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POSITIVE AND STABLE TIME-VARYING CONTINUOUS-TIME LINEAR SYSTEMS AND ELECTRICAL CIRCUITS

The positivity and stability of a class of time-varying continuous-time linear systems and electrical circuits are addressed. Sufficient conditions for the positivity and asymptotic stability of the system are established. It is shown that there exists a large class of positive and asymptotically stable electrical circuits with time-varying parameters. Examples of positive electrical circuits are presented.

KEYWORDS: positive, linear, time-varying, system, electrical circuit, stability, test

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [1, 5]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc..

The positivity and stability of fractional time varying discrete-time linear systems have been addressed in [7, 10, 11] and the stability of continuous-time linear systems with delays in [12]. The fractional positive linear systems have been analyzed in [3, 4, 14-17].The positive electrical circuits and their reachability have been considered in [6, 9] and the controllability and observability in [2]. The stability and stabilization of positive fractional linear systems by state-feedbacks have been analyzed in [13, 14]. The Hurwitz stability of Metzler matrices has been investigated in [14, 15, 18].

In this paper positivity and stability of a class of time-varying continuous-time linear systems and electrical systems will be addressed.

The paper is organized as follows. In section 2 the solution to the scalar time-varying linear system and some stability tests of positive continuous-time linear systems are recalled. Sufficient conditions for the positivity and asymptotic stability of a class of time-varying continuous-time linear systems and electrical systems are established in section 3. The positive and asymptotically stable

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electrical circuits with time-varying parameter are addressed in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix, T – denotes the transposition of matrix (vector).

2. PRELIMINARIES

Consider the scalar time-varying continuous-time linear system

$$\dot{x}(t) = -a(t)x(t) + b(t)u(t)\,,\ t\in[0,+\infty) \tag{2.1}$$

where $x(t)$ and $u(t)$ are the state and input of the system and $a(t)$, $b(t)$ are continuous-time functions.

Lemma 2.1. The solution of (2.1) for given initial condition $x_0 = x(0)$ and input $u(t)$ has the form

$$x(t) = e^{-\int a(t)dt}x_0 + \int_0^t e^{-\int a(t-\tau)dt}b(\tau)u(\tau)d\tau\,. \tag{2.2}$$

Proof. Using (2.2) and (2.1) we obtain

$$-a(t)x(t) + b(t)u(t) = -a(t)\left[e^{-\int a(t)dt}x_0 + \int_0^t e^{-\int a(t-\tau)dt}b(\tau)u(\tau)d\tau\right] + b(t)u(t)$$

$$= -a(t)e^{-\int a(t)dt}x_0 - a(t)\int_0^t e^{-\int a(t-\tau)dt}b(\tau)u(\tau)d\tau + b(t)u(t) = \dot{x}(t).$$

□

Consider the autonomous continuous-time linear system with constant coefficients

$$\dot{x}(t) = Ax(t)\,, \tag{2.3}$$

where $x(t)\in\Re^n$ is the state vector and $A = [a_{ij}]\in M_n$.

Theorem 2.1. [15] The positive system (2.3) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0\,, \tag{2.4}$$

are positive, i.e. $a_k > 0$ for $k = 0,1,\ldots,n-1$.

2) All principal minors M_k, $k = 1,\ldots,n$ of the matrix $-A$ are positive, i.e.

$$M_1 = -a_{11} > 0,\ \ M_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \ldots, M_n = \det[-A] > 0 \tag{2.5}$$

