The problem of state constraints in designing the discrete time sliding mode controller

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Abstract: In this paper we study the problem of state constraints in discrete time sliding mode control. We present a sufficient condition for the strategy that drives the representative point monotonically to the sliding hyperplane in finite time. The advantage of this strategy is that disturbances do not have to fulfill the matching conditions. Our approach is based on the so-called reaching law technique.

Keywords: variable structure systems, sliding mode control, discrete time systems, state constraints, reaching law

1. Introduction

In recent years, sliding mode control has become a popular regulation technique as a result of its computational efficiency and robustness guarantee [1–13]. In the first place this method was analyzed on continuous systems [14, 15]. Further, discrete time systems [16–18] were considered due to their wide application in practice. In order to apply the sliding mode control technique we start with choosing parameters of the sliding hyperplane. These parameters determine the eigenvalues of the closed-loop system matrix, which is connected with the system dynamics. The main goal is to control the system in such a way that the representative point will reach that predefined hyperplane and remain in its certain neighborhood. The control signal is usually computed using one of two well-known methods. In the first one the control law, which guarantees the stable sliding motion is introduced. The other one involves the so-called reaching law technique. In this approach the control signal is computed in order to satisfy a predefined evolution of the sliding variable. Originally, the reaching law method was introduced for continuous time systems [19], and then developed for discrete time ones [20]. Recently, a large number of new reaching laws were presented [21–30].

The sliding mode control can be divided in two phases: the reaching phase and the sliding phase. The first one refers to the evolution of the representative point from the initial position to the predefined sliding hyperplane. The second one is connected with the motion of the state point along the sliding manifold to the destination point. During the control process external disturbances may interfere with the behavior of the representative point and cause large values of state variables or oscillations. In the sliding phase, the impact of disturbances implicates the quasi-sliding mode i.e. the representative point cannot move precisely along the sliding hyperplane, but it can stay in a certain neighborhood of that hyperplane. In this paper we define the quasi-sliding mode [24, 31, 32], i.e. we do not require crossing the sliding hyperplane in each consecutive step.

The issue of limiting state variables is an important problem in the sliding mode control due to its frequent occurrence in practice [33–40]. Usually, during the sliding mode controller design, the problem of constraining the state variables is omitted in favor of limiting the control signal. Our work focuses on finding the best strategy for limiting all of the state variables. In particular, we can select the state variables that are crucial to our system and constrain only these ones.

This paper is an extension of the article [41] and is organized as follows. Section 2 presents the sliding mode controller design based on the reaching law technique. Moreover, the state constraints problem is analyzed. The sufficient condition for the monotonic, finite time convergence to the sliding hyperplane is introduced in Section 3. Section 4 deals with the selection of the convergence rate of the representative point to the sliding hyperplane. In Section 5 the sufficient condition is extended to the form, which is easier to apply. A simulation example is presented in Section 6. Section 7 comprises the conclusions of this paper.

2. Sliding mode controller design

Let us consider the discrete time system affected by unknown, bounded external disturbances. Its dynamics is described by the following state equation

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) + \boldsymbol{d}(k), \tag{1}$$

where $\boldsymbol{x}(k) = [x_1(k), ..., x_n(k)]^{\mathrm{T}}$ is the state vector, $\boldsymbol{A}$ is the state matrix of the dimension $n \times n$, $\boldsymbol{b}$ is the input distribution vector of dimension $n \times 1$, $u(k)$ is a scalar control signal, $\boldsymbol{d}(k) = [d_1(k), ..., d_n(k)]^{\mathrm{T}}$ is the external disturbance vector and $\max |d_i(k)| = D_i$ for any $i \in \{1, ..., \mathrm{n}\}$. Let us observe, that our

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Artykuł recenzowany
nadesłany 19.09.2017 r., przyjęty do druku 27.11.2017 r.

