Control and Cybernetics

vol. **48** (2019) No. 3

The relation between four input-four output transfer function matrix entries and the RGA entries*

by

Aref Shahmansoorian

EE Department, Imam Khomeini International University
Qazvin, Iran
shahmansoorian@eng.ikiu.ac.ir

Abstract: It is shown that four input-four output transfer functions with identical RGA entries in zero frequency exist and such transfer functions can be parameterized. Four input-four output transfer function entries are parameterized with respect to entries of RGA. This parameterization is useful for independent loops design, where the compensator is designed such that RGA has desired value. Finally a numerical criterion for being close to triangularity for compensated transfer function matrix is proposed

Keywords: RGA, input-output pairing, compensator design, four input-four output systems

1. Introduction

One of important issues in decentralized control of multivariate systems is the choice of controllable variables (outputs) and changeable (manipulated) variables (inputs). After this step, it is necessary to select the appropriate input and output pairs. This means that an input or a set of inputs is used to control an output or a set of outputs, this assignment being referred to as "input/output pairing" (see Maciejowski, 1989). The issue of interaction between the loops and the stability of all loops, as well as maintaining the stability of the whole system in the event of the opening of some loops (integrity) and the performance that can be obtained from the controller are considered in the selection of input and output pairs.

Several tools have been proposed to solve the problem and also to measure the interaction such as the relative gain array (RGA), the Niederlinski Index, the Structural Singular Value, and the Henkel Interaction Index Matrix (HIIA) (see Bristol, 1966; Wittenmark and Salgado, 2002, or Skogestad and Postlethwaite, 2005). However, the most widely used of these is RGA, which was introduced by Bristol (1966), commonly used in the industry.

*Submitted: February 2019; Accepted: August 2019

The RGA for the $G(s)$ square transform function is defined as follows (here, $\circ*$ denotes the element-by-element multiplication):

$$\Lambda(G) = G \circ *(G^{-1})^T.$$

This indicator is used for zero frequency, but in some cases, non-zero frequencies have also been considered, see Khaki-Sedigh and Moaveni (2009). The properties of RGA are given in Maciejowski (1989), Skogestad and Postlethwaite (2005), Khaki-Sedigh and Moaveni (2009), and in Hovd and Skogestad (1992). The block RGA is also defined in Manousioutakis, Savage and Arkun (1986) and in Kariwala, Forbes and Meadows (2003). RGA has been generalized to non-square systems by Chang and Yu (1990) using generalized inverse matrices. In Kadhim, Birk and Arranz (2016), a method was presented for estimating the dynamical RGA for nonlinear systems. The design of the control structure of an oil separation tower was carried out using RGA in Khalilipour et al. (2016). In Jain and Babu (2016), the sensitivity of RGA entries to uncertainties of inputs and time constants and delays has been investigated.

The most important property of RGA is that the sum of elements of each row and each column is one. Algebraically, there is no other relationship between the RGA entries, and it is assumed that they can be any arbitrary values such that the sum of the entries of each row and each column is one. But it has been shown in Shahmansoorian (2013) that there is no three input-three output system whose RGA entries are equal at zero frequency. In Section 2 of the present paper, it is shown that four input-four output systems with identical RGA entries at zero frequency exist, and the structure of such systems is expressed in a parametric manner. In Section 3 of the paper, the transfer function matrix entries are expressed in terms of RGA entries. These relationships are useful in the design of independent loop closing, so that the compensator is designed such that RGA have an optimal value. Finally, in Section 4, with the introduction of a lemma, a numerical criterion for being close to triangularity for the compensated transfer function matrix is defined.

2. The four input-four output system with identical RGA entries

It has been shown in Shahmansoorian (2013) that there is no three input-three output system with identical RGA entries (1/3) at zero frequency. Here, it will be shown that there are four input-four output systems with identical RGA enteries (1/4), and the general structure of such systems will be expressed parametrically. These systems have the same interaction for all input and output pairs and there are possible 24 input-output pairs. All of these scenarios can be considered in terms of other functional and design aspects and the choice can be performed of the best. Throughout this paper, it was assumed that the transfer function matrices and RGA entries in the zero frequency are non-zero.

All calculations are carried out at zero frequency, although these relationships also exist at non-zero frequencies.

