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ON THE DECOMPOSITION OF FAMILIES OF QUASINORMAL OPERATORS

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Abstract. The canonical injective decomposition of a jointly quasinormal family of operators is given. Relations between the decomposition of a quasinormal operator T and the decomposition of a partial isometry in the polar decomposition of T are described. The decomposition of pairs of commuting quasinormal partial isometries and its applications to pairs of commuting quasinormal operators is shown. Examples are given.

Keywords: multiple canonical decomposition, quasinormal operators, partial isometry.

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1. INTRODUCTION

Let $L(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. If $T \in L(H)$, then T^* stands for the adjoint of T. By $\mathcal{N}(T)$ and $\mathcal{R}(T)$ we denote the kernel and the range space of T respectively.

By a subspace we always understand a closed subspace. The orthogonal complement of a subspace $H_0 \subset H$ is denoted by $H_0^\perp$ or $H \ominus H_0$. The *commutant* of $T \in L(H)$ denoted by T' is the algebra of all operators commuting with T. A subspace $H_0 \subset H$ is T *hyperinvariant,* when it is invariant for every $S \in T'$. An orthogonal projection onto H_0 is denoted by P_{H_0}. A subspace H_0 reduces operator $T \in L(H)$ (or is reducing for T) if and only if $P_{H_0} \in T'$. An operator is called *completely non unitary* (*non normal, non isometric etc.*) if there is no non trivial subspace reducing it to a unitary operator (normal operator, isometry). Such an operator is also called *pure.*

Let $\mathcal{W}$ denote some property of an operator $T \in L(H)$ (like being unitary, normal, isometry etc.) If there is a decomposition $H = H_1 \oplus H_2$ such that H_1, H_2 reduce operator T and $T|_{H_1}$ has the property $\mathcal{W}$ while $T|_{H_2}$ is pure, then

$$T = T|_{H_1} \oplus T|_{H_2}$$

is called *a $\mathcal{W}$ canonical decomposition* of T. The classical example of a canonical decomposition is the Wold decomposition [11]. Wold showed that any isometry can be decomposed into a unitary and a completely non unitary operators. An isometry $S \in L(H)$ is called *a unilateral shift* if $H = \bigoplus_{n\geq 0} S^n(\mathcal{N}(S^*))$. A completely non unitary isometry is a unilateral shift.

An operator $T \in L(H)$ is called *quasinormal*, if it is normal on $\overline{\mathcal{R}(T)}$ (i.e. $TT^*T = T^*TT$). The class of quasinormal operators has been introduced by Brown in [2]. Quasinormal operators are subnormal. Every quasinormal operator T can be decomposed to $N \oplus S \otimes A$, where N is normal, S is a unilateral shift and A is a positive, injective operator. For a quasinormal operator T holds $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. Thus $\mathcal{N}(T)$ reduces a quasinormal operator. Recall that $|T| = \sqrt{T^*T}$. An operator W is *a partial isometry,* if the restriction $W|_{\mathcal{N}(W)^\perp}$ is an isometry. Every $T \in L(H)$ has the polar decomposition $T = W|T|$, where W is a partial isometry. Operator T is quasinormal if and only if $W|T| = |T|W$. Recall after [7] that a family of operators $\mathcal{A} \subset L(H)$ is called *jointly quasinormal,* if $ST^*T = T^*TS$ holds for all $S, T \in \mathcal{A}$.

Assume that, for some property $\mathcal{W}$, there are $\mathcal{W}$ canonical decompositions of operators $T_1, T_2 \in L(H)$. If there is a decomposition $H = H_{11} \oplus H_{12} \oplus H_{21} \oplus H_{22}$, where each H_{ij} reduces operators T_1, T_2 for $i, j = 1, 2$, such that: $T_1|_{H_{ij}}$ has the property $\mathcal{W}$ for $i = 1$, is pure for $i = 2$ and $T_2|_{H_{ij}}$ has the property $\mathcal{W}$ for $j = 1$ and is pure for $j = 2$, then

$$T_k = T_k|_{H_{11}} \oplus T_k|_{H_{12}} \oplus T_k|_{H_{21}} \oplus T_k|_{H_{22}} \quad \text{for} \quad k = 1, 2$$

is called *a $\mathcal{W}$ multiple canonical decomposition.* Multiple canonical decompositions can be defined in a similar way for families of operators. Recall that operators $T_1, T_2 \in L(H)$ *doubly commute* if $T_1, T_1^* \in T_2'$. Recall after [4] that a family of doubly commuting operators has a multiple canonical decomposition if each operator in the family has a (single) canonical decomposition. However, the doubly commutativity assumption is rather strong. Only a normal operator can doubly commute with itself. On the other hand, a pair T, T has a multiple canonical decomposition if T has a canonical decomposition.

