multiplying i-th row of $\mathbf{A_n}^{-1}$ by the last column of $\mathbf{A_n}$ for each i, $1 \leqslant i \leqslant n$), which is equivalent to

$$\mathbf{A_n}^{-1}\begin{bmatrix}1\\1\\ \vdots\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\ \vdots\\0\\-1\end{bmatrix} \quad \text{or more evidently} \quad \mathbf{A_n}\begin{bmatrix}0\\0\\ \vdots\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\1\\ \vdots\\1\\-1\end{bmatrix}.$$

Similarly, from the relations (which are easy to verify):

$$\mathbf{A_n}\begin{bmatrix}2\\2\\ \vdots\\2\\2\end{bmatrix} = \begin{bmatrix}2n\\2(n-1)\\ \vdots\\2\\2\end{bmatrix} \quad \Leftrightarrow \quad \mathbf{A_n}^{-1}\begin{bmatrix}2n\\2(n-1)\\ \vdots\\2\\2\end{bmatrix} = \begin{bmatrix}2\\2\\ \vdots\\2\\2\end{bmatrix}$$

we obtain the convolution type identity

$$a_{i(n+1)} + \sum_{k=i}^{n}(n-k+1)\,a_{ik} = 2\,a_{i(n+1)} + \sum_{k=i}^{n-1}(n-k)\,a_{ik} = (2n)!!.$$

Note that from (9), (10) and (14) we easily deduce our main formula

$$\begin{aligned}\int \sec^{2n+1} x\,dx &= \frac{1}{(2n)!!}(\cot x)\sum_{k=1}^{n} a_{1k}\sec^{2(n-k)+3} x - \frac{1}{(2n)!!}\left(\sum_{k=1}^{n} a_{1k}\right)\csc x \\ &+ \frac{1}{(2n)!!}a_{1(n+1)}\int \sec x\,dx = \frac{1}{(2n)!!}(\cot x)\sum_{k=1}^{n} a_{1k}\sec^{2(n-k)+3} x - \frac{(2n-1)!!}{(2n)!!}\csc x \\ &+ \frac{(2n-1)!!}{(2n)!!}\ln(\sec x + \tan x), \qquad (15)\end{aligned}$$

i.e.,

$$\begin{aligned}\int \sec^{2n+1} x\,dx &= \\ &= \frac{\csc x}{(2n)!!}\sum_{k=1}^{n} a_{1k}\sec^{2(n-k+1)} x + \frac{(2n-1)!!}{(2n)!!}\left(\ln(\sec x + \tan x) - \csc x\right), \qquad (16)\end{aligned}$$

where a_{1k} are defined by expressions (11)-(13). For example, we obtain

$$\begin{aligned}\int \sec^{7} x\,dx &= \\ &= \csc x\left(\frac{1}{6}\sec^6 x + \frac{1}{24}\sec^4 x + \frac{5}{48}\sec^2 x\right) + \frac{5}{16}\big(\ln(\sec x + \tan x) - \csc x\big).\end{aligned}$$

Parker presented in [1] the alternative formula (obtained by integrating by parts the respective integral):

$$\int \sec^{2n+1} x\,dx = \frac{\binom{2n}{n}}{4^n}\left[\log(\sec x + \tan x) + \sum_{k=1}^{n}\frac{4^k}{2k\binom{2k}{k}}\sec^{2k} x \sin x\right] \qquad (17)$$

which is compatible with our formula (16) and implies the interesting trigonometric identity

$$\begin{aligned}1 + \sin^2 x \sum_{k=1}^{n}\frac{4^k}{2k\binom{2k}{k}}\sec^{2k} x &= \frac{(2(n-1))!!}{(2n-1)!!}\sec^{2n} x + \sum_{k=1}^{n-1}\frac{4^{k-1}}{(4k^2-1)\binom{2(k-1)}{k-1}}\sec^{2k} x \\ &= \frac{(2(n-1))!!}{(2n-1)!!}\sec^{2n} x + \sum_{k=0}^{n-2}\frac{4^k}{(4(k+1)^2-1)\binom{2k}{k}}\sec^{2(k+1)} x, \qquad (18)\end{aligned}$$