Given that the RGA is independent of the scale of rows and columns of the matrix, it is assumed that the system transfer function is as follows:

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & g_{22} & g_{23} & g_{24} \\ 1 & g_{32} & g_{33} & g_{34} \\ 1 & g_{42} & g_{43} & g_{44} \end{bmatrix}. \quad (1)$$

The conditions are checked, under which the RGA of system (1) is as follows:

$$\Lambda = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}. \quad (2)$$

According to the definition of RGA, it can be said that G^{-1} is as follows:

$$G^{-1} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/(4g_{22}) & 1/(4g_{32}) & 1/(4g_{42}) \\ 1/4 & 1/(4g_{23}) & 1/(4g_{33}) & 1/(4g_{43}) \\ 1/4 & 1/(4g_{24}) & 1/(4g_{34}) & 1/(4g_{44}) \end{bmatrix}. \quad (3)$$

Given the equality of $GG^{-1} = I$, the following set of equations is obtained:

$$\begin{aligned} &g_{22} + g_{23} + g_{24} = -1\ , \quad 1/g_{22} + 1/g_{23} + 1/g_{24} = -1 \\ &g_{32} + g_{33} + g_{34} = -1\ , \quad 1/g_{32} + 1/g_{33} + 1/g_{34} = -1 \\ &g_{42} + g_{43} + g_{44} = -1\ , \quad 1/g_{42} + 1/g_{43} + 1/g_{44} = -1 \\ &g_{22}/g_{32} + g_{23}/g_{33} + g_{24}/g_{34} = -1\ , \quad g_{32}/g_{22} + g_{33}/g_{23} + g_{34}/g_{24} = -1 \\ &g_{22}/g_{42} + g_{23}/g_{43} + g_{24}/g_{44} = -1\ , \quad g_{42}/g_{22} + g_{43}/g_{23} + g_{44}/g_{24} = -1 \\ &g_{32}/g_{42} + g_{33}/g_{43} + g_{34}/g_{44} = -1\ , \quad g_{42}/g_{32} + g_{43}/g_{33} + g_{44}/g_{34} = -1 \end{aligned} \quad (4)$$

It can be concluded from the two equations of the first row in the set of equations (4), that one of g_{22}, g_{23}, g_{24} entries is equal to -1 and the other two are also symmetric. These conditions are also established in the third and fourth lines of G. Therefore, the general structure of the system is one out of the following three alternative (by rotating -1's in three columns) structures:

Alternative 1:

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \alpha & -\alpha \\ 1 & -1 & \beta & -\beta \\ 1 & -1 & \gamma & -\gamma \end{bmatrix}$$

Alternative 2:

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \alpha & -\alpha \\ 1 & \beta & -1 & -\beta \\ 1 & -1 & \gamma & -\gamma \end{bmatrix}$$

Alternative 3:

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \alpha & -\alpha \\ 1 & \beta & -1 & -\beta \\ 1 & \gamma & -\gamma & -1 \end{bmatrix}$$

In the first alternative, the matrix is singular and therefore this alternative is not possible. Hence, the second and the third alternatives are investigated.

Alternative 2: Matrix entries should establish the following six equations

$$\begin{aligned} -1/\beta - \alpha + \alpha/\beta \quad &= -1 \\ -\beta - 1/\alpha + \beta/\alpha \quad &= -1 \\ 1 + 2\alpha/\gamma \quad &= -1 \\ 1 + 2\gamma/\alpha \quad &= -1 \\ -\beta - 1/\gamma + \beta/\gamma \quad &= -1 \\ -1/\beta - \gamma + \gamma/\beta \quad &= -1. \end{aligned} \tag{5}$$

The general solution to equations (5) is $\gamma = -\alpha,\ \beta = 1$. Thus, the second alternative is ($\theta \neq 0$):

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \theta & -\theta \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\theta & \theta \end{bmatrix}. \tag{6}$$

Alternative 3:

$$\begin{aligned} -1/\beta - \alpha + \alpha/\beta &= -1 \\ -\beta - 1/\alpha + \beta/\alpha &= -1 \\ -1/\gamma - \alpha/\gamma + \alpha &= -1 \\ -\gamma - \gamma/\alpha + 1/\alpha &= -1 \\ \beta/\gamma + 1/\gamma + \beta &= -1 \\ \gamma/\beta + \gamma + 1/\beta &= -1. \end{aligned} \tag{7}$$

These equations have two solutions, namely $\gamma = \alpha = 1,\ \beta = -1$ and $\gamma = \alpha = -1,\ \beta = 1$. Both of these solutions lead to a matrix G, which is a special case of the second alternative, considered before.