In the present paper we show that, although there does not need to exist a normal canonical decomposition of a jointly quasinormal family of operators, there is an injective canonical decomposition. We also give a generalization of this decomposition to a pair of commuting quasinormal partial isometries. The generalization is not a canonical decomposition. In the last paragraph we give some applications of this decomposition to arbitrary pairs of commuting quasinormal operators.

2. DECOMPOSITIONS OF A QUASINORMAL OPERATOR

By the model given by Brown a quasinormal operator has a normal canonical decomposition. Let $T \in L(H)$ be a quasinormal operator. Recall after [10] that if T is additionally a contraction, then the subspace $\mathcal{N}(I - T^*T)$ is the maximal subspace reducing T to an isometry. Note that a subspace reduces operator T if and only if it reduces αT, for $\alpha \in \mathbb{C}\backslash\{0\}$. If $|\alpha| \neq 1$, then $\mathcal{N}(I - T^*T) \neq \mathcal{N}(I - (\alpha T)^*(\alpha T))$. Since αT

is not a contraction for sufficiently large $|\alpha|$, it is not known whether $\mathcal{N}(I-(\alpha T)^*(\alpha T))$ reduces T. In this section the existence of the maximal subspace reducing a bounded quasinormal operator to an isometry will be proved. As a consequence, any eigenspace of $|T|$ reduces operator T to an isometry weighted by the corresponding eigenvalue. Moreover, such a subspace is $|T|$ hyperinvariant.

Remark 2.1. Let $A \in L(H)$ be a positive operator. For $x \in \mathcal{N}(I-A)$ we have $x = Ax \in \mathcal{R}(A)$. Consequently, the subspace $\mathcal{N}(I-A)$ is orthogonal to $\mathcal{N}(A)$. An arbitrary $x \in \mathcal{N}(A-A^2)$ can be decomposed to $x = (x-Ax)+Ax \in \mathcal{N}(A) \oplus \mathcal{N}(I-A)$. It follows that $\mathcal{N}(A-A^2) \subset \mathcal{N}(I-A) \oplus \mathcal{N}(A)$. Since the reverse inclusion is obvious, we have $\mathcal{N}(A-A^2) = \mathcal{N}(I-A) \oplus \mathcal{N}(A)$.

For arbitrary $x \in \mathcal{N}(I-A^2)$, we have

$$0 \leq (A(x-Ax), x-Ax) = (Ax-x, x-Ax) = -\|x-Ax\|^2 \leq 0.$$

Consequently, $\mathcal{N}(I-A^2) = \mathcal{N}(I-A)$.

A decomposition of an operator T is called T *hyperinvariant,* when subspaces in the corresponding decomposition of the Hilbert space $H = \bigoplus_i H_i$ are T hyperinvariant. It follows that H_i and $H \ominus H_i$ are T hyperinvariant and consequently subspaces H_i reduce every operator in T'.

Proposition 2.2. *Let $T \in L(H)$ and $|T| = \sqrt{T^*T}$. Then the decomposition*

$$H = \overline{\mathcal{R}(|T|-T^*T)} \oplus \mathcal{N}(I-|T|) \oplus \mathcal{N}(T)$$

is $|T|$ hyperinvariant.

