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On asymptotic approach to reliability improvement of multi-state systems with components quantitative and qualitative redundancy: „*m* out of *n*" systems

Keywords

reliability improvement, limit reliability functions

Abstract

The paper is composed of two parts, in this part the multi-state homogeneous „*m* out of *n*" systems with reserve components are defined and their multi-state limit reliability functions are determined. In order to improve of the reliability of these systems the following methods are used: (i) a warm duplication of components, (ii) a cold duplication of components, (iii) a mixed duplication of components, (iv) improving the reliability of components by reducing their failure rate. Next, the effects of the systems' reliability different improvements are compared.

1. Introduction

Presented paper is continuation of a work about reliability improvement of large system. In the first part of this work are defined the component's and system's multi-state reliability functions and next the asymptotic approach are brought forward. There are presented results concerned with improvement of large series and parallel systems, their multi-state limit reliability functions in case when the systems have reserve components and in case when the reliability of components is improved by reducing their failure rate. As the main result are found the forms of reducing their failure rate factor for both kinds of large systems.

2. Reliability improvement of a multi-state „*m* out of *n*" system

Definition 2.1. A multi-state system is called an „*m* out of *n*" system if its lifetime in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = T_{(n-m+1)}(u),\ m = 1,2,...,n,\ u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m th* maximal order statistics in the sequence of the component lifetimes $T_1(u), T_2(u),..., T_n(u)$.

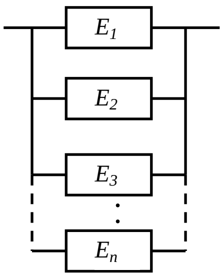

Figure 1. The scheme of a homogeneous „m out of n" system

The above definition means that the multi-state „*m* out of *n*" system is in the state subset $\{u,u+1,...,z\}$ if and only if at least *m* out of *n* its components is in this state subset and it is a multi-state parallel system if $m = 1$ and it is a multi-state series system if $m = n$.

Definition 2.2. A multi-state „*m* out of *n*" system is called homogeneous if its component lifetimes $T_i(u)$ in the state subsets have an identical distribution function

$$F_i(t,u) = F(t,u),\ u = 1,2,...,z,\ t \in (-\infty,\infty),\ i = 1,2,...,n.$$

The reliability function of the homogeneous multi-state „*m* out of *n*" system is given either by

$$\boldsymbol{R}_n^{(m)}(t,\cdot) = [1,\boldsymbol{R}_n^{(m)}(t,1),...,\boldsymbol{R}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{R}_n^{(m)}(t,u) = 1 - \sum_{i=0}^{m-1}\binom{n}{i}[R(t,u)]^i [F(t,u)]^{n-i},$$
$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

or by

$$\overline{\boldsymbol{R}}_n^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,1),..., \overline{\boldsymbol{R}}_n^{(\overline{m})}(t,z)],$$

where

$$\overline{\mathbf{R}}_n^{(\overline{m})}(t,u) = \sum_{i=0}^{\overline{m}}\binom{n}{i}[F(t,u)]^i [R(t,u)]^{n-i}, \ t \in (-\infty,\infty),$$
$\overline{m} = n - m$, $u = 1,2,...,z$.

Definition 2.3. A multi-state system is called an „*m* out of *n*" system with a hot reserve of its components if its lifetime $T^{(1)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(1)}(u) = T_{(n-m+1)}(u), \ m = 1,2,...,n, \ u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m-th* maximal order statistics in the sequence of the component lifetimes

$$T_i(u) = \max_{1\le j\le 2}\{T_{ij}(u)\}, \ i = 1,2,..,n, \ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state „*m* out of *n*" system with a hot reserve of its components is given either by

$$\boldsymbol{IR}^{(1)}{}_n^{(m)}(t,\cdot) = [1, \boldsymbol{IR}^{(1)}{}_n^{(m)}(1,z),..., \boldsymbol{IR}^{(1)}{}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{IR}^{(1)}{}_n^{(m)}(t,u) = 1 - \sum_{i=0}^{m-1}\binom{n}{i}[1-(F(t,u))^2]^i [F(t,u)]^{2(n-i)}, \quad (1)$$
$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

or by

$$\boldsymbol{I\overline{R}}^{(1)}{}_n^{(\overline{m})}(t,\cdot) = [1, \boldsymbol{I\overline{R}}^{(1)}{}_n^{(\overline{m})}(t,1),..., \boldsymbol{I\overline{R}}^{(1)}{}_n^{(\overline{m})}(t,z)],$$

where

$$\boldsymbol{I\overline{R}}^{(1)}{}_n^{(\overline{m})}(t,u)$$

$$= \sum_{i=0}^{\overline{m}}\binom{n}{i}[(F(t,u))]^{2i}[1-(F(t,u))^2]^{(n-i)}, \quad (2)$$
$\overline{m} = n - m$, $t \in (-\infty,\infty)$, $u = 1,2,...,z$.

Lemma 2.1.
case 1: If

(i) $\boldsymbol{I\mathfrak{R}}^{(1)(m)}(t,u) = 1 - \sum_{i=0}^{m-1}\frac{[V(t,u)]^i}{i!}\exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(1)}{}_n^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a hot reserve of its components defined by (16),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) m = constant ($m/n \to 0$, as $n \to \infty$),

then

$$\lim_{n\to\infty}\boldsymbol{IR}^{(1)}{}_n^{(m)}(a_n(u)t + b_n(u)) = \boldsymbol{I\mathfrak{R}}^{(1)(m)}(t,u),$$
$t \in C_{\boldsymbol{I\mathfrak{R}}}$, $u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty} n[1-F^2(a_n(u)t + b_n(u))] = V(t,u), \ t \in C_V,$$
$u = 1,2,...z$,

case 2: If

(i) $\boldsymbol{I\mathfrak{R}}^{(1)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-v(t,u)} e^{-\frac{x^2}{2}}dx$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(1)}{}_n^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a hot reserve of its components defined by (16),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $m/n \to \mu$, $0 < \mu < 1$, as $n \to \infty$,

then

$$\lim_{n\to\infty}\boldsymbol{IR}^{(1)}{}_n^{(m)}(a_n(u)t + b_n(u)) = \boldsymbol{I\mathfrak{R}}^{(1)(\mu)}(t,u),$$
$t \in C_{\boldsymbol{I\mathfrak{R}}}$, $u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty}\frac{(n+1)[1-F^2(a_n(u)t + b_n(u))] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}} = v(t,u),$$
$u = 1,2,...,z$.