Therefore, all four input-four output systems, whose RGA entries have the same values (1/4) have the following general structure, which can be always obtained by means of such operations as transposition or inverting or permutation of rows or columns or appropriate scaling:

$$G = \begin{bmatrix} 1 & -1 & \theta & -\theta \\ 1 & -1 & -\theta & \theta \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}. \tag{8}$$

3. Expression of the entries of the four input-four output transfer function in terms of RGA entries

In this section, the entries of the four input-four output system in a special case are expressed in terms of its RGA entries. Suppose that the transfer function matrix should be found so that RGA is as follows:

$$\Lambda = \begin{bmatrix} 1 & a & b & -a-b \\ a & 1 & -a-b & b \\ b & -a-b & 1 & a \\ -a-b & b & a & 1 \end{bmatrix}. \tag{9}$$

Since the RGA is independent of the scaling of the rows and the columns, assume

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & x_1 & x_2 & x_3 \\ 1 & x_4 & x_5 & x_6 \\ 1 & x_7 & x_8 & x_9 \end{bmatrix}. \tag{10}$$

Given that the RGA is in the form of (9), G^{-1} is obtained as follows:

$$G^{-1} = \begin{bmatrix} 1 & a & b & -a-b \\ a & 1/x_1 & (-a-b)/x_4 & b/x_7 \\ b & (-a-b)/x_2 & 1/x_5 & a/x_8 \\ -a-b & b/x_3 & a/x_6 & (a+b)/x_9 \end{bmatrix}. \tag{11}$$

By multiplying the first row of G by the second column of G^{-1} and also by multiplying the second row of G by the first column of G^{-1}, we obtain

$$a + 1/x_1 - (a+b)/x_2 + b/x_3 = 0 \tag{12}$$

$$1 + ax_1 + bx_2 - (a+b)x_3 = 0. \tag{13}$$

Resulting from these two equations, we get

$$\frac{1+ax_1}{x_1} = \frac{1+ax_1}{x_2x_3}. \tag{14}$$

A solution to equation (14) is

$$x_1 = -1/a, \tag{15}$$

which leads to

$$x_3 = \frac{b}{a+b} x_2. \tag{16}$$

By multiplying the third row of G by the first columns of G^{-1}, we obtain

$$1 + ax_4 + bx_5 - (a+b)x_6 = 0. \tag{17}$$

By multiplying the first row of G by the third columns of G^{-1}, we obtain

$$\frac{1 + bx_5}{x_5} + \frac{ax_4 - (a+b)x_6}{x_4 x_6} = 0. \tag{18}$$

Now, $ax_4 - (a+b)x_6$ is equal to $-(1 + bx_5)$ from Equation (17). Applying this in Equation (18) yields

$$\frac{(1 + bx_5)}{x_5} - \frac{(1 + bx_5)}{x_4 x_6} = 0. \tag{19}$$

A solution to this equation is

$$x_5 = -1/b. \tag{20}$$

By applying $x_5 = -1/b$ in Equation (17) we get

$$x_6 = \frac{a}{a+b} x_4. \tag{21}$$

By multiplying the second row of G by the third column of G^{-1} and applying $x_1 = -1/a$ and $x_5 = -1/b$ in Equations (16) and (21), we obtain

$$x_4 = \frac{a + b + abx_2}{ab(x_2 - 1)}, \tag{22}$$

and from Equation (21),

$$x_6 = \frac{a + b + abx_2}{b(a+b)(x_2 - 1)}. \tag{23}$$

By multiplying the first row of G^{-1} by the second column of G and applying $x_1 = -1/a$, and using equation (22), we obtain

$$x_7 = \frac{a + b + abx_2}{a(a+b)(x_2 - 1)}. \tag{24}$$

By multiplying the first row of G^{-1} by the third column of G and applying $x_5 = -1/b$, we obtain

$$x_8 = \frac{ax_2}{(a+b)}. \tag{25}$$

By multiplying the first row of G^{-1} by the fourth column of G and applying Equations (23) and (16), we get

$$x_9 = \frac{(a+b+abx_2)x_2}{(a+b)^2(x_2-1)}. \tag{26}$$

The values of $x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9$, which are derived from Equations (15), (16), (20), (22), (23), (24), (25), and (26) are obtained on the basis of x_2, and the equality $GG^{-1} = I$ is established. Therefore, the obtained solution is acceptable, and in this manner the sixteen entries of the transfer function are expressed on the basis of RGA entries.