3) The diagonal entries of the matrices

$$A_{n-k}^{(k)} \text{ for } k = 1,\ldots,n-1 \tag{2.6a}$$

are negative, where $A_{n-k}^{(k)}$ are defined as follows:

$$A_n^{(0)} = A = \begin{bmatrix} a_{11}^{(0)} & \ldots & a_{1,n}^{(0)} \\ \vdots & \ldots & \vdots \\ a_{n,1}^{(0)} & \ldots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} a_{11}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & A_{n-1}^{(0)} \end{bmatrix},\ A_{n-1}^{(0)} = \begin{bmatrix} a_{22}^{(0)} & \ldots & a_{2,n}^{(0)} \\ \vdots & \ldots & \vdots \\ a_{n,2}^{(0)} & \ldots & a_{n,n}^{(0)} \end{bmatrix},$$
$$b_{n-1}^{(0)} = [a_{12}^{(0)} \ \ldots \ a_{1,n}^{(0)}],\ c_{n-1}^{(0)} = \begin{bmatrix} a_{21}^{(0)} \\ \vdots \\ a_{n,1}^{(0)} \end{bmatrix} \tag{2.6b}$$

and

$$A_{n-k}^{(k)} = A_{n-k}^{(k-1)} - \frac{c_{n-k}^{(k-1)} b_{n-k}^{(k-1)}}{a_{k+1,k+1}^{(k-1)}} = \begin{bmatrix} a_{k+1,k+1}^{(k)} & \ldots & a_{k+1,n}^{(k)} \\ \vdots & \ldots & \vdots \\ a_{n,k+1}^{(k)} & \ldots & a_{n,n}^{(k)} \end{bmatrix} = \begin{bmatrix} a_{k+1,k+1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & A_{n-k-1}^{(k)} \end{bmatrix},$$
$$A_{n-k-1}^{(k)} = \begin{bmatrix} a_{k+2,k+2}^{(k)} & \ldots & a_{k+2,n}^{(k)} \\ \vdots & \ldots & \vdots \\ a_{n,k+2}^{(k)} & \ldots & a_{n,n}^{(k)} \end{bmatrix},\ b_{n-k-1}^{(k)} = [a_{k+1,k+2}^{(k)} \ \ldots \ a_{k+1,n}^{(k)}],\ c_{n-k-1}^{(k)} = \begin{bmatrix} a_{k+2,k+1}^{(k)} \\ \vdots \\ a_{n,k+1}^{(k)} \end{bmatrix} \tag{2.6c}$$

for $k = 1,\ldots,n-1$.

4) All diagonal entries of the upper (lower) triangular matrix

$$\widetilde{A}_u = \begin{bmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \ldots & \widetilde{a}_{1,n} \\ 0 & \widetilde{a}_{22} & \ldots & \widetilde{a}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \widetilde{a}_{n,n} \end{bmatrix},\ \widetilde{A}_l = \begin{bmatrix} \widetilde{a}_{11} & 0 & \ldots & 0 \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_{n,1} & \widetilde{a}_{n,2} & \ldots & \widetilde{a}_{n,n} \end{bmatrix} \tag{2.7}$$

are negative, i.e. $\widetilde{a}_{kk} < 0$ for $k = 1,\ldots,n$ and the matrices $\widetilde{A}$ has been obtained from the matrix A by the use of elementary row operation [5, 14].

The elementary row operations for time-varying systems are the following:

1) Multiplication of the ith row by a real number $c(t)$. This operation will be denoted by $L[i \times c(t)]$.
2) Addition to the ith row (column) of the jth row (column) multiplied by a real number $c(t)$. This operation will be denoted by $L[i + j \times c(t)]$.
3) Interchange of the ith and jth rows (columns). This operation will be denoted by $L[i, j]$.

3. POSITIVE TIME-VARYING CONTINUOUS-TIME LINEAR SYSTEMS

Consider the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.1a}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{3.1b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A(t) \in \Re^{n \times n}$, $B(t) \in \Re^{n \times m}$, $C(t) \in \Re^{p \times n}$, $D(t) \in \Re^{p \times m}$ are real matrices with entries depending continuously on time and $\det A(t) \neq 0$ for $t \in [0,+\infty)$.

Definition 2.1. The system (3.1) is called positive if $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \in [0,+\infty)$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \in [0,+\infty)$.
It is assumed that $A(t) \in M_n$ with negative diagonal entries and nonnegative off diagonal entries for all $t \in [0,+\infty)$.