case 3: If

(i) $\boldsymbol{I\overline{\mathfrak{R}}}^{(1)(\overline{m})}(t,u) = \sum_{i=0}^{m}\frac{[\overline{V}(t,u)]^i}{i!}\exp[-\overline{V}(t,u)]$,

$\overline{m} = n - m$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{I\overline{R}}^{(1)}{}_{n}^{(\overline{m})}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a hot reserve of its components defined by (17),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $n - m = \overline{m}$ = constant $(m/n \to 1 \text{ as } n \to \infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{I\overline{R}}^{(1)}{}_{n}^{(\overline{m})}(a_n(u)t + b_n(u)) = \boldsymbol{I\overline{\mathfrak{R}}}^{(1)(\overline{m})}(t,u),$$

$t \in C_{I\overline{\mathfrak{R}}}$, $u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u)]^2 = \overline{V}(t,u), t \in C_{\overline{V}},$$

$u = 1,2,...,z.$

Proposition 2.1. If components of the homogeneous multi-state „*m* out of *n*" system with a hot reserve of its components have multi-state exponential reliability functions

and

case 1 m = constant,

$$a_n(u) = \frac{1}{\lambda(u)},\ b_n(u) = \frac{1}{\lambda(u)}\log 2n,\ u = 1,2,...,z,$$

then

$$\boldsymbol{I\mathfrak{R}}^{(1)(m)}(t,u) = 1 - \sum_{i=0}^{m-1}\frac{\exp[-it]}{i!}\exp[-\exp[-t]],$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

case 2 $m/n \to \mu$, $0 < \mu < 1$, $n \to \infty$,

$$a_n(u) = \frac{\sqrt{\mu}}{\lambda(u)2\sqrt{n+1}},\ b_n(u) = \frac{1}{\lambda(u)}\sqrt{1-\mu},$$

$u = 1,2,...,z$,

then

$$\boldsymbol{I\mathfrak{R}}^{(1)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{x^2}{2}}dx,$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

case 3 $n - m = \overline{m}$ = constant, ($m/n \to 1$, $n \to \infty$),

$$a_n(u) = \frac{1}{\sqrt{n}\lambda(u)},\ b_n(u) = 0,\ u = 1,2,...,z,$$

then

$$I\overline{\mathfrak{R}}^{(1)(\overline{m})}(t,u) = 1,\ t < 0,$$

$$I\overline{\mathfrak{R}}^{(1)(\overline{m})}(t,u) = \sum_{i=0}^{n-m}\frac{t^{2i}}{i!}\exp[-t^2],\ t \geq 0,\ u = 1,2,...,z,$$

is its limit reliability function.

Proof:
case 1: Since for all fixed u, we have

$$a_n(u)t + b_n(u) \to \infty \text{ as } n \to \infty.$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[1 - F^2(a_n(u)t + b_n(u))]$$

$$= \lim_{n\to\infty} n[2\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$-\exp[-2\lambda(u)(a_n(u)t + b_n(u))]]$$

$$= \lim_{n\to\infty} 2n\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$[1 - \frac{1}{2}\exp[-\lambda(u)(a_n(u)t + b_n(u))]]$$

$$= \lim_{n\to\infty} \exp[-t][2n\exp[-\lambda(u)b_n(u)]$$

$$-n\exp[-t]\exp[-2\lambda(u)b_n(u)]]$$

$$= \lim_{n\to\infty} \exp[-t][2n\frac{1}{n} - n\frac{1}{n^2}\exp[-t]]$$

$$= \exp[-t],\ t \in (-\infty,\infty),\ u = 1,2,...,z,$$

which by *case* 1 in Lemma 2.1 completes the proof.

case 2: Since for all fixed u, we have

$$a_n(u)t + b_n(u) \to \infty \text{ as } n \to \infty,$$

moreover

$$1 - F^2(a_n(u)t + b_n(u))$$

$$= 2\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$-\exp[-2\lambda(u)(a_n(u)t + b_n(u))]$$

$$= 2[1-\lambda(u)(a_n(u)t+b_n(u))$$

$$+\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2]$$

$$-[1-2\lambda(u)(a_n(u)t+b_n(u))$$

$$+\frac{1}{2}4\lambda^2(u)(a_n(u)t+b_n(u))^2]+o(\frac{1}{(n+1)})$$

$$=1-\lambda^2(u)(a_n(u)t+b_n(u))^2 \;\; +o(\frac{1}{(n+1)}),$$

next

$$\nu(t,u)=\lim_{n\to\infty}\frac{(n+1)[1-F^2(a_n(u)t+b_n(u))]]-m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$=\lim_{n\to\infty}\frac{(n+1)[-\frac{\sqrt{\mu(1-\mu)}}{\sqrt{n+1}}t+\mu-o(\frac{1}{\sqrt{n+1}})]-m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$=\lim_{n\to\infty}\frac{-\sqrt{\mu(1-\mu)}t+o(1)}{\sqrt{\mu(1-\mu)}}=-t\,,\; t\in(-\infty,\infty),$$

$u = 1,2,...,z,$

which by *case* 2 in Lemma 2.1 completes the proof.

case 3: Since for all fixed u, we have

$$a_n(u)t+b_n(u)=\frac{t}{\lambda(u)\sqrt{n}}<0 \;\text{ for } t<0$$

and

$$a_n(u)t+b_n(u)=\frac{t}{\lambda(u)\sqrt{n}}\geq 0 \;\text{ for } t\geq 0\,,$$

then

$$F^2(a_n(u)t+b_n(u))=0\,,\; t<0$$

and

$$F^2(a_n(u)t+b_n(u))$$

$$=[1-\exp[-\lambda(u)(a_n(u)t+b_n(u))]]^2$$

$$=[1-\exp[-\frac{t}{\sqrt{n}}]]^2\,,\; t\geq 0.$$

Therefore

$$V(t,u)=\lim_{n\to\infty}n[F(a_n(u)t+b_n(u))]^2 = 0\,, t<0,$$

$u = 1,2,...,z,$

and

$$V(t,u)=\lim_{n\to\infty}n[F(a_n(u)t+b_n(u))]^2$$

$$=\lim_{n\to\infty}n[1-\exp[-\frac{t}{\sqrt{n}}]]^2$$

$$=\lim_{n\to\infty}n[\frac{t}{\sqrt{n}}+o(\frac{1}{\sqrt{n}})]^2 = t^2\,,\; t\geq 0,$$