Note 1: x_2 is the design parameter and various solutions can be obtained by choosing it. Each solution is, in fact, a set of solutions, because by inverting or scaling rows and columns or by transposing the transfer function matrix, we still do obtain the same RGA (provided $x_2 \neq 1$).

Note 2: The values provided constitute just one solution and there exist other solutions, and, if necessary, they can be calculated and selected according to design goals.

Example 1 Distillation tower system (Kariwala, Forbes and Meadows, 2003)

$$G(s) = \begin{bmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-1.2s}}{(31.6s+1)(20s+1)} & \frac{-0.25e^{-1.4s}}{(21s+1)} & \frac{-0.49e^{-6s}}{(22s+1)^2} \\ \frac{-4.17e^{-5s}}{(45s+1)} & \frac{6.93e^{-1.02s}}{(44.6s+1)} & \frac{-0.05e^{-6s}}{(34.5s+1)^2} & \frac{1.53e^{-3.8s}}{(48s+1)} \\ \frac{1.73e^{-18s}}{(13s+1)^2} & \frac{5.11e^{-12s}}{(13.3s+1)^2} & \frac{4.61e^{-1.01s}}{(18.5s+1)} & \frac{-5.49e^{-1.5s}}{(15s+1)} \\ \frac{-11.2e^{-2.6s}}{(43s+1)(6.53s+1)} & \frac{14(10s+1)e^{-0.02s}}{(45s+1)(17.4s^2+3s+1)} & \frac{0.1e^{-0.05s}}{(31.6s+1)(5s+1)} & \frac{4.49e^{-0.6s}}{(48s+1)(6.3s+1)} \end{bmatrix} \tag{27}$$

RGA at zero frequency is as follows

$$\Lambda = \begin{bmatrix} 2.8119 & -0.9601 & -0.2556 & -0.5964 \\ -4.0958 & 4.2281 & 0.0189 & 0.8553 \\ 0.0903 & 0.0945 & 1.1815 & -0.3662 \\ 2.1914 & -2.3562 & 0.0554 & 1.1072 \end{bmatrix}. \tag{28}$$

It can be seen that some entries of the main diagonal are larger than one and some of the elements outside the main diagonal are negative and large. Assume that the required RGA of the compensated system is as follows:

$$\Lambda_1 = \begin{bmatrix} 1 & -0.1 & -0.1 & 0.2 \\ -0.1 & 1 & 0.2 & -0.1 \\ -0.1 & 0.2 & 1 & -0.1 \\ 0.2 & -0.1 & -0.1 & 1 \end{bmatrix}. \tag{29}$$

Assuming that $a = b = -0.1$ and $x_2 = 5$, from Equations (15), (16), (20), (22), (23), (24), (25), and (26), the compensated transfer function matrix is obtained as follows

$$G_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 10 & 5 & 2.5 \\ 1 & -3.75 & 10 & -1.875 \\ 1 & -1.875 & 2.5 & -4.6875 \end{bmatrix}. \quad (30)$$

Given $K = G^{-1}(0)G_1$, the compensator is obtained in the following form (see Fig. 1):

$$K = \begin{bmatrix} 1.5515 & 10.9511 & 5.7540 & 4.0643 \\ 0.6174 & 6.5500 & 2.9900 & 2.4519 \\ 1.4923 & -4.3559 & 3.2624 & -2.8485 \\ 2.1345 & 6.5729 & 5.5142 & 1.5125 \end{bmatrix} \quad (31)$$

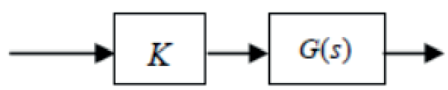

Figure 1. The control structure including the compensator

This is a concrete, definite answer, and by selecting other values for x_2 other answers can be obtained. In addition, in general, the entries of the first row and those of the first column of transfer function matrix do not have to be equal to 1, and by scaling the rows and columns they can be changed.

4. A numerical criterion for "being close to triangularity" for the compensated system

In this section, by introducing a lemma, a numerical criterion for being close to triangularity for the compensated system is presented.

Lemma 1 *Let $\Lambda = [\lambda_{ij}]_{n\times n}$ be the RGA of a transfer function matrix. Then, $mI + (1-m)\Lambda$ is the+ RGA of another transfer function matrix. (I is the identity matrix).*

Proof We just prove that the sum of the entries of each row and each column is equal to one. We prove this for the i^{th} row. The proof for columns is similar.

