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APPLICATION OF THE HOMOTOPY PERTURBATION METHOD FOR THE SYSTEMS OF VOLTERRA INTEGRAL EQUATIONS

Summary. In this paper the convergence of homotopy perturbation method for the systems of Volterra integral equations of the second kind is proved. Estimation of errors of approximate solutions obtained by taking the partial sum of the series is also elaborated in the paper.

ZASTOSOWANIE HOMOTOPIJNEJ METODY PERTURBACYJNEJ DO UKŁADÓW RÓWNAŃ CAŁKOWYCH TYPU VOLTERRY

Streszczenie. W artykule wykazano zbieżność homotopijnej metody perturbacyjnej dla układów równań całkowych Volterry drugiego rodzaju. Podano także oszacowanie błędu rozwiązania przybliżonego uzyskanego jako suma częściowa tworzonego w metodzie szeregu.

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1. Introduction

The current paper is a continuation of our previous work [9] in which we investigated the systems of Fredholm integral equations. At present we propose to apply the homotopy perturbation method for solving the systems of Volterra integral equations. We prove in the paper the convergence of homotopy perturbation method for systems of Volterra integral equations of the second kind. Moreover, the formulas for estimating the error of approximate solution are elaborated.

Methods using the ideas of homotopy were already applied for solving some type of integral equations (see for example [1, 2, 4–8, 10–12]). Application of homotopy perturbation method for solving the systems of Volterra integral equations is described also in paper [3]. However convergence of the method or estimation of the error of approximate solution were not discussed in this paper.

2. Systems of Volterra integral equations

We consider the system of equations of the form

$$u_i(x) - \lambda \sum_{j=1}^{n} \int_a^x K_{ij}(x,t)\, u_j(t)\, dt = f_i(x), \tag{1}$$

for $i = 1, 2, \ldots, n$, where $x \in [a,b]$, $\lambda \in \mathbb{C}$, functions $K_{ij} \in C([a,b] \times [a,b])$ and $f_i \in C[a,b]$ are known, whereas the functions u_i are sought. The above system of equations can be written in the matrix form

$$\mathbf{U}(x) - \lambda \int_a^x \mathbf{K}(x,y)\, \mathbf{U}(t)\, dt = \mathbf{F}(x), \tag{2}$$

where

$$\mathbf{K}(x,t) = \begin{bmatrix} K_{11}(x,t) & K_{12}(x,t) & \ldots & K_{1n}(x,t) \\ K_{21}(x,t) & K_{22}(x,t) & \ldots & K_{2n}(x,t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1}(x,t) & K_{n2}(x,t) & \ldots & K_{nn}(x,t) \end{bmatrix}$$

and

$$\mathbf{U}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}, \qquad \mathbf{F}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}.$$

According to the homotopy perturbation method (for details see, for example, [8]) we define operators L and N in the following way

$$L(\mathbf{V}) = \mathbf{V}, \qquad N(\mathbf{V}) = -\lambda \int_a^x \mathbf{K}(x,t)\,\mathbf{V}(t)\,dt. \tag{3}$$

By using the above operators we obtain the homotopy operator for the system of Volterra integral equations of the second kind

$$H(\mathbf{V},p) = \mathbf{V}(x) - \mathbf{U}_0(x) + p\Big(\mathbf{U}_0(x) - \mathbf{F}(x) - \lambda \int_a^x \mathbf{K}(x,t)\,\mathbf{V}(t)\,dt\Big). \tag{4}$$

According to the method, in the next step we search for the solution of operator equation $H(\mathbf{V},p) = 0$ in the form of power series

$$\mathbf{V}(x) = \sum_{k=0}^{\infty} p^k\,\mathbf{V}_k(x), \tag{5}$$

where $\mathbf{V}_k(x) = [v_{1,k}(x), v_{2,k}(x), \dots, v_{n,k}(x)]^T$. In order to determine the functions $\mathbf{V}_j$ we substitute relation (5) into equation $H(\mathbf{V},p) = 0$ and we get (under assumption that the series is convergent which will be discussed later):

$$\sum_{k=0}^{\infty} p^k\,\mathbf{V}_k(x) = \mathbf{U}_0(x) + p\big(\mathbf{F}(x) - \mathbf{U}_0(x)\big) + \sum_{k=1}^{\infty} p^k\,\lambda \int_a^x \mathbf{K}(x,t)\,\mathbf{V}_{k-1}(t)\,dt. \tag{6}$$

By comparing the expressions with the same powers of parameter p, we receive the relations

$$\mathbf{V}_0(x) = \mathbf{U}_0(x), \tag{7}$$

$$\mathbf{V}_1(x) = \mathbf{F}(x) - \mathbf{U}_0(x) + \lambda \int_a^x \mathbf{K}(x,t)\,\mathbf{V}_0(t)\,dt, \tag{8}$$

$$\mathbf{V}_k(x) = \lambda \int_a^x \mathbf{K}(x,t)\,\mathbf{V}_{k-1}(t)\,dt, \qquad k \geqslant 2. \tag{9}$$

Now we proceed to discussing the convergence of series (5).