Theorem 3.1. The time-varying linear system (3.1) with upper triangular form

$$A_u(t) = \begin{bmatrix} -a_{11}(t) & a_{12}(t) & \dots & a_{1,n}(t) \\ 0 & -a_{22}(t) & \dots & a_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -a_{n,n}(t) \end{bmatrix} \in M_n(t), \tag{3.2a}$$

or lower triangular form

$$A_l(t) = \begin{bmatrix} -a_{11}(t) & 0 & \dots & 0 \\ a_{21}(t) & -a_{22}(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \dots & -a_{n,n}(t) \end{bmatrix} \in M_n(t) \tag{3.2b}$$

with negative diagonal entries for $t \in [0,+\infty)$ and

$$B(t) \in \Re_+^{n \times m},\ C(t) \in \Re_+^{p \times n},\ D(t) \in \Re_+^{p \times m},\ t \in [0,+\infty) \tag{3.3}$$

is positive and asymptotically stable.

Proof. For the matrices $A(t)$ and $B(t)$ using (3.1a) and (3.2) we obtain

$$\dot{x}_n(t) = -a_{nn}(t)x_n(t) + \sum_{k=1}^{m} b_{nk}(t)u_k(t) \tag{3.4}$$

where

$$x(t) = [x_1(t) \ \dots \ x_n(t)]^T,\ u(t) = [u_1(t) \ \dots \ u_m(t)]^T,\ B(t) = \begin{bmatrix} b_{11}(t) & \dots & b_{1,m}(t) \\ \vdots & \dots & \vdots \\ b_{n,1}(t) & \dots & b_{n,m}(t) \end{bmatrix}. \tag{3.5}$$

By Lemma 2.1 the solution of (3.4) has the form

$$x_n(t) = e^{-\int a_{n,n}(t)dt} x_{n0} + \sum_{k=1}^{m} \int_0^t e^{-\int a_{n,n}(t)(t-\tau)dt} b_{nk}(\tau)u_k(\tau)d\tau \tag{3.6}$$

and $x_n(t) \in \Re_+$, $t \in [0,+\infty)$ for all $x_{0n} \in \Re_+$ and $x_k(t) \in \Re_+$ for $t \in [0,+\infty)$.
Similarly, form (3.1a) and (3.2) we obtain

$$\dot{x}_{n-1}(t) = e^{-\int a_{n-1,n-1}(t)dt} x_{n-1,0} + \int_0^t e^{-\int a_{n-1,n-1}(t)(t-\tau)dt} [a_{n-1,n}(\tau)x_n(\tau) + \sum_{k=1}^{m} b_{n-1,k}(\tau)u_k(\tau)]d\tau . \tag{3.7}$$

From (3.6) we have $x_{n-1}(t) \in \Re_+$ for $t \in [0,+\infty)$ since $x_n(t) \in \Re_+$ for $t \in [0,+\infty)$.
Continuing this procedure we obtain

$$x_k(t) \in \Re_+ \text{ for } k = 1,2,\ldots,n \text{ and } t \in [0,+\infty) \tag{3.8}$$

and any nonnegative initial conditions and inputs.
From (3.1b) it follows that $y(t) \in \Re_+^p$, $t \in [0,+\infty)$ if the conditions (3.2) and (3.3) are satisfied for any nonnegative initial conditions and all nonnegative inputs.
If the matrix (3.2) has negative diagonal entries then its all eigenvalues are negative function for $t \in [0,+\infty)$ and from (2.2) for $u(t) = 0$ it follows that $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \Re_+^n$. □

Remark 3.1. To check the asymptotic stability of the time-varying continuous-time linear system (2.1) the Theorem 2.1 can be used.
The system is asymptotically stable if one of the equivalent conditions of Theorem 2.1 is satisfied for all $t \in [0,+\infty)$.