system does not have to fulfill the matching conditions, i.e. the external disturbance vector affects the whole system dynamics and not only the control signal. We design the discrete time sliding mode controller in order to obtain finite time, monotonic convergence of the representative point to the sliding hyperplane

$$s(k) = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}(k) = 0, \quad (2)$$

where $\boldsymbol{c}^{\mathrm{T}} = [c_1, \ldots, c_{n-1}, 1]$. Vector $\boldsymbol{c}$ is selected in such a way that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b} \neq 0$. The quasi-sliding mode is defined likewise in [31, 32]. This definition says that crossing the sliding hyperplane is not required in each consecutive step. We select the following reaching law in order to compute the control signal

$$s(k+1) = s(k) - K\operatorname{sgn}[s(k)] + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k), \quad (3)$$

where K is a positive, real number and the function $\operatorname{sgn}(x)$ is given as follows

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{for} \quad x < 0 \\ 0, & \text{for} \quad x = 0 \\ 1, & \text{for} \quad x > 0 \end{cases} \quad (4)$$

From (1), (2) and (3) we can compute the following control signal

$$u(k) = \left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}\right)^{-1} \left\{ -\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k) - K\operatorname{sgn}\left[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)\right]\right\}. \quad (5)$$

In order to obtain finite time convergence to the sliding hyperplane, we require that

$$|s(k)| - |s(k+1)| \geq \lambda, \quad (6)$$

is fulfilled during the reaching phase, where λ is a real, positive constant. Analyzing signs of $s(k)$, $s(k+1)$ and using (3) we can rewrite (6) in the following form

$$K - \boldsymbol{c}^{\mathrm{T}} \boldsymbol{d}(k) \operatorname{sgn}[s(k)] \geq \lambda. \quad (7)$$

Let us denote the biggest influence on the sliding variable caused by external disturbances by

$$D_{\max} = |c_1| D_1 + \ldots + |c_{n-1}| D_{n-1} + D_n. \quad (8)$$

Hence, in order to fulfill (7) we require

$$K - D_{\max} \geq \lambda. \quad (9)$$

Further in this paper we will modify the parameter K in order to include the influence of external disturbances.

2.1. State Constraints

The purpose of this paper is to limit each state variable $x_i(k)$, $i \in \{1, \ldots, n\}$ for any $k \in \mathbb{N}$. We want to immunize our system to external disturbances. Therefore, we alter the parameter K. Let us assume that the absolute value of the state variable $x_i(0)$ is limited by r_i for any $i \in \{1, \ldots, n\}$. We will calculate the parameter K so that if the absolute value of the state variable $x_i(k)$ is limited by r_i, then the absolute value of the state variable $x_i(k+1)$ is also limited by r_i, i.e.

$$-r_i \leq x_i(k+1) \leq r_i. \quad (10)$$

We assume that $D_i < ½\, r_i$ for any $i \in \{1, \ldots, n\}$. From (1) and (5) we get

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) - \boldsymbol{b}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + \\ + \boldsymbol{b}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k) - \boldsymbol{b}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}K\operatorname{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)] + \boldsymbol{d}(k). \quad (11)$$

Let us define the matrix $\boldsymbol{G}$ as follows

$$\boldsymbol{G} = \boldsymbol{A} + \boldsymbol{b}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}(-\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{c}^{\mathrm{T}}) \quad (12)$$

and the vector $\boldsymbol{e}_i$ ($\dim \boldsymbol{e}_i = 1 \times \mathrm{n}$) as the versor of the i-th axis of a Cartesian coordinate system, i.e. the i-th element of the vector $\boldsymbol{e}_i$ is equal to one, while remaining elements of this vector are equal to zero. Then we can write the state equation for the i-th variable

$$x_i(k+1) = \boldsymbol{e}_i\{\boldsymbol{G}\boldsymbol{x}(k) - K(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{b}\operatorname{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)] + \boldsymbol{d}(k)\}. \quad (13)$$