Theorem 1. *Let the functions $K_{ij}(x,t)$ and $f_i(x)$ for $i,j \in \{1,2,\dots,n\}$, appearing in system (1), be continuous in regions $\Omega_1 = [a,b] \times [a,b]$ and $\Omega = [a,b]$, respectively. Furthermore, as the initial approximation $\mathbf{U}_0$ let us choose a vector of functions continuous in interval $[a,b]$. Then series (5), in which the functions $\mathbf{V}_k$ are determined by means of relations (7)–(9), is uniformly convergent in interval $[a,b]$ for each $p \in [0,1]$ to the uniquely determined solution $\mathbf{V}(x)$, which is a vector of functions continuous in $[a,b]$.*

Proof. Proof of this theorem runs analogically as the proof of Theorem 1 from paper [9]. □

Remark 2. We note that for the discussed systems of Volterra integral equations Remarks 2–4 from paper [9] remain true.

Theorem 3. *Error of the n-order approximate solution can be estimated in the following way*

$$\begin{aligned} E_n &\leqslant B \left(e^{|\lambda|\, M\,(b-a)} - \sum_{k=0}^{n-1} \frac{\big(|\lambda|\, M\,(b-a)\big)^k}{k!} \right) \leqslant \\ &\leqslant B \, \frac{(|\lambda|\, M\,(b-a))^n}{(n+1)!} \, \big(n + \exp(|\lambda|\, M\,(b-a))\big), \end{aligned} \tag{10}$$

where $E_n := \sup_{x\in[a,b]} \|\mathbf{U}(x) - \widehat{\mathbf{U}}_n(x)\|$, $B := N_0 + N_1 + |\lambda|\, M\, N_0\,(b-a)$ *and the constants* M, N_1 *and* N_0 *are such that*

$$\|\mathbf{K}(x,t)\| \leqslant M \quad \wedge \quad \|\mathbf{F}(x)\| \leqslant N_1 \quad \wedge \quad \|\mathbf{U}_0(x)\| \leqslant N_0 \qquad \forall\, x,t \in [a,b]. \tag{11}$$

Proof. Proof of this theorem runs analogically as the proof of Theorem 5 from paper [9]. □

3. Example

Now, in the example we use the investigated method for solving the following system of Volterra integral equations of the second kind

$$\begin{aligned} u_1(x) &= e^x \cosh(2\,x) - \int_0^x e^{x-t}\, u_1(t)\, dt - \int_0^x e^{x+t}\, u_2(t)\, dt, \\ u_2(x) &= (x+1)\, e^x + \sinh x - \int_0^x e^{x+t}\, u_1(t)\, dt - \int_0^x e^{x-t}\, u_2(t)\, dt, \end{aligned}$$

where $x \in [0, \frac{1}{2}]$. Solution of the above system is given by the functions

$$u_{d1}(x) = e^{-x}, \qquad u_{d2}(x) = e^x.$$

By taking the zero initial approximation $\mathbf{U}_0(x) = (0,0)^T$ and next by using relations (7)–(9) we get successively

$$\begin{aligned}
&\mathbf{V}_0(x) = \mathbf{U}_0(x) = (0,0)^T,\\
&\mathbf{V}_1(x) = \big(e^x \cosh(2\,x),\ (x+1)\,e^x + \sinh x\big)^T,\\
&\mathbf{V}_2(x) = \big(-e^x\big(e^x(1+x) + \cosh x\big)\sinh x,\ -\frac{1}{8}\,e^{5x} - \big(x+\frac{3}{8}\big)\,e^x + \frac{1}{2}\,(x+1)\,e^{-x}\big)^T,\\
&\quad\vdots
\end{aligned}$$

Obtained approximate solution $\widehat{\mathbf{U}}_n = \big(\widehat{u}_{1,n}, \widehat{u}_{2,n}\big)^T$ can be compared with the exact solution by taking the difference of these functions and expanding it into the series. Thus, for $n=5$ we obtain

$$\begin{aligned}
\widehat{u}_{1,5}(x) - e^{-x} &= 3.05311\,10^{-16}\,x^3 + 5.89806\,10^{-16}\,x^4 - 0.266667\,x^5 - 0.8\,x^6 -\\
&\quad - 1.50317\,x^7 + O(x^8),\\
\widehat{u}_{2,5}(x) - e^{x} &= 1.11022\,10^{-16} + 2.22045\,10^{-16}\,x + 3.88578\,10^{-16}\,x^2 -\\
&\quad - 5.82867\,10^{-16}\,x^3 - 1.11022\,10^{-16}\,x^4 - 0.266667\,x^5 -\\
&\quad - 0.755556\,x^6 - 1.46825\,x^7 + O(x^8),
\end{aligned}$$