$u = 1,2,...,z,$

which by *case* 3 in Lemma 2.1 completes the proof. □

Corollary 2.1. The reliability function of exponential „m out of n" system with a hot reserve of its components is given by
case 1

$$\mathrm{IR}^{(1)}{}_n^{(m)}(t,u)\cong 1-\sum_{i=0}^{m-1}\frac{\exp[-i(\lambda(u)t-\log 2n)]}{i!}$$

$$\exp[-\exp[-\lambda(u)t+\log 2n]\,, \tag{3}$$

$t\in(-\infty,\infty),\; u = 1,2,...,z.$

case 2

$$\boldsymbol{IR}^{(1)}{}_n^{(\mu)}(t,u)\cong 1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\Im}e^{-\frac{x^2}{2}}dx \tag{4}$$

where

$$\Im=\frac{2\lambda(u)\sqrt{n+1}}{\sqrt{\mu}}t-\frac{2\sqrt{n+1}\sqrt{1-\mu}}{\sqrt{\mu}}\,,\; t\in(-\infty,\infty), \tag{5}$$

$u = 1,2,...,z.$

case 3

$$\boldsymbol{IR}^{(1)}{}_n^{(\overline{m})}(t,u)=1\,,\; t<0,$$

$$\boldsymbol{IR}^{(1)}{}_n^{(\overline{m})}(t,u)\cong\sum_{i=0}^{n-m}\frac{[\lambda(u)\sqrt{n}t]^{2i}}{i!}\exp[-\lambda^2(u)nt^2]\,, \tag{6}$$

$t\geq 0\,,\; u = 1,2,...,z.$

Definition 2.4. A multi-state system is called an „*m* out of *n*" system with a cold reserve of its components if its lifetime $T^{(2)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(2)}(u) = T_{(n-m+1)}(u), m = 1,2,...,n,\ u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m-th* maximal order statistics in the sequence of the component lifetimes

$$T_i(u) = \sum_{j=1}^{2} T_{ij}(u),\ i = 1,2,..,n,\ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components.

The reliability function of the homogeneous multi-state „*m* out of *n*" system with a cold reserve of its components is given either by

$$\boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,\cdot) = [1,\ \boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,1),...,\ \boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,z)],$$

where

$$\boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,u)$$

$$= 1 - \sum_{i=0}^{m-1} \binom{n}{i} [1 - F(t,u) * F(t,u)]^{i}$$

$$= \cdot [F(t,u) * F(,u)]^{n-i} \quad (7)$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

or by

$$\boldsymbol{I\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,\cdot) = [1,\ \mathrm{I}\boldsymbol{\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,1),...,\boldsymbol{I\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,z)],$$

where

$$\boldsymbol{I\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,u)$$

$$= \sum_{i=0}^{\overline{m}} \binom{n}{i} [F(t,u) * F(t,u)]^{i} [1 - F(t,u) * F(t,u)]^{n-i}, \quad (8)$$

$t \in (-\infty,\infty)$, $\overline{m} = n - m$, $u = 1,2,...,z$.

Lemma 2.2.
case 1: If

(i) $\boldsymbol{I\mathfrak{R}}^{(2)}{}^{(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \frac{[V(t,u)]^{i}}{i!} \exp[-V(t,u)]$,
$u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a cold reserve of its components defined by (24),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) m = constant ($m/n \to 0$, as $n \to \infty$),

then

$$\lim_{n\to\infty} \boldsymbol{IR}^{(2)}{}_{n}^{(m)}(a_n(u)t + b_n(u)) = \boldsymbol{I\mathfrak{R}}^{(1)}{}^{(m)}(t,u),\ t \in C_{\boldsymbol{I\mathfrak{R}}},$$

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]]$$

$$= V(t,u),\ t \in C_V,\ u = 1,2,...z,$$

case 2: If

(i) $\boldsymbol{I\mathfrak{R}}^{(2)}{}^{(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-v(t,u)} e^{-\frac{x^2}{2}} dx$,
$u = 1,2,...,z$, is non-degenerate reliability function,

(i) $\boldsymbol{IR}^{(2)}{}_{n}^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a cold reserve of its components defined by (24),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $m/n \to \mu$, $0 < \mu < 1$, as $n \to \infty$,

then

$$\lim_{n\to\infty} \boldsymbol{IR}^{(2)}{}_{n}^{(m)}(a_n(u)t + b_n(u)) = \boldsymbol{I\mathfrak{R}}^{(2)}{}^{(\mu)}(t,u),$$

$t \in C_{\boldsymbol{I\mathfrak{R}}}$,

if and only if

$$\lim_{n\to\infty} \frac{(n+1)[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= v(t,u),\ u = 1,2,...,z.$$

case 3: If

(i) $\boldsymbol{I\overline{\mathfrak{R}}}^{(2)}{}^{(\overline{m})}(t,u) = \sum_{i=0}^{m} \frac{[\overline{V}(t,u)]^{i}}{i!} \exp[-\overline{V}(t,u)]$,
$\overline{m} = n - m$, u = 1,2,...,z, is non-degenerate reliability function,

(ii) $\boldsymbol{I\overline{R}}^{(2)}{}_{n}^{(\overline{m})}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a cold reserve of its components defined by (25),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $n - m = \bar{m}$ = constant $(m/n \to 1 \text{ as } n \to \infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{I\bar{R}}^{(2)}{}_n^{(\bar{m})}(a_n(u)t + b_n(u)) = \boldsymbol{I\bar{\mathfrak{R}}}^{(2)(\bar{m})}(t,u),$$

$t \in C_{\boldsymbol{I\bar{\mathfrak{R}}}}$,

if and only if

$$\lim_{n\to\infty} n[F(a_n(u)t + b_n(u) * F(a_n(u)t + b_n(u)] = \bar{V}(t,u),$$

$t \in C_{\bar{V}}$, $u = 1,2,...,z$.