Example 3.1. Consider the time-varying continuous-time linear system (2.1) with the matrices

$$A_l(t) = \begin{bmatrix} -e^{-t} & 0 & 0 \\ 1 & -1 & 0 \\ e^{-t} & 0 & -e^{-t} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 2 + 2.2e^{-t} + \sin t \\ 1 + 1.2e^{-t} \\ e^{-t} \end{bmatrix}, \tag{3.9}$$
$$C(t) = [0.1 \quad 1 + 0.5\sin t \quad 2e^{-t}], \quad D(t) = [0].$$

From (3.9) it follows that the system is positive and asymptotically stable since $A_l(t) \in M_3(t)$, $B(t) \in \Re_+^3$, $C(t) \in \Re_+^{1\times 3}$ for $t \in [0,+\infty)$.
From (3.9) we have

$$\begin{aligned} \dot{x}_1(t) &= -e^{-t}x_1(t) + (2 + 2.2e^{-t} + \sin t)u(t), \\ \dot{x}_2(t) &= x_1(t) - x_2(t) + (1 + 1.2e^{-t})u(t), \\ \dot{x}_3(t) &= e^{-t}x_1(t) - e^{-t}x_3(t) + e^{-t}u(t). \end{aligned} \tag{3.10}$$

Using Lemma 2.1 we can find in sequence the positive solution of the equation (3.10).

4. POSITIVE TIME-VARYING LINEAR CIRCUITS

First let us consider a simple time-varying electrical circuit shown in Fig. 4.1 with given resistance $R(t)$, inductance $L(t)$ depending on time *t*, and source voltage $e(t)$.

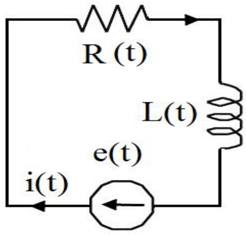

Fig. 4.1. Electrical circuit

Using Kirchhoff's law, we can write the equation

$$e(t) = \left[R(t) + \frac{dL(t)}{dt} \right] i(t) + L(t) \frac{di(t)}{dt}, \tag{4.1}$$

which can be written in the form

$$\frac{di(t)}{dt} = -a(t)i(t) + b(t)e(t), \tag{4.2a}$$

where

$$a(t) = \frac{1}{L(t)} \left[R(t) + \frac{dL(t)}{dt} \right], \quad b(t) = \frac{1}{L(t)}. \tag{4.2b}$$

Using the formula (2.2) we can find the solution $i(t) \in \Re_+$, $t \in [0,+\infty)$ to the equation (4.2a) for given positive resistance $R(t)$ positive inductance $L(t)$ and nonnegative source voltage $e(t)$.

Therefore, the electrical circuit is a positive and asymptotically stable system if the resistance and inductance are positive functions of *t* and $e(t) \in \Re_+$, $t \in [0,+\infty)$.

Now let us consider electrical circuit shown on Fig. 4.2 with given conductances $G_k(t)$, $k = 0,1,...,n$ depending on time *t*, inductances L_i, $i = 2,4,...,n_2$, capacitances C_j, $j = 1,3,...,n_1$ and source voltages $e_1(t), e_2(t),..., e_n(t)$. We shall show that this electrical circuit is a positive time-varying linear system.

Using the Kirchhoff's law we can write the equations

$$e_1(t) = \frac{C_k}{G_k(t)} \frac{du_k(t)}{dt} + u_k(t) \text{ for } k = 1,3,\ldots,n_1, \tag{4.3a}$$

$$e_1(t)+e_k(t)=L_k\frac{di_k(t)}{dt}+\frac{i_k(t)}{G_k(t)}+u_k(t) \text{ for } k=2,4,\ldots,n_2, \tag{4.3b}$$