We want to calculate the largest time-varying K which guarantees the fulfillment of (10). Using (13) and the substitution $\boldsymbol{g}_i = \boldsymbol{e}_i\boldsymbol{G}$ we get

$$\boldsymbol{g}_i\boldsymbol{x}(k) - r_i + d_i(k) \leq K(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{e}_i\boldsymbol{b}\operatorname{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)] \leq \boldsymbol{g}_i\boldsymbol{x}(k) + \\ + r_i + d_i(k). \quad (14)$$

The value of $d_i(k)$ is unknown, therefore we have to require that the following conditions are satisfied

$$\boldsymbol{g}_i\boldsymbol{x}(k) - r_i + D_i \leq K(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{e}_i\boldsymbol{b}\operatorname{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)] \\ \leq \boldsymbol{g}_i\boldsymbol{x}(k) + r_i - D_i. \quad (15)$$

Let us observe, that when $\boldsymbol{e}_i\boldsymbol{b} = 0$ for $i \in \{1, \ldots, n\}$, then K has no influence on the behavior of the i-th state variable and (15) is true if

$$-r_i + D_i \leq \boldsymbol{g}_i\boldsymbol{x}(k) \leq r_i - D_i. \quad (16)$$

Consider the case when $\boldsymbol{e}_i\boldsymbol{b} \neq 0$. Using (15) and analyzing the sign of the expression $(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{e}_i\boldsymbol{b}\,\mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]$ we obtain that for each $i \in \{1, \ldots, n\}$ the largest possible K (denoted by K_i) is given by the equation

$$K_i = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\,\mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]\,\boldsymbol{g}_i\boldsymbol{x}(k) + \\ + |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|(r_i - D_i). \tag{17}$$

We can observe that K_i is a function of k. Therefore, from now on, we will write $K_i(k)$ instead of K_i in order to emphasize that it is a time-varying coefficient. From (17) we conclude that in order to calculate (5) it is necessary to know only the maximum absolute value of external disturbances and the current state. Therefore, the measurement of external disturbances is not needed.

3. Sufficient condition

In this section we will formulate and prove two theorems which specify the sufficient condition for $K_i(k) - D_{\max} \geq \lambda$ for any $i \in \{1, \ldots, n\}$ and $k \in \mathbb{N} \cup \{0\}$.

Theorem 1. *Denote by g_{ij} the expression in the i-th row and j-th column of the matrix $\boldsymbol{G}$. In order to obtain $K_i(k) - D_{\max} > 0$ in each consecutive step it is sufficient that inequalities*

$$|g_{i1}|r_1 + \ldots + |g_{in}|r_n < r_i - D_i - \frac{D_{max}}{\left|\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\right|} \tag{18}$$

are satisfied for any $i \in \{1, \ldots, n\}$.

Proof. From (17) we observe that if

$$|\boldsymbol{g}_i\boldsymbol{x}(k)| < r_i - D_i - \frac{D_{\max}}{\left|\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\right|}, \tag{19}$$

then $K_i(k) - D_{\max} > 0$. Hence, our goal is to satisfy inequalities (19) for any $i \in \{1, \ldots, n\}$. Let us derive the greatest possible value of the left-hand side of (19)

$$\max|\boldsymbol{g}_i\boldsymbol{x}(k)| = |g_{i1}\,\mathrm{sgn}(g_{i1})r_1 + \ldots + g_{in}\,\mathrm{sgn}(g_{in})r_n| = \\ = |g_{i1}|r_1 + \ldots + |g_{in}|r_n. \tag{20}$$

Using (19) and (20) we obtain that if (18) is true, then $K_i(k) - D_{\max} > 0$. This ends the proof. ■

In order to prove the finite time convergence of the representative point to the sliding hyperplane we will determine the parameter λ_i such that $K_i(k) - D_{\max} \geq \lambda_i > 0$.

Theorem 2. *Assume that* (18) *is satisfied. Then $K_i(k) - D_{\max} \geq \lambda_i > 0$ in each consecutive step. Parameter*

$$\lambda_i = |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\,[r_i - D_i - (|g_{i1}|r_1 + \ldots + |g_{in}|r_n)] - D_{\max} \tag{21}$$

for $i \in \{1, \ldots, n\}$.