Proposition 3.2. If components of the homogeneous multi-state „m out of n" system with a cold reserve of its components have multi-state exponential reliability functions
and
case 1 m = constant,

$$a_n(u) = \frac{1}{\lambda(u)}, \quad \frac{\exp[\lambda(u)b_n(u)]}{\lambda(u)b_n(u)} = n, \quad u = 1,2,...,z,$$

then

$$\boldsymbol{I\mathfrak{R}}^{(2)(m)}(t,u) = 1 - \sum_{i=0}^{m-1}\frac{\exp[-it]}{i!}\exp[-\exp[-t]],$$

$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

case 2 $m/n \to \mu$ $0 < \mu < 1$, $n \to \infty$,

$$a_n(u) = \frac{\sqrt{\mu}}{\lambda(u)\sqrt{n+1}}, \quad b_n(u) = \frac{\sqrt{1-\mu}}{\lambda(u)}, \quad u = 1,2,...,z,$$

then

$$\boldsymbol{I\mathfrak{R}}^{(2)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{x^2}{2}}dx, \quad t \in (-\infty,\infty),$$

$u = 1,2,...,z$,

case 3 $n - m = \bar{m}$ = constant $(m/n \to 1,\ n \to \infty)$,

$$a_n(u) = \frac{\sqrt{2}}{\sqrt{n}\lambda(u)}, \quad b_n(u) = 0, \quad u = 1,2,...,z,$$

then

$$\mathrm{I}\boldsymbol{\bar{\mathfrak{R}}}^{(2)(\bar{m})}(t,u) = 1, \quad t < 0,$$

$$\boldsymbol{I\bar{\mathfrak{R}}}^{(2)(\bar{m})}(t,u) = \sum_{i=0}^{n-m}\frac{t^{2i}}{i!}\exp[-t^2], \quad t \geq 0, \quad u = 1,2,...,z,$$

is its limit reliability function.

Proof:
case 1: Since for all fixed u, we have

$$a_n(u)t + b_n(u) \to \infty \text{ as } n \to \infty, \quad t \in (-\infty,\infty).$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$= \lim_{n\to\infty} n[\frac{1+t}{\exp[\lambda(u)b_n(u)]}$$

$$+ \frac{\lambda(u)b_n(u)}{\exp[\lambda(u)b_n(u)]}]\exp[-t]$$

$$= \exp[-t], \quad t \in (-\infty,\infty), \quad u = 1,2,...z,$$

which by *case* 1 in Lemma 2.2 completes the proof.

case 2: Since for all fixed u, we have

$$a_n(u)t + b_n(u) = \frac{\sqrt{\mu}}{\lambda(u)\sqrt{n+1}}t + \frac{1}{\lambda(u)}\sqrt{1-\mu}$$

$$\to \frac{1}{\lambda(u)}\sqrt{1-\mu} > 0 \text{ as } n \to \infty$$

and

$$1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))$$

$$= [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$= [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$[1 - \lambda(u)(a_n(u)t + b_n(u)) +$$

$$\frac{1}{2}\lambda^2(u)(a_n(u)t + b_n(u))^2 - o(\frac{1}{n+1})]$$

$$= 1 - \frac{1}{2}\lambda^2(u)(a_n(u)t + b_n(u))^2$$

$$- o(\frac{1}{n+1})],\ t \in (-\infty, \infty).$$

Therefore

$$v(t,u)$$

$$= \lim_{n\to\infty} \frac{(n+1)[1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \lim_{n\to\infty} \frac{(n+1)[-\frac{\sqrt{\mu(1-\mu)}}{\sqrt{n+1}}t + \mu - o(\frac{1}{n+1})] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \lim_{n\to\infty} \frac{-\sqrt{\mu(1-\mu)}t + o(1)}{\sqrt{\mu(1-\mu)}} = -t,\ t \in (-\infty, \infty),$$

which by *case* 2 in Lemma 2.2 completes the proof.

case 3: Since for all fixed u, we have

$$a_n(u)t + b_n(u) = \frac{\sqrt{2}t}{\lambda(u)\sqrt{n}} < 0 \text{ for } t < 0$$

and

$$a_n(u)t + b_n(u) = \frac{t\sqrt{2}}{\lambda(u)\sqrt{n}} \geq 0 \text{ for } t \geq 0,$$

then

$$F(a_n(u)t + b_n(u)) = 0 \text{ for } t < 0$$

and

$$F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))$$

$$= [1 - [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]]$$

$$= 1 - (1 + \frac{t\sqrt{2}}{\sqrt{n}})\exp[-\frac{t\sqrt{2}}{\sqrt{n}}]$$

$$= \frac{t^2}{n} + o(\frac{1}{n}),\ t \geq 0.$$

Therefore

$$v(t,u) = \lim_{n\to\infty} n[F(a_n(u)t + b_n(u))$$

$$* F(a_n(u)t + b_n(u))]$$

$$= \lim_{n\to\infty} n[1 - (1 + \frac{t\sqrt{2}}{\sqrt{n}})\exp[-\frac{t\sqrt{2}}{\sqrt{n}}]]$$

$$= \lim_{n\to\infty} n[\frac{t^2}{n} + o(\frac{1}{n})] = t^2,\ t \geq 0,\ u = 1,2,...z,$$

which by *case* 3 in Lemma 2.2 completes the proof. □

Corollary 2.2. The reliability function of exponential „m out of n" system whit a cold reserve of its components is given by

case 1

$$\boldsymbol{IR}^{(2)(m)}_{\ n}(t,u) \cong 1 - \sum_{i=0}^{m-1} \frac{\exp[-i\lambda(u)(t - b_n(u))]}{i!}$$

$$\exp[-\exp[-\lambda(u)t + \lambda(u)b_n(u)], \quad (9)$$

$t \in (-\infty, \infty)$, $u = 1,2,...,z$.

case 2

$$\boldsymbol{IR}^{(2)(\mu)}_{\ n}(t,u) \cong 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Im} e^{-\frac{x^2}{2}} dx \quad (10)$$

where

$$\Im = \frac{\lambda(u)\sqrt{n+1}}{\sqrt{\mu}} t - \frac{\sqrt{n+1}\sqrt{1-\mu}}{\sqrt{\mu}},\ t \in (-\infty, \infty). \quad (11)$$

case 3

$$\boldsymbol{IR}^{(2)(\overline{m})}_{\ n}(t,u) = 1,\ t < 0,$$

$$\boldsymbol{IR}^{(2)(\overline{m})}_{\ n}(t,u)]$$

$$\cong \sum_{i=0}^{n-m} \frac{[\lambda(u)\sqrt{n}t / \sqrt{2}]^{2i}}{i!} \exp[-\lambda^2(u)nt^2 / 2], \quad (12)$$

$t \geq 0$, $u = 1,2,...,z$.