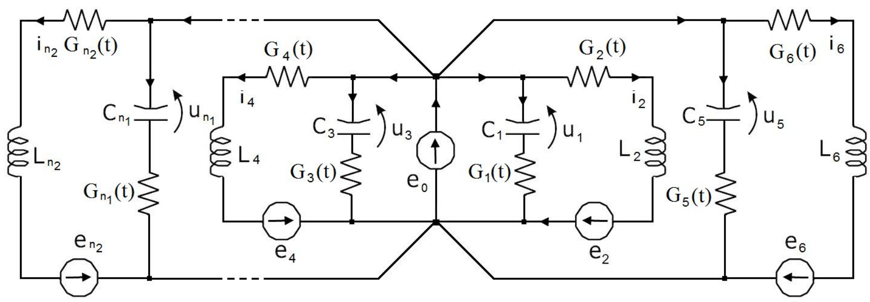

Fig. 4.2. Electrical circuit.

which can be written in the form

$$\frac{d}{dt}\begin{bmatrix} u(t) \\ i(t) \end{bmatrix}=A(t)\begin{bmatrix} u(t) \\ i(t) \end{bmatrix}+B(t)e(t), \tag{4.4a}$$

where

$$u(t)=\begin{bmatrix} u_1(t) \\ u_3(t) \\ \vdots \\ u_{n_1}(t) \end{bmatrix},\ i(t)=\begin{bmatrix} i_2(t) \\ i_4(t) \\ \vdots \\ i_{n_2}(t) \end{bmatrix},\ e(t)=\begin{bmatrix} e_1(t) \\ e_3(t) \\ \vdots \\ e_n(t) \end{bmatrix},\ (n=n_1+n_2) \tag{4.4b}$$

and

$$A(t)=\operatorname{diag}\left[-\frac{G_1(t)}{C_1},-\frac{G_3(t)}{C_3},\ldots,-\frac{G_{n_1}(t)}{C_{n_1}},-\frac{1}{G_2(t)L_2},-\frac{1}{G_4(t)L_4},\ldots,-\frac{1}{G_{n_2}(t)L_{n_2}}\right],$$

$$B(t)=\begin{bmatrix} B_1(t) \\ B_2 \end{bmatrix},\ B_1(t)=\begin{bmatrix} \frac{G_1(t)}{C_1} & 0 & 0 & \ldots & 0 \\ \frac{G_3(t)}{C_3} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \frac{G_{n_1}(t)}{C_{n_1}} & 0 & 0 & \ldots & 0 \end{bmatrix},\ B_2=\begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & \ldots & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{n_2}} & 0 & 0 & \ldots & \frac{1}{L_{n_2}} \end{bmatrix}. \tag{4.4c}$$

The electrical circuit is positive time-varying linear system since all diagonal entries of the matrix $A(t)$ are negative functions of $t\in[0,+\infty)$ and the matrix B

has nonnegative entries for $t \in [0,+\infty)$. The solution $\begin{bmatrix} u(t) \\ i(t) \end{bmatrix}$ of the equation (4.4) can be found using Lemma 2.1.

The considerations can be easily extended to the positive electrical circuits with time-varying inductances $L(t)$ and capacitances $C(t)$ as follows.

Let $\Psi(t) = L(t)i(t)$ then using the equality

$$\frac{d\Psi(t)}{dt} = \frac{dL(t)}{dt}i(t) + L(t)\frac{di(t)}{dt} \tag{4.5}$$

and the Kirchhoff's laws we can write the state equation of the form (4.4). Similarly, let $q(t) = C(t)u(t)$ then using the equality

$$\frac{dq(t)}{dt} = \frac{dC(t)}{dt}u(t) + C(t)\frac{du(t)}{dt} \tag{4.6}$$

and the Kirchhoff's laws we can write the state equation of the form (4.4).

5. CONCLUDING REMARKS

The positivity and asymptotic stability of a class of time-varying continuous-time linear systems and electrical circuits have been addressed. Sufficient conditions for the positivity and asymptotic stability of the system have been established. It has been shown that there exists a large class of positive and asymptotically stable electrical circuits with time-varying parameters. The considerations have been illustrated by positive and asymptotically stable electrical circuits. The consideration can be extended to fractional time-varying linear systems and fractional electrical circuits.

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