Proof. One can observe that if

$$\boldsymbol{g}_i\boldsymbol{x}(k) = \mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}]\,\mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]\,(|g_{i1}|r_1 + \ldots + |g_{in}|r_n), \tag{22}$$

then (17) has the following form

$$K_i(k) = |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\,[r_i - D_i + (|g_{i1}|r_1 + \ldots + |g_{in}|r_n)]. \tag{23}$$

Otherwise, if

$$\boldsymbol{g}_i\boldsymbol{x}(k) = -\mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}]\,\mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]\,(|g_{i1}|r_1 + \ldots + |g_{in}|r_n), \tag{24}$$

then

$$K_i(k) = |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\,[r_i - D_i - (|g_{i1}|r_1 + \ldots + |g_{in}|r_n)]. \tag{25}$$

Noting that $|g_{ij}|r_j > 0$ for any $j \in \{1, \ldots, n\}$ we conclude that

$$r_i - D_i + (|g_{i1}|r_1 + \ldots + |g_{in}|r_n) > r_i - D_i - (|g_{i1}|r_1 + \ldots + |g_{in}|r_n). \tag{26}$$

Using (26) one can observe that if $|\boldsymbol{g}_i\boldsymbol{x}(k)| \leq |g_{i1}|r_1 + \ldots + |g_{in}|r_n$, then

$$K_i(k) = |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\,[r_i - D_i - (|g_{i1}|r_1 + \ldots + |g_{in}|r_n)]. \tag{27}$$

It can be seen from (27) that

$$K_i(k) - D_{\max} \geq |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\,[r_i - D_i - (|g_{i1}|r_1 + \ldots + |g_{in}|r_n)] - D_{\max}. \tag{28}$$

Furthermore, multiplying (18) by $|\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|$ we obtain

$$|\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\,[r_i - D_i - (|g_{i1}|r_1 + \ldots + |g_{in}|r_n)] - D_{\max} > 0. \tag{29}$$

Hence, the representative point arrives to the neighborhood of the sliding hyperplane in finite time and the parameter λ_i is of the form (21). This ends the proof. ■

4. Maximum admissible value of the convergence rate

In this section we will formulate a theorem, which states that choosing a smaller value of $K(k)$ does not affect the fulfillment of the state constraints.

Theorem 3. *Assume that $K_i(k) > D_{\max}$, $i \in \{1, \ldots, n\}$ is defined by* (17). *Then for any $K(k) \in (D_{\max}; K_i(k)]$ the limit condition is fulfilled for the i-th state variable.*

Proof. Let us define the parameter $K_\varepsilon(k)$ as follows

$$K_\varepsilon(k) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\ \mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]\ \boldsymbol{g}_i\boldsymbol{x}(k) + |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|(r_i - D_i - \varepsilon). \tag{30}$$

For any $K(k) \in (D_{\max}; K_i(k)]$ there exists $\varepsilon \geq 0$ such that $K_\varepsilon(k) = K(k)$. Note that $K_\varepsilon(k) < K_i(k)$. From (13) and (30) we have

$$\boldsymbol{x}(k+1) = \boldsymbol{G}\boldsymbol{x}(k) - (\boldsymbol{e}_i\boldsymbol{b})^{-1}\boldsymbol{b}\boldsymbol{g}_i\boldsymbol{x}(k) + \\ + \boldsymbol{d}(k) - \mathrm{sgn}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})\ \mathrm{sgn}\ [\boldsymbol{c}^{\mathrm{T}}\ \boldsymbol{x}(k)]\ |(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\boldsymbol{b}(r_i - D_i - \varepsilon). \tag{31}$$

Multiplying both sides of (31) by $\boldsymbol{e}_i$ we obtain

$$x_i(k+1) = -\mathrm{sgn}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})\ \mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]\ \mathrm{sgn}(\boldsymbol{e}_i\boldsymbol{b})(r_i - D_i - \varepsilon) + d_i(k). \tag{32}$$