Definition 2.5. A multi-state series system is called an „*m* out of *n*" system with a mixed reserve of its components if its lifetime $T^{(3)}(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T^{(3)}(u) = T_{(n-m+1)}(u), m = 1,2,...,n,\ u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m-th* maximal order statistics in the sequence of the component lifetimes

$$T_i(u) = \{\max_{1\le i\le s_1 n}\{\max_{1\le j\le 2}\{T_{ij}(u)\}\},\ \max_{s_1 n+1\le i\le n}\{\sum_{j=1}^{2} T_{ij}(u)\}\},$$
$$i = 1,2,..,n.,\ u = 1,2,...,z,$$

where $T_{i1}(u)$ are lifetimes of components in the basic system and $T_{i2}(u)$ are lifetimes of reserve components and s_1, s_2, where $s_1 + s_2 = 1$ are fractions of the components with hot and cold reserve, respectively.

The reliability function of the homogeneous multi-state „*m* out of *n*" system with a mixed reserve of its components is given either by

$$\boldsymbol{IR}^{(3)}{}_n^{(m)}(t,\cdot) = [1,\ \boldsymbol{IR}^{(3)}{}_n^{(m)}(t,1),...,\ \boldsymbol{IR}^{(3)}{}_n^{(m)}(t,z)],$$

where

$$\boldsymbol{IR}^{(3)}{}_n^{(m)}(t) = 1 - \sum_{i=0}^{m-1}\binom{n}{i}[1-(F(t,u))^2]^{s_1 i}$$
$$[1-F(t,u)*F(t,u)]^{s_2 i}[F(t,u)]^{2(n-i)s_1}$$
$$[F(t,u)*F(t,u)]^{(n-i)s_2}, \quad (13)$$
$$t \in (-\infty,\infty),\ u = 1,2,...,z,$$

or by

$$\boldsymbol{I\bar{R}}^{(3)}{}_n^{(\bar{m})}(t,\cdot) = [1,\ \boldsymbol{I\bar{R}}^{(3)}{}_n^{(\bar{m})}(t,1),...,\ \boldsymbol{I\bar{R}}^{(3)}{}_n^{(\bar{m})}(t,z)],$$

where

$$\boldsymbol{I\bar{R}}^{(3)}{}_n^{(\bar{m})}(t) = \sum_{i=0}^{\bar{m}}\binom{n}{i}[F(t,u)]^{2s_1 i}[F(t,u)*F(t,u)]^{s_2 i}$$
$$[1-(F(t,u))^2]^{(n-i)s_1}[1-F(t,u)*F(t,u)]^{(n-i)s_2}, \quad (14)$$
$$t \in (-\infty,\infty),\ \bar{m} = n-m,\ u = 1,2,...,z.$$

Lemma 2.3.
case 1: If

(i) $\boldsymbol{I\Re}^{(3)(m)}(t,u) = 1 - \sum_{i=0}^{m-1}\frac{[V(t,u)]^i}{i!}\exp[-V(t,u)]$, $u = 1,2,...,z$, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(3)}{}_n^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system whit a mixed reserve of its components defined by (30),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) m = constant ($m/n \to 0$, as $n \to \infty$),

then

$$\lim_{n\to\infty}\boldsymbol{IR}^{(3)}{}_n^{(m)}(a_n(u)t+b_n(u)) = \boldsymbol{I\Re}^{(3)(m)}(t,u),$$
$$t \in C_{\boldsymbol{I\Re}},\ u = 1,2,...,z,$$

if and only if

$$\lim_{n\to\infty} n[s_1[1-[F(a_n(u)t+b_n(u))]^2]$$
$$+ s_2[1-F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))]]$$
$$= V(t,u),\ t \in C_V,\ u = 1,2,...z,$$

case 2: If

(i) $\boldsymbol{I\Re}^{(3)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-v(t,u)} e^{-\frac{x^2}{2}}dx$, u = 1,2,...,z, is non-degenerate reliability function,

(ii) $\boldsymbol{IR}^{(3)}{}_n^{(m)}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system whit a mixed reserve of its components defined by (30),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $m/n \to \mu$, $0 < \mu < 1$, przy $n \to \infty$,

then

$$\lim_{n\to\infty}\boldsymbol{IR}^{(3)}{}_n^{(m)}(a_n(u)t+b_n(u)) = \boldsymbol{I\Re}^{(3)(\mu)}(t,u),$$
$$t \in C_{\boldsymbol{I\Re}},\ u = 1,2,...,z,$$

if and only if

$$\lim_{n\to\infty}\frac{(n+1)[s_1[1-[F(a_n(u)t+b_n(u))]^2]}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$
$$\frac{s_2[1-F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))]]-m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$
$$= v(t,u),\ u = 1,2,...,z.$$

case 3: If

(i) $\boldsymbol{I\overline{\mathfrak{R}}}^{(3)(\overline{m})}(t,u) = \sum_{i=0}^{m} \frac{[\overline{V}(t,u)]^i}{i!} \exp[-\overline{V}(t,u)]$, $\overline{m} = n - m$, $u = 1,2,...,z$, is non-degenerat reliability function,

(ii) $\boldsymbol{I\overline{R}}^{(3)(\overline{m})}_{n}(t,u)$ is the reliability function of non-degenerate multi-state „*m* out of *n*" system with a mixed reserve of its components defined by (31),

(iii) $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, $u = 1,2,...,z$,

(iv) $n - m = \overline{m}$ = constant $(m/n \to 1 \; as \; n \to \infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{I\overline{R}}^{(3)(\overline{m})}_{n}(a_n(u)t + b_n(u)) = \boldsymbol{I\overline{\mathfrak{R}}}^{(3)(\overline{m})}(t,u),$$
$t \in C_{\mathrm{I}\overline{\mathfrak{R}}}$, $u = 1,2,...,z$,

if and only if

$$\lim_{n\to\infty} n[s_1[F(a_n(u)t + b_n(u))]^2$$
$$+ s_2[F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]]$$
,
$$= \overline{V}(t,u)$$
$t \in C_{\overline{V}}$, $u = 1,2,...,z$.