since we have

$$\frac{1}{(2n)!!}a_{1,n-k+1} = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{(2k-1)(2k+1)}\cdot\frac{(2k-2)!!}{(2k-3)!!}$$
$$= 4^{-n}\binom{2n}{n}\cdot\frac{1}{(4k^2-1)4^{-k+1}\binom{2(k-1)}{k-1}}$$

for every $k = 1,2,\ldots,n-1$. Setting $x = 0$ in (18) gives the binomial-coefficient identity

$$1-\frac{1}{2n}\cdot\frac{4^n}{\binom{2n}{n}} = \sum_{k=0}^{n-2}\frac{4^k}{(4(k+1)^2-1)\binom{2k}{k}}, \tag{19}$$

which is the recursive relation for one of the following sequences of real numbers: either for $\left\{\frac{4^k}{\binom{2k}{k}}\right\}_{k=0}^{\infty}$ or better for $\left\{\frac{4^k}{(2k+1)(2k+3)\binom{2k}{k}}\right\}_{k=0}^{\infty}$ (see also similar type relations (20), (21) and (24) given below).

Let us notice that (19) is equivalent to (14), and we get it analytically. From (17) we obtain one more formula observed by Parker

$$\lim_{x\to 0}\frac{\int_0^x \sec^{2n+1} y\,dy}{\sin x} = 4^{-n}\binom{2n}{n}\left[1+\sum_{k=1}^{n}\frac{4^k}{2k\binom{2k}{k}}\right].$$

Using de l'Hospital's rule we get

$$\lim_{x\to 0}\frac{\int_0^x \sec^{2n+1} y\,dy}{\sin x} = \lim_{x\to 0}\frac{\sec^{2n+1} x}{\cos x} = 1$$

which easily implies the purely algebraic form of the given Parker's formula above

$$1+\sum_{k=1}^{n}\frac{4^k}{2k\binom{2k}{k}} = \frac{4^n}{\binom{2n}{n}}. \tag{20}$$

In [4], among many similar types of formulae, the following one is also given

$$1+\sum_{k=0}^{n-1}\frac{4^k}{(2k+1)\binom{2k}{k}} = \frac{4^n}{\binom{2n}{n}}. \tag{21}$$

But this is hardly surprising, since the sums on the right hand side of equalities (20) and (21) for fixed $n\in\mathbb{N}$ are termwise the same

$$\frac{4^k}{(2k+1)\binom{2k}{k}} = \frac{2\cdot 4^k}{(2k+2)\frac{2k+1}{k+1}\binom{2k}{k}} = \frac{2\cdot 4^k}{(2k+2)\binom{2k+1}{k+1}} = \frac{4^{k+1}}{(2k+2)\binom{2k+2}{k+1}}. \tag{22}$$

Similarly we obtain

$$\frac{4^k}{(4(k+1)^2-1)\binom{2k}{k}} = \frac{2\cdot 4^k}{(2k+2)(2k+3)\frac{2k+1}{k+1}\binom{2k}{k}} = \frac{4^{k+1}}{2(k+1)(2(k+1)+1)\binom{2k+2}{k+1}},$$

which means that identity (19) is equivalent to the identity

$$\sum_{k=1}^{n-1}\frac{4^k}{2k(2k+1)\binom{2k}{k}} = 1-\frac{4^n}{2n\binom{2n}{n}}$$

(cf. [3, Theorem 4.3]).