Therefore, using the equation $D_i = \max |d_i(k)|$ we conclude that

$$x_i(k+1) \in [-\mathrm{sgn}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})\ \mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]\ \mathrm{sgn}(\boldsymbol{e}_i\boldsymbol{b})(r_i - D_i - \varepsilon) - D_i \\ -\mathrm{sgn}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})\ \mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]\ \mathrm{sgn}(\boldsymbol{e}_i\boldsymbol{b})(r_i - D_i - \varepsilon) + D_i]. \tag{33}$$

One can observe that $-\mathrm{sgn}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})\ \mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)]\ \mathrm{sgn}(\boldsymbol{e}_i\boldsymbol{b})$ can only be equal to –1, 0 or 1. Furthermore, the maximum value of ε has to be smaller or equal to $2r_i - 2D_i$. Hence,

$$-r_i + \varepsilon \leq x_i(k+1) \leq r_i - \varepsilon. \tag{34}$$

Therefore, the constraint of the state variable x_i is satisfied. This ends the proof. ■

Our goal is to satisfy all of the state constraints and select the largest value of $K(k)$. Hence,

$$K(k) = \min\{K_1(k), \ldots, K_n(k)\}. \tag{35}$$

From (35) and inequalities $K_i - D_{\max} > \lambda_i$ for any $i \in \{1, \ldots, n\}$ we conclude that $K(k) - D_{\max} \geq \lambda = \min\{\lambda_1, \ldots, \lambda_n\}$. Therefore, the finite time convergence to the sliding hyperplane, without violating the state constraints is obtained.

5. Improvement of the sufficient condition

In this section we will weaken the sufficient condition presented in this paper. Let us observe that it is possible that

$$|\boldsymbol{g}_i\boldsymbol{x}| > r_i - D_i - \frac{D_{\max}}{\left|\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}\left(\boldsymbol{e}_i\boldsymbol{b}\right)^{-1}\right|} \text{ and } K_i - D_{\max} > 0.$$

We can rewrite (17) in the following form

$$K_i = \mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\ \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\ \boldsymbol{c}^{\mathrm{T}}\ \boldsymbol{x}(k)\boldsymbol{g}_i\boldsymbol{x}(k)]\ |\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}| \cdot |\boldsymbol{g}_i\boldsymbol{x}(k)| + \\ + |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|(r_i - D_i). \tag{36}$$

Hence, if $\mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)\boldsymbol{g}_i\boldsymbol{x}(k)] = 1$, then

$$K_i - D_{\max} \geq |\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\ \boldsymbol{b})^{-1}|\ (r_i - D_i) - D_{\max} > 0. \tag{37}$$

Denote by X_i^+ and X_i^- subsets of the state space in which

$$\mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)\boldsymbol{g}_i\boldsymbol{x}(k)] = 1, \tag{38}$$

$$\mathrm{sgn}[\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(k)\boldsymbol{g}_i\boldsymbol{x}(k)] = -1 \tag{39}$$