Since Λ is an RGA matrix, so,

$$\sum_{j=1}^{n} \lambda_{ij} = 1. \tag{32}$$

The sum of the entries of the i^{th} row of $mI + (1-m)\Lambda$ is

$$m + (1-m)\sum_{j=1}^{n} \lambda_{ij} = m + (1-m) = 1. \tag{33}$$

Now, if the RGA of a system is Λ, the compensator is designed so that the RGA of the compensated system has the form of $mI + (1-m)\Lambda$, $0 \leq m \leq 1$. In this case, m can be considered as a measure of being close to triangularity for the compensated system. The closer m is to one, the closer is the compensated system to triangularity. Of course, in general, the RGA matrix of the compensated system does not necessarily have the structure like $mI + (1-m)\Lambda$, but the advantage of this choice is that it does not change the entries of the main diagonal elements that are equal to 1, and it causes that the diagonal elements larger than one get closer to 1. Additionally, this choice decreases the entries away from the main diagonal of RGA and reduces the RGA-number due to multiplication by $(1-m)$. It is important to note that even if the RGA matrix is the identity matrix, it cannot be concluded that the system is triangular. Only for 2×2 systems $RGA{=}I$ implies that the transfer function matrix is triangular. Nevertheless, $RGA{=}I$ is preferred, because otherwise the plant cannot be triangular, see Skogestad and Postlethwaite (2005). How close the compensated system is to the triangular state at zero frequency or the other frequencies, needs to be investigated.

5. Conclusion

It is shown that there are four input-four output systems with identical RGA entries at zero frequency, and the general structure of such systems is expressed in the parametric form. The matrix entries of the transfer function of the four input-four output system are expressed in terms of the RGA entries. These relationships are useful in the design of independent loops, so that the compensator is designed such that the RGA has a desirable shape before the closing of the loops. The criterion for being close to triangularity for the compensated transfer function matrix is defined by presenting a lemma.

6. References

Bristol, E. H. (1966) On a new measure of interactions for multivariable process control. *IEEE Transactions on Automatic Control*, **AC-11**, 133-134.

Chang, J. and Yu, C. (1990) The relative gain for nonsquare multivariable systems. *Chemical Engineering Science*, **45**, 5, 1309-1323.

Hovd, M. and Skogestad, S. (1992) Use of simple frequency-dependent tools for control system analysis, structure selection and design. *Automatica*, **28**, 5, 989-996.

Jain, A. and Babu, B. V. (2016) Sensitivity of Relative Gain Array for Processes with Uncertain Gains and Residence Times. *IFAC-Papers OnLine* **49** (1), 486-491.

Kadhim, A. M. H., Birk, W. and Arranz, M. C. (2016) Dynamic Relative Gain Array Estimation using Local Polynomial Approximation Approach. *Modeling, Identification and Control*, **37**, 4, 247-259.

Kariwala, V., Forbes, J. F. and Meadows, E. S. (2003) Block relative gain: properties and pairing rules. *Industrial & Engineering Chemistry Research*, **42**, 20, 4664-4574.

Khaki-Sedigh, A. and Moaveni, B. (2009) Control configuration selection for multivariable plants. In: **LNCIS 391**, Springer Verlag.

Khalilipour, M. M., Sadeghi, J., Shahraki, F. and Razaghi, K. (2016) Designing Non-Square Multivariate Control Structures in the Process of Isolation of Crude Oil: The Effect of Capacity Change. *Isolation Engineering Science Journal*, **8**, 2.

Maciejowski, J. M. (1989) *Multivariable Feedback Design*. Addison-Wesley.

Manousioutakis, V., Savage, R. and Arkun, Y. (1986) Synthesis of decentralized processes control structure using the concept of block relative gain. *AIChE Journal*, **32**, 991-1003.

Shahmansoorian, A. (2013) Relation Between Transfer Function Entries and the RGA Entries and Its Application in Decentralized Controllers Design. *Journal of Control,* **7**, 3, Autumn 2013.

Skogestad, S. and Postlethwaite, I. (2005) *Multivariable Feedback Control Analysis and Design*. Wiley, Second edition.

Wittenmark, B. and Salgado, M. E. (2002) Hankel-norm based interaction measure for input-output pairing. *The 2002 IFAC World Congress*. Barcelona, Spain.