Proposition 2.3. If components of the homogeneous multi-state „*m* out of *n*" system with a mixed reserve of its components have multi-state exponential reliability functions
and
case 1 m = constant,

$$a_n(u) = \frac{1}{\lambda(u)}, \; \frac{\exp[\lambda(u)b_n(u)]}{2s_1 + s_2\lambda(u)b_n(u)} = n, \; u = 1,2,...,z,$$

then

$$\boldsymbol{I\mathfrak{R}}^{(3)(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \frac{\exp[-it]}{i!} \exp[-\exp[-t]],$$
$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

case 2 $m/n \to \mu$ $0 < \mu < 1$, $n \to \infty$,

$$a_n(u) = \frac{\sqrt{\mu/2}}{\lambda(u)\sqrt{(2s_1 + s_2)(n+1)}},$$

$$b_n(u) = \frac{1}{\lambda(u)} \sqrt{\frac{2(1-\mu)}{2s_1 + s_2}}, \; u = 1,2,...,z,$$

then

$$\boldsymbol{I\mathfrak{R}}^{(3)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx,$$
$t \in (-\infty,\infty)$, $u = 1,2,...,z$,

case 3 $n - m = \overline{m}$ = constant ($m/n \to 1$, $n \to \infty$),

$$a_n(u) = \frac{\sqrt{2}}{\lambda(u)\sqrt{(2s_1 + s_2)n}}, \; b_n(u) = 0, \; u = 1,2,...,z,$$

then

$$\boldsymbol{I\overline{\mathfrak{R}}}^{(3)(\overline{m})}(t,u) = 1, \; t < 0,$$

$$\boldsymbol{I\overline{\mathfrak{R}}}^{(3)(\overline{m})}(t,u) = \sum_{i=0}^{n-m} \frac{t^{2i}}{i!} \exp[-t^2], \; t \ge 0, \; u = 1,2,...,z,$$

is its limit reliability function.

Proof:
case 1: Since for all fixed u, we have

$$a_n(u)t + b_n(u) \to \infty \text{ as } n \to \infty \text{ for } t \in (-\infty,\infty),$$

and

$$1 - [F(a_n(u)t + b_n(u))]^2$$
$$= 2\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$
$$- \exp[-2\lambda(u)(a_n(u)t + b_n(u))], \; t \in (-\infty,\infty),$$

$$1 - F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))$$
$$= [1 + \lambda(u)(a_n(u)t + b_n(u))]$$
$$\exp[-\lambda(u)(a_n(u)t + b_n(u))], \; t \in (-\infty,\infty).$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[s_1[1 - [F(a_n(u)t + b_n(u))]^2]$$
$$+ s_2[1 - F(a_n(u)t + b_n(u))$$
$$* F(a_n(u)t + b_n(u))]]$$
$$= \lim_{n\to\infty}[ns_1[2\exp[-\lambda(u)(a_n(u)t + b_n(u))]$$

$$-\exp[-2\lambda(u)(a_n(u)t+b_n(u))]]$$

$$+ns_2[1+\lambda(u)(a_n(u)t+b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t+b_n(u))]]$$

$$=\lim_{n\to\infty}[ns_1 2\exp[-\lambda(u)(a_n(u)t+b_n(u))]$$

$$[1-1/2\exp[-\lambda(u)(a_n(u)t+b_n(u))]]$$

$$+ns_2\lambda(u)b_n(u)[1+\frac{1+\lambda(u)a_n(u)t}{\lambda(u)b_n(u)}]$$

$$\exp[-\lambda(u)(a_n(u)t+b_n(u))]]$$

$$=\lim_{n\to\infty}\exp[-t][ns_1 2\exp[-\lambda(u)b_n(u)]$$

$$[1-\frac{1}{2}\exp[-t]\exp[-\lambda(u)b_n(u)]]$$

$$+ns_2\lambda(u)b_n(u)[1+\frac{1+t}{\lambda(u)b_n(u)}]$$

$$\exp[-\lambda(u)b_n(u)]]$$

$$=\lim_{n\to\infty}\exp[-t][ns_1 2\exp[-\lambda(u)b_n(u)]$$

$$-ns_1\exp[-t]\exp[-2\lambda(u)b_n(u)]$$

$$+ns_2\lambda(u)b_n(u)\exp[-\lambda(u)b_n(u)]$$

$$+ns_2[1+t]\exp[-\lambda(u)b_n(u)]]$$

$$=\lim_{n\to\infty}\exp[-t]$$

$$[1-\frac{s_1}{n(2s_1+\lambda(u)b_n(u)s_2)^2}\exp[-t]$$

$$+\frac{s_2}{(2s_1+\lambda(u)b_n(u)s_2)}(1+t)]$$

$$=\exp[-t],\ t\in(-\infty,\infty),\ u=1,2,...,z,$$

which by *case* 1 in Lemma 2.3 completes the proof.

case 2: Since for all fixed u, we have

$$a_n(u)t+b_n(u)\to\frac{1}{\lambda(u)}\sqrt{\frac{2(1-\mu)}{2s_1+s_2}}>0 \text{ as } n\to\infty,$$
$$t\in(-\infty,\infty).$$

and

$$1-[F(a_n(u)t+b_n(u))]^2$$

$$=2\exp[-\lambda(u)(a_n(u)t+b_n(u))]$$

$$-\exp[-2\lambda(u)(a_n(u)t+b_n(u))]$$

$$=2[1-\lambda(u)(a_n(u)t+b_n(u))$$

$$+\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2]$$

$$-[1-2\lambda(u)(a_n(u)t+b_n(u))$$

$$+\frac{1}{2}4\lambda^2(u)(a_n(u)t+b_n(u))^2]+o(\frac{1}{n+1})$$

$$=1-\lambda^2(u)(a_n(u)t+b_n(u))^2+o(\frac{1}{n+1}),$$
$$t\in(-\infty,\infty),$$

$$1-F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))$$

$$=[1+\lambda(u)(a_n(u)t+b_n(u))]$$

$$[1-\lambda(u)(a_n(u)t+b_n(u))$$

$$+\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2-o(\frac{1}{n+1})]$$

$$=1-\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2-o(\frac{1}{n+1})],$$
$$t\in(-\infty,\infty).$$