respectively, and

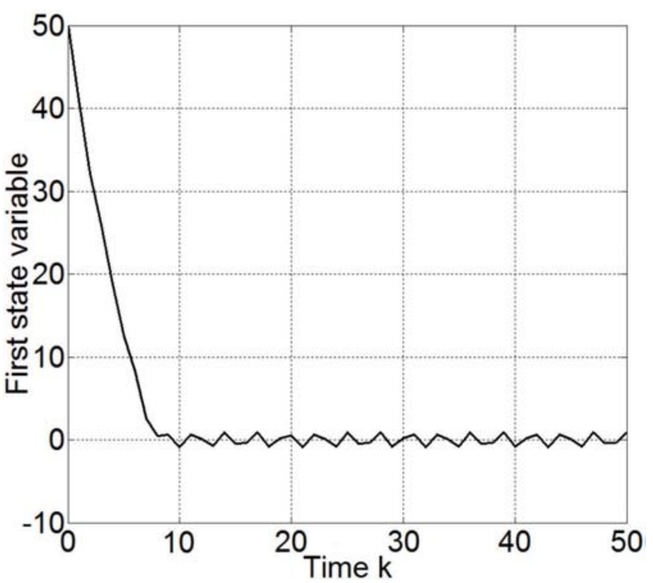

Fig. 1. First state variable
Rys. 1. Pierwsza zmienna stanu

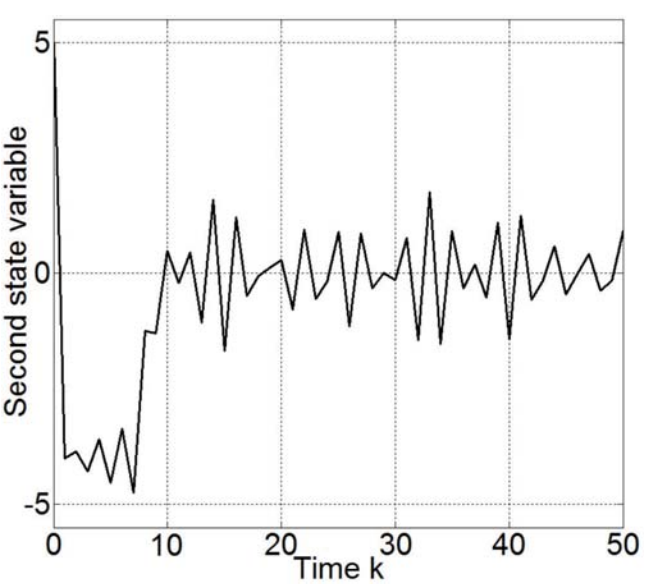

Fig. 2. Second state variable
Rys. 2. Druga zmienna stanu

$$|x_i| \le r_i \tag{40}$$

for the i-th state variable. X_i^- is the only set in which $K_i - D_{max}$ may take a negative value.

Theorem 4. *In order to obtain* $K_i - D_{max} \ge \lambda > 0$ *and* $|x_i| \le r_i$ *it is sufficient that inequalities*

$$\max_{x \in X_i^- \cup \mathbf{0}} |\boldsymbol{g}_i \boldsymbol{x}| < r_i - D_i - \frac{D_{max}}{\left|\boldsymbol{c}^{T}\boldsymbol{b}\left(\boldsymbol{e}_i\boldsymbol{b}\right)^{-1}\right|} \tag{41}$$

and

$$\max_{x \in X_i^+} |\boldsymbol{g}_i \boldsymbol{x}| \le r_i - D_i + \frac{D_{max}}{\left|\boldsymbol{c}^{T}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\right|} \tag{42}$$

are satisfied for any $i \in \{1, ..., n\}$.

Proof. Let $\boldsymbol{x} \in X_i^-$, i.e. $\mathrm{sgn}[\boldsymbol{c}^T \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1} \boldsymbol{c}^T \boldsymbol{x}(k)\boldsymbol{g}_i\boldsymbol{x}] = -1$. Hence, (36) is of the form

$$K_i = -|\boldsymbol{c}^T \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\cdot|\boldsymbol{g}_i\boldsymbol{x}| + |\boldsymbol{c}^T \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|(r_i - D_i). \tag{43}$$

Therefore, if

$$\max_{x \in X_i^-} |\boldsymbol{g}_i \boldsymbol{x}| < r_i - D_i - \frac{D_{max}}{\left|\boldsymbol{c}^{T}\boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}\right|},$$

then $K_i - D_{max} > 0$ for any $i \in \{1, ..., n\}$. Moreover, if $\boldsymbol{x} = \mathbf{0}$ then $K_i = |\boldsymbol{c}^T \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|(r_i - D_i)$. Using (41) one can observe that in this case $K_i - D_{max} > 0$. If $\boldsymbol{x} \notin X_i^-$, then $K_i > |\boldsymbol{c}^T \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|(r_i - D_i)$. From previous considerations $K_i - D_{max} > 0$ for any $i \in \{1, ..., n\}$.