From (19) and (21) we also deduce the relation

$$3+\sum_{k=0}^{n-1}\frac{4^k}{(2k+3)\binom{2k}{k}} = \frac{4^n}{\binom{2n}{n}} + \frac{4^{n+1}}{(n+1)\binom{2n+2}{n+1}} = \frac{2n+3}{2n+1}\cdot\frac{4^n}{\binom{2n}{n}}. \tag{23}$$

We note that the formula

$$\sum_{k=1}^{n}\frac{4^k}{2k\binom{2k}{k}} = \sum_{k=0}^{n-1}\frac{4^k}{(2k+1)\binom{2k}{k}}, \tag{24}$$

which follows from (22), is connected with the formula obtained by P. Bundschuh (see Aufgabe 811, Elemente der Mathematik):

$$2n\sum_{k=0}^{n}\binom{2k}{k}\frac{4^{-k}}{2(n-k)-1} = (2n+1)\sum_{k=0}^{n-1}\binom{2k}{k}\frac{4^{-k}}{2(n-k)+1}.$$

At last, from (18) and (20) we get the limit

$$\lim_{x\to 0}\frac{1-\frac{(2(n-1))!!}{(2n-1)!!}\sec^{2n}x-\sum\limits_{k=0}^{n-2}\frac{4^k}{(4(k+1)^2-1)\binom{2k}{k}}\sec^{2(k+1)}x}{\sin^2 x} =$$
$$= -\sum_{k=1}^{n}\frac{4^k}{2k\binom{2k}{k}} = 1-\frac{4^n}{\binom{2n}{n}}. \tag{25}$$

Moreover, one can easily notice that there exists the following limit

$$\lim_{x\to 0}\frac{1-\frac{(2(n-1))!!}{(2n-1)!!}\sec^{2}x-\sum\limits_{k=0}^{n-2}\frac{4^k}{(4(k+1)^2-1)\binom{2k}{k}}\sec^{2(n-k)}x}{\sin^2 x} = \sum_{k=1}^{n}\frac{4^k}{2k\binom{2k}{k}}-(n+1).$$

Unfortunately, we were not able to determine this limit analytically for every $n\in\mathbb{N}$, $n\geqslant 2$. We evaluated the correctness of this formula only by hand calculations for $n=2,\dots,5$ and by numerical calculations for $n=2,\dots,100$.

3. Conclusions

1. The standard reference for the binomial-coefficient identities is the Gould's paper [2], where equations (2.9), (2.20), (2.21), (2.22) and (2.23) involve such sums. The appearance of the identity (due to Hjortnaes [13], see also the new paper by Jakob Ablinger [14]):

$$\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4\binom{2n}{n}}$$

in Apéry's proof of the irrationality of $\zeta(3)$ (see [15]) stimulated something of a vogue for the infinite series involving the reciprocals of central binomial coefficients starting in the late 1970s (see [16, 17]), and some of this spilled over into works on the finite sums. In this connection, it is helpful to look at Sprugnoli [3], who gives the additional references to older papers (though he missed Parker [1]: the above formula (20), that appears at the end of Parker's note, is the Sprugnoli's Theorem 4.1 - which is also the Gould's equation (2.9)).

2. During the review process of this paper, we discovered, by using Parker's formula (20), the following convolution type identity (see [18, 19]), which holds for each $n\in\mathbb{N}$:

$$a_n\sum_{k=1}^{n-1}\frac{1}{a_k a_{n-k}}=\frac{n}{2}-1-\frac{n(n+1)}{2^{2n+1}}\sum_{k=1}^{n}\binom{2n}{n+k}\frac{(-1)^k-1}{k^2}, \tag{26}$$

where $a_n:=\binom{2n}{n}$ for every $n\in\mathbb{N}$.

3. The method of generating the combinatoric identities, proposed in this paper, has just been sketched by us and it may still give a lot of satisfaction to the readers interested in continuing or even in generalizing this subject matter.

Moreover, it should be emphasized that the calculation method is used in this paper in terms of basic consideration. The theoretical foundations indicates the remarkable similarity, or even the full compliance, with the consideration carried out in paper [20]. It refers to the relations between the categorical structures and infinitesimal calculus. We plan to refer to this topic in a separate paper.

Exploring the literature, one can also find some other methods referring to the method presented in this paper, e.g. the methods using the determinants of matrices which are discussed in [21–25].

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