Therefore

$$s_1[1-[F(a_n(u)t+b_n(u))]^2]$$

$$+s_2[F(a_n(u)t+b_n(u))*F(a_n(u)t+b_n(u))]$$

$$=s_1[1-\lambda^2(u)(a_n(u)t+b_n(u))^2]$$

$$+s_2[1-\frac{1}{2}\lambda^2(u)(a_n(u)t+b_n(u))^2]+o(\frac{1}{n+1})$$

$$= 1 - \frac{2s_1 + s_2}{2}\lambda^2(u)(a_n(u)t + b_n(u))^2 \ + o(\frac{1}{n+1})$$

$$= -\frac{\sqrt{\mu(1-\mu)}}{\sqrt{n+1}}t + \mu - o(\frac{1}{n+1}), \ \ t \in (-\infty, \infty),$$

next

$$\nu(t,u) = \lim_{n\to\infty} \frac{(n+1)[s_1[1-[F(a_n(u)t+b_n(u))]^2]}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$+ \frac{s_2[1-F(a_n(u)t+b_n(u)) * F(a_n(u)t+b_n(u))]] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \lim_{n\to\infty} \frac{(n+1)[-\frac{\sqrt{\mu(1-\mu)}}{\sqrt{n+1}}t + \mu - o(\frac{1}{n+1})] - m}{\sqrt{\frac{m(n-m+1)}{n+1}}}$$

$$= \lim_{n\to\infty} \frac{-\sqrt{\mu(1-\mu)}t + o(1)}{\sqrt{\mu(1-\mu)}} = -t, \ t \in (-\infty, \infty),$$
u = 1,2,..., z,

which by *case* 2 in Lemma 2.3 completes the proof.

case 3: Since for all fixed u, we have

$$a_n(u)t + b_n(u) = \frac{\sqrt{2}t}{\lambda(u)\sqrt{(2s_1+s_2)n}} < 0, \ t < 0$$

and

$$a_n(u)t + b_n(u) = \frac{t\sqrt{2}}{\lambda(u)\sqrt{(2s_1+s_2)n}} \geq 0, \ t \geq 0,$$

then

$$F(a_n(u)t + b_n(u)) = 0, \ t < 0,$$

and for $t \geq 0$

$$[F(a_n(u)t + b_n(u))]^2$$

$$= [1 - \exp[-\lambda(u)(a_n(u)t + b_n(u))]]^2,$$

$$F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))$$

$$= [1 - [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]].$$

Therefore

$$V(t,u) = \lim_{n\to\infty} n[s_1[F(a_n(u)t + b_n(u))]^2$$

$$+ s_2[F(a_n(u)t + b_n(u)) * F(a_n(u)t + b_n(u))]]$$

$$= \lim_{n\to\infty}[ns_1[1 - \exp[-\lambda(u)(a_n(u)t + b_n(u))]]^2$$

$$+ ns_2[1 - [1 + \lambda(u)(a_n(u)t + b_n(u))]$$

$$\exp[-\lambda(u)(a_n(u)t + b_n(u))]]]$$

$$= \lim_{n\to\infty}[ns_1[\lambda(u)(a_n(u)t + b_n(u))]^2$$

$$+ ns_2[\lambda^2(u)(a_n(u)t + b_n(u))^2$$

$$-\frac{1}{2}\lambda^2(u)(a_n(u)t + b_n(u))^2]]$$

$$= \lim_{n\to\infty} n[\frac{2s_1 + s_2}{2}(\lambda(u)a_n(u)t)^2]$$

$$= \lim_{n\to\infty} n[\frac{t^2}{n}] = t^2, \ t \geq 0, \ t \in (-\infty, \infty), \ u = 1,2,...,z,$$

which by *case* 3 in Lemma 2.3 completes the proof. □

Corollary 2.3. The reliability function of exponential „m out of n" system with a mixed reserve of its components is given by
case 1

$$\boldsymbol{IR}^{(3)}{}^{(m)}_{n}(t,u) \cong 1 - \sum_{i=0}^{m-1} \frac{\exp[-i\lambda(u)(t - b_n(u))]}{i!}$$

$$\exp[-\exp[-\lambda(u)t + \lambda(u)b_n(u)], \quad (15)$$
$t \in (-\infty, \infty)$, u = 1,2,...,z.

case 2

$$\boldsymbol{IR}^{(3)}{}^{(\mu)}_{n}(t,u) \cong 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\Im} e^{-\frac{x^2}{2}}dx, \quad (16)$$

where

$$\Im = \frac{\lambda(u)\sqrt{(2s_1+s_2)(n+1)2}}{\sqrt{\mu}}t - \frac{2\sqrt{n+1}\sqrt{1-\mu}}{\sqrt{\mu}}, \quad (17)$$

$t \in (-\infty, \infty)$, $u = 1,2,...,z$.

case 3

$$\boldsymbol{IR}^{(3)}{}_{n}^{(\overline{m})}(t,u) = 1, \ t < 0,$$

$$\boldsymbol{IR}^{(3)}{}_{n}^{(\overline{m})}(t,u) \cong \sum_{i=0}^{n-m} \frac{[\lambda(u)\sqrt{(2s_1+s_2)nt}/\sqrt{2}]^{2i}}{i!}$$

$$\exp[-\lambda^2(u)(2s_1+s_2)nt^2/2], \quad (18)$$

$t \geq 0$ $u = 1,2,...,z$.