Proving the fact that $K_i - D_{max} \ge \lambda > 0$ is similar to the proof of the Theorem 2.

We have to guarantee that the minimal possible convergence rate, that satisfies the i-th state constraint which is equal to

$$\boldsymbol{c}^T \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1} \mathrm{sgn}[\boldsymbol{c}^T \boldsymbol{x}(k)]\ \boldsymbol{g}_i\boldsymbol{x}(k) - |\boldsymbol{c}^T \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|(r_i - D_i). \tag{44}$$

has to be lower or equal to D_{max}. Otherwise, the selected $K(k)$ does not have to satisfy at least one of the state constraints, even if it would drive the representative point monotonically to the sliding hyperplane in finite time.

If $\boldsymbol{x} \in X_i^-$, then (44) is negative, so we can only consider our condition in the set X_i^+. Therefore, we want to satisfy inequalities

$$|\boldsymbol{c}^T \boldsymbol{b}(\boldsymbol{e}_i\boldsymbol{b})^{-1}|\ (\boldsymbol{g}_i\boldsymbol{x}(k) - r_i + D_i) \le D_{max} \tag{45}$$

for any $i \in \{1, ..., n\}$. We can rewrite (45) in the form of (42), which ends the proof. ■

6. Simulation Example

Consider the system given by (1), where

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ 1.1 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{c}^T = \begin{bmatrix} 10 & 1 \end{bmatrix}. \tag{46}$$

In order to apply the strategy that allows us to constrain both state variables at level $r_1 = 50$ and $r_2 = 5$, we start from calculating the matrix

$$\boldsymbol{G} = \begin{bmatrix} 0.9909 & 0.0909 \\ 0.0909 & 0.0909 \end{bmatrix}. \tag{47}$$