Proof. Since the operator $|T|-T^*T$ is self-adjoint, then

$$H = \overline{\mathcal{R}(|T|-T^*T)} \oplus \mathcal{N}(|T|-T^*T).$$

By Remark 2.1, we obtain the decomposition

$$\mathcal{N}(|T|-T^*T) = \mathcal{N}(|T|) \oplus \mathcal{N}(I-|T|).$$

Obviously commutants of $|T|$ and $I-|T|$ are equal. Since the kernel of an operator is a hyperinvariant subspace, then $\mathcal{N}(I-|T|)$ and $\mathcal{N}(|T|) = \mathcal{N}(T)$ are $|T|$ hyperinvariant. The orthogonal complement of a subspace, which is hyperinvariant for a self-adjoint operator is also a hyperinvariant subspace. Thus $\overline{\mathcal{R}(|T|-T^*T)}$ is $|T|$ hyperinvariant as well. □

As a corollary we obtain the following decomposition.

Proposition 2.3. *Let $T \in L(H)$ be a quasinormal operator. There is a decomposition*

$$H = \overline{\mathcal{R}(T^*T-T^{*2}T^2)} \oplus \mathcal{N}(I-T^*T) \oplus \mathcal{N}(T),$$

where the subspaces are the maximal reducing operator T, such that:

(i) $T|_{\overline{\mathcal{R}(T^*T-T^{*2}T^2)}}$ *is a completely non isometric, injective, quasinormal operator,*

(ii) $T|_{\mathcal{N}(I-T^*T)}$ *is an isometry,*
(iii) $T|_{\mathcal{N}(T)} = 0$.

Moreover, each of these subspaces is $|T|$ hyperinvariant.

Proof. Applying Proposition 2.2 to the operator T^*T we obtain the decomposition

$$H = \overline{\mathcal{R}(T^*T - (T^*T)^2)} \oplus \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T^*T).$$

Since $|T^*T| = T^*T$, the decomposition is T^*T hyperinvariant. Commutants of T^*T and its square root $|T|$ are equal. Thus the decomposition is also $|T|$ hyperinvariant. Since $\mathcal{N}(T) = \mathcal{N}(T^*T)$ and by quasinormality $(T^*T)^2 = T^{*2}T^2$ the decomposition is equivalent to

$$H = \overline{\mathcal{R}(T^*T - T^{*2}T^2)} \oplus \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T).$$

Since T is quasinormal, it commutes with $|T|$. Consequently, subspaces obtained in the decomposition reduce T.

Since for an isometry $T^*T = I$, then a subspace reducing T, reduces the operator to an isometry if and only if it is a subspace of $\mathcal{N}(I - T^*T)$. On the other hand we have shown that $\mathcal{N}(I - T^*T)$ reduces T. Thus it is the maximal subspace reducing T to an isometry. Since $\mathcal{N}(T)$ reduces T, it is the maximal subspace reducing T to a zero operator. Since $\overline{\mathcal{R}(T^*T - T^{*2}T^2)}$ is the orthogonal complement of $\mathcal{N}(T) \oplus \mathcal{N}(I - T^*T)$, it is the maximal subspace reducing T to a completely non isometric, injective, quasinormal operator. □

By the proposition above, we obtain the following decomposition.

Theorem 2.4. *Let $T \in L(H)$ be a quasinormal operator, where H is a separable Hilbert space. There is a decomposition*

$$H = \bigcap_{\lambda\in\Lambda} \overline{\mathcal{R}(\lambda|T| - T^*T)} \oplus \bigoplus_{\lambda\in\Lambda} \mathcal{N}(\lambda^2 I - T^*T),$$

where Λ is the set of all eigenvalues of $|T|$. The subspaces are the maximal reducing operator T such that:

(i) $|T|\big|_{\bigcap_{\lambda\in\Lambda}\overline{\mathcal{R}(|T|-\lambda T^*T)}}$ *has no eigenvectors,*
(ii) $\lambda^{-1}T|_{\mathcal{N}(\lambda^2 I-T^*T)}$ *is an isometry for $\lambda \neq 0$,*
(iii) $T|_{\mathcal{N}(T)} = 0$.

Moreover, each of the subspaces is $|T|$ hyperinvariant.

Since H is separable, there is only countably many eigenvalues of $|T|$. Hence, in the decomposition above, there is countably many subspaces and the orthogonal sum can be used.