Proposition 2.3. If components of the homogeneous multi-state „*m* out of *n*" system have improved component reliability functions i.e. its components failure rates $\lambda(u)$ is reduced by a factor $\rho(u)$, $\rho(u) \in <0,1>$, $u = 1,2,...,z$,
and
case 1 m = constant,

$$a_n(u) = \frac{1}{\lambda(u)\rho(u)}, \ b_n(u) = \frac{\log n}{\lambda(u)\rho(u)}, \ u = 1,2,...,z,$$

then

$$\boldsymbol{I\mathfrak{R}}^{(4)(m)}(t,u) = 1 - \sum_{i=0}^{m-1} \frac{\exp[-it]}{i!} \exp[-\exp[-t]],$$

$t \in (-\infty,\infty)$,

case 2 $m/n \to \mu$, $0 < \mu < 1$, $n \to \infty$,

$$a_n(u) = \frac{1}{\lambda(u)\rho(u)\sqrt{n+1}}\sqrt{\frac{1-\mu}{\mu}}, \ b_n(u) = \frac{\log(1/\mu)}{\lambda(u)\rho(u)},$$

$u = 1,2,...,z$,

then

$$\boldsymbol{I\mathfrak{R}}^{(4)(\mu)}(t,u) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx, \ t \in (-\infty,\infty),$$

case 3 $n - m = \overline{m}$ constant ($m/n \to 1$, $n \to \infty$),

$$a_n(u) = \frac{1}{n\lambda(u)\rho(u)}, \ b_n(u) = 0, \ u = 1,2,...,z,$$

then

$$\mathrm{I}\overline{\boldsymbol{\mathfrak{R}}}^{(4)(\overline{m})}(t,u) = 1, \ t < 0,$$

$$\mathrm{I}\overline{\boldsymbol{\mathfrak{R}}}^{(4)(\overline{m})}(t,u) = \sum_{i=0}^{n-m} \frac{t^i}{i!} \exp[-t], \ t \geq 0,$$

is its limit reliability function.

Corollary 2.3. The reliability function of exponential „*m* out of *n*" system with improved reliability functions of its components is given by
case 1

$$\boldsymbol{IR}^{(4)}{}_{n}^{(m)}(t,u) \cong 1 - \sum_{i=0}^{m-1} \frac{\exp[-i\lambda(u)\rho(u)t - \log n)]}{i!}$$

$$\exp[-\exp[-\lambda(u)\rho(u)t + \log n], \quad (19)$$

$t \in (-\infty, \infty)$, $u = 1,2,...,z$.

case 2

$$\boldsymbol{IR}^{(4)}{}_{n}^{(\mu)}(t,u) \cong 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Im} e^{-\frac{x^2}{2}} dx, \quad (20)$$

where

$$\Im = \frac{\lambda(u)\rho(u)\sqrt{(n+1)\mu}}{\sqrt{1-\mu}}t - \frac{\sqrt{n+1}\sqrt{\mu}}{\sqrt{1-\mu}}\log(1/\mu), \quad (21)$$

$t \in (-\infty, \infty)$, $u = 1,2,...,z$.

case 3

$$\boldsymbol{IR}^{(4)}{}_{n}^{(\overline{m})}(t,u) = 1, \ t < 0,$$

$$\boldsymbol{IR}^{(4)}{}_{n}^{(\overline{m})}(t,u)$$

$$\cong \sum_{i=0}^{n-m} \frac{[\lambda(u)\rho(u)nt]^i}{i!} \exp[-\lambda(u)\rho(u)nt], \quad (22)$$

$t \geq 0$, $u = 1,2,...,z$.

3. Comparison of reliability improvement effects

The comparisons of the limit reliability functions of the systems with different kinds of reserve and such systems with improved components allow us to find the value of the components decreasing failure rate factor $\rho(u)$, which warrants an equivalent effect of the system reliability improvement.

3.1. The "*m* out of *n*" system

The comparison of the system reliability improvement effects in the case of the reservation to the effects in the case its components reliability improvement may be obtained by solving with respect to the factor $\rho(u) = \rho(t,u)$ the following equations

$$\boldsymbol{I}\boldsymbol{\Re}^{(4)^{(m)}}\left((t - b_n(u)) / a_n(u)\right)$$

$$= \boldsymbol{I}\boldsymbol{\Re}^{(k)^{(m)}}\left((t - b_n(u)) / a_n(u)\right), \qquad (23)$$

$u = 1,2,...,z$, $k = 1,2,3$.

The factors $\rho(u) = \rho(t,u)$ decreasing components failure rates of the homogeneous exponential multi-state „*m* out of *n*” system equivalent with the effects of hot, cold and mixed reserve of its components as a solution of the comparisons (23) are respectively given by

k = 1

case 1 $\rho(u) = \rho(t,u) = 1 - \frac{\ln 2}{\lambda(u)t}$, $u = 1,2,...,z$,

case 2 $\rho(u) = \frac{2\sqrt{1-\mu}}{\mu} - \frac{2(1-\mu) + \mu \log \mu}{\lambda(u)\mu t}$,
$u = 1,2,...,z$,

case 3 $\rho(u) = \rho(t,u) = \lambda(u)t$, $u = 1,2,...,z$,

k = 2

case 1 $\rho(u) = 1 - \frac{\lambda(u)b_n(u) - \log n}{\lambda(u)t}$

$$= 1 - \frac{\log \lambda(u)b_n(u)}{\lambda(u)t}, \quad u = 1,2,...,z,$$

case 2 $\rho(u) = \frac{\sqrt{1-\mu}}{\mu} - \frac{1-\mu + \mu \log \mu}{\lambda(u)\mu t}$, $u = 1,2,...,z$,

case 3 $\rho(u) = \rho(t,u) = \frac{\lambda(u)t}{2}$, $u = 1,2,...,z$,

k = 3

case 1 $\rho(u) = 1 - \frac{\lambda(u)b_n(u) - \log n}{\lambda(u)t}$

$$= 1 - \frac{\log(2s_1 + s_2\lambda(u)b_n(u))}{\lambda(u)t}, \quad u = 1,2,...,z,$$

case 2

$$\rho(u) = \frac{\sqrt{2(2s_1 + s_2)}\sqrt{1-\mu}}{\mu} - \frac{2(1-\mu) + \mu \log \mu}{\lambda(u)\mu t},$$
$u = 1,2,...,z$,

case 3 $\rho(u) = \rho(t,u) = \frac{(2s_1 + s_2)\lambda(u)t}{2}$, $u = 1,2,...,z$.

4. Conclusion

Proposed in the paper application of the limit multi-state reliability functions for reliability of large systems evaluation and improvement simplifies calculations. The methods may be useful not only in the technical objects operation processes but also in their new processes designing, especially in their optimization. The case of series, parallel (in part 1) and *„m* out of *n”* systems composed of components having exponential reliability functions with double reserve of their components is considered only. It seems to be possible to extend the results to the systems having other much complicated reliability structures and components with different from the exponential reliability function. Further, it seems to be reasonable to elaborate a computer programs supporting calculations and accelerating decision making, addressed to reliability practitioners.

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