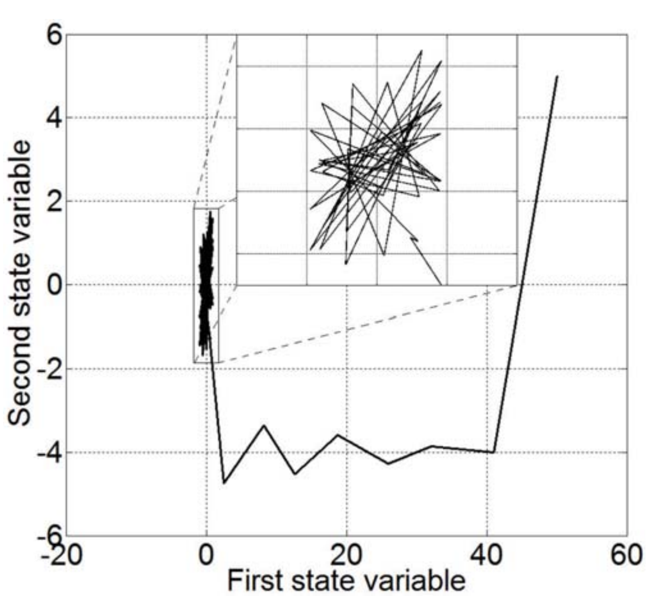

Fig. 3. State trajectory
Rys. 3. Trajektoria stanu

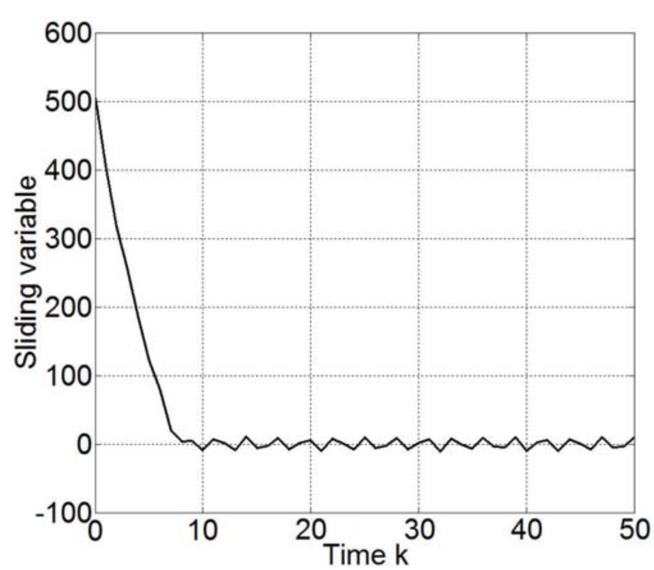

Fig. 4. Sliding variable
Rys. 4. Zmienna ślizgowa

We select the initial point at the intersection of lines that describe the upper limits of the state constraints, i.e. $\boldsymbol{x}(0) = [50\ 5]^{\mathrm{T}}$. External disturbances are $d_1(k) = \sin\ (k \cdot 40\ \mathrm{rad})$ and $d_2(k) = \sin\ (\mathrm{k} \cdot 30\ \mathrm{rad})$. One can observe, that in this example the matching conditions are not satisfied. In this example we modified the reaching law (3) to the form

$$s(k+1) = s(k) - \min\{K(k),\ |s(k)|\}\ \mathrm{sgn}[s(k)] + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}(k). \quad (48)$$

Let us observe, that in the special case if external disturbances are equal to zero, then the representative point is driven precisely on the sliding hyperplane. After calculations we obtain that (41) and (42) are true. Moreover,

$$\max_{x \in X_i^+} \left|\boldsymbol{g}_i \boldsymbol{x}\right| > r_i - D_i - \frac{D_{\max}}{\left|\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{e}_i \boldsymbol{b})^{-1}\right|}. \quad (49)$$

Hence, we conclude that in our example (18) is not satisfied, which shows that in this paper the previous sufficient condition was weakened. From Fig. 1 one can observe that the first state variable is always smaller than 50. In addition, after a certain period of time it reaches the neighborhood of its demand value and remains in it. The evolution of the second state variable is shown in Fig. 2. Starting from its initial value 5 it decreases, but is always greater than its minimal admissible value equal to –5. Further, the second state variable increases to its demand value. Figure 3 presents the trajectory of both state variables. In our example $r_2 - D_2 = 4$. Therefore, the control strategy causes that the second state variable may take a value from the interval [–4, 4], without the influence of external disturbances. Hence, in the presence of perturbations ($D_2 = 1$), the second state variable in the worst case will reach the value –5. From Fig. 4 we can observe that the monotonic convergence to the sliding hyperplane in finite time is obtained. The representative point is not driven precisely on the sliding hyperplane, due to the occurrence of disturbances. During the sliding phase the sliding variable may take a value from the interval $[-D_{\max},\ D_{\max}]$.

7. Conclusions

In this paper the issue of limiting the state variables in discrete time sliding mode control influenced by perturbations was analyzed. In our system external disturbances did not have to fulfill the matching conditions, which is beneficial from the practical point of view. Sufficient condition for the fastest, monotonic convergence of the representative point to the sliding hyperplane in finite time was introduced. A simulation example was presented in order to illustrate theoretical considerations. In the future we are going to improve our approach by involving the impact of the initial conditions and modify constant state constraints to the function form.

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Ograniczenie zmiennych stanu w dyskretnym sterowaniu ślizgowym

Streszczenie: W artykule przeanalizowany został problem ograniczenia zmiennych stanu w dyskretnym sterowaniu ślizgowym. Do zaprojektowania regulatora zastosowano regułę osiągania ruchu ślizgowego. Zaprezentowano warunek dostateczny na monotoniczną zbieżność stanu obiektu do płaszczyzny ślizgowej w skończonym czasie. Zaletą przedstawionej metody jest to, że zakłócenia nie muszą spełniać warunków dopasowania.

Słowa kluczowe: sterowanie o zmiennej strukturze, sterowanie ślizgowe, układy czasu dyskretnego, ograniczenia zmiennych stanu, reguła osiągania ruchu ślizgowego

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