whereas for $n=15$ we receive

$$\begin{aligned}
\widehat{u}_{1,15}(x) - e^{-x} &= 1.70974\,10^{-14}\,x + 5.32907\,10^{-14}\,x^2 + 9.97813\,10^{-14}\,x^3 +\\
&\quad + 1.28487\,10^{-13}\,x^4 + 1.39859\,10^{-13}\,x^5 + 1.19243\,10^{-13}\,x^6 +\\
&\quad + 9.53675\,10^{-14}\,x^7 + \ldots - 7.20038\,10^{-6}\,x^{19} -\\
&\quad - 0.0000155848\,x^{20} + O(x^{21}),\\
\widehat{u}_{2,15}(x) - e^{x} &= 1.11022\,10^{-16} + 3.33067\,10^{-16}\,x - 7.77156\,10^{-16}\,x^2 -\\
&\quad - 5.30131\,10^{-15}\,x^3 - 8.76382\,10^{-15}\,x^4 - 1.86032\,10^{-14}\,x^5 -\\
&\quad - 1.21337\,10^{-14}\,x^6 - 9.88313\,10^{-15}\,x^7 + \ldots\\
&\quad - 7.127\,10^{-6}\,x^{19} - 0.0000154338\,x^{20} + O(x^{21}).
\end{aligned}$$

As it can be seen, with the increasing number of calculated terms the approximate solution is more and more close to the exact solution.

In Table 1 there are presented the errors ($\|u_{di} - \widehat{u}_{i,n}\| = \sup_{x\in[0,1]} |u_{di}(x) - \widehat{u}_{i,n}(x)|$) which occur in approximating the exact solution by the successive approximate solutions. Distributions of error in the entire interval $[0,1]$ for $n=5$ and $n=10$ are displayed in Figures 1 and 2. Obtained results confirm that the method is fast convergent. Thus, computing just a few (a dozen or so) first terms of the series ensures a very good approximation of the exact solution.

Table 1

Errors of the exact solution approximation

n	$\|u_{d1} - \widehat{u}_{1,n}\|$	$\|u_{d2} - \widehat{u}_{2,n}\|$	n	$\|u_{d1} - \widehat{u}_{1,n}\|$	$\|u_{d2} - \widehat{u}_{2,n}\|$
1	1.9376	1.3455	6	$1.1622\,10^{-2}$	$1.1379\,10^{-2}$
2	1.1559	1.1651	7	$2.2639\,10^{-3}$	$2.2223\,10^{-3}$
3	0.5507	0.5205	8	$3.8554\,10^{-4}$	$3.7933\,10^{-4}$
4	0.1867	0.1817	9	$5.8339\,10^{-5}$	$5.7498\,10^{-5}$
5	$5.1138\,10^{-2}$	$4.9804\,10^{-2}$	10	$7.9421\,10^{-6}$	$7.8388\,10^{-6}$

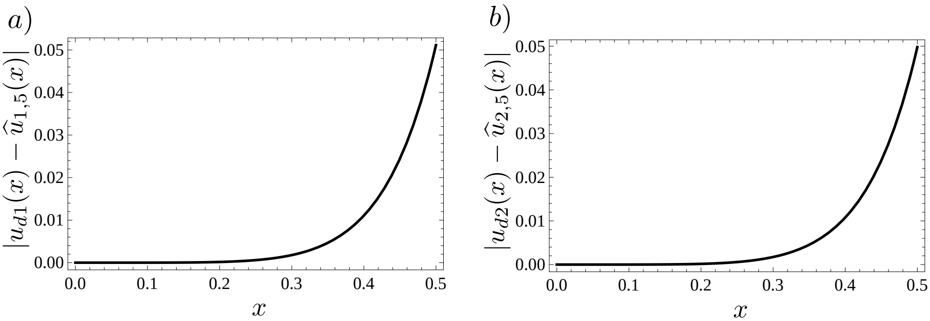

Fig. 1. Distribution of error of the exact solution approximation for $n = 5$
Rys. 1. Rozkład błędu rozwiązania przybliżonego dla $n = 5$

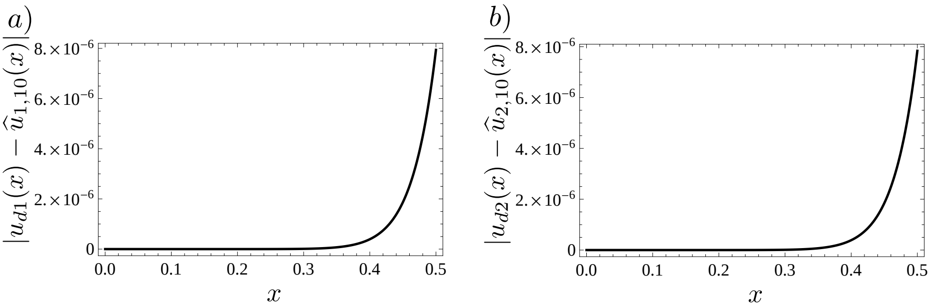

Fig. 2. Distribution of error of the exact solution approximation for $n = 10$
Rys. 2. Rozkład błędu rozwiązania przybliżonego dla $n = 10$

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