Proof. Note that if $\mathcal{N}(T) \neq \{0\}$, then $\lambda = 0$ is an eigenvalue of $|T|$ and $\mathcal{N}(T) = \mathcal{N}(|T|)$ is a summand of the decomposition. Since T is quasinormal, then $\mathcal{N}(T)$ reduces T. Obviously it is the maximal subspace reducing T to a zero operator.

For any $\alpha \in \mathbb{C} \setminus \{0\}$ denote by $\lambda = \frac{1}{|\alpha|}$. Proposition 2.3 applied to the operator αT shows that $\mathcal{N}(I - |\alpha|^2 T^*T) = \mathcal{N}(\lambda^2 I - T^*T)$ is a $|T|$ hyperinvariant subspace and reduces the operator αT to an isometry. Consequently, $\lambda^{-1}T|_{\mathcal{N}(\lambda^2 I - T^*T)} = |\alpha|T|_{\mathcal{N}(\lambda^2 I - T^*T)}$ is an isometry. Assume that for some subspace $L \subset H$ reducing T the operator $\lambda^{-1}T|_L$ is an isometry. Then $(\lambda^{-1}T)^*(\lambda^{-1}T)|_L = I|_L$ implies that $L \subset \mathcal{N}(I - \lambda^{-2}T^*T) = \mathcal{N}(\lambda^2 I - T^*T)$. Consequently, $\mathcal{N}(\lambda^2 I - T^*T)$ is the maximal subspace reducing T such that $\lambda^{-1}T$ is an isometry.

If $\mathcal{N}(\lambda^2 I - T^*T) \neq 0$, then λ is an eigenvalue of $|T|$. Subspaces $\mathcal{N}(\lambda^2 I - T^*T)$ are orthogonal, since they are different eigenspaces of $|T|$.

By Proposition 2.2 applied to the operator αT, we obtain

$$\overline{\mathcal{R}(|\alpha||T| - |\alpha|^2 T^*T)} = H \ominus (\mathcal{N}(I - |\alpha||T|) \oplus \mathcal{N}(\alpha T)). \tag{2.1}$$

Note that $\big||\alpha|^2 T^*T\big| = |\alpha|^2 T^*T$ and since T is quasinormal, $(T^*T)^2 = T^{*2}T^2$. Thus by Proposition 2.2 applied to the operator $|\alpha|^2 T^*T$, we obtain

$$\overline{\mathcal{R}(|\alpha|^2 T^*T - |\alpha|^4 T^{*2}T^2)} = H \ominus (\mathcal{N}(I - |\alpha|^2 T^*T) \oplus \mathcal{N}(|\alpha|^2 T^*T)). \tag{2.2}$$

Since $\alpha \neq 0$, then $\mathcal{N}(|\alpha|^2 T^*T) = \mathcal{N}(\alpha T) = \mathcal{N}(T)$. By Remark 2.1 applied to $A = |\alpha||T|$, we obtain $\mathcal{N}(I - |\alpha||T|) = \mathcal{N}(I - |\alpha|^2|T|^2) = \mathcal{N}(I - |\alpha|^2 T^*T)$. Consequently, the right hand sides of equalities (2.1) and (2.2) are equal. It follows that

$$\overline{\mathcal{R}(|\alpha|^2 T^*T - |\alpha|^4 T^{*2}T^2)} = \overline{\mathcal{R}(|\alpha||T| - |\alpha|^2 T^*T)} = \overline{\mathcal{R}(\lambda|T| - T^*T)}.$$

By the equality above and Proposition 2.3 applied to the operator αT, we have that

$$H \ominus \mathcal{N}(\lambda^2 I - T^*T) = \overline{\mathcal{R}(\lambda|T| - T^*T)} \oplus \mathcal{N}(T).$$

Note that either $\mathcal{N}(T)$ is $\{0\}$ or is an eigenspace of $|T|$. Thus

$$H \ominus \bigoplus_{\lambda \in \Lambda} \mathcal{N}(\lambda^2 I - T^*T) = \bigcap_{\lambda \in \Lambda} \overline{\mathcal{R}(\lambda|T| - T^*T)}.$$

It is a $|T|$ hyperinvariant subspace, since it is an intersection of such subspaces. By the construction above, the orthogonal complement of $\bigcap_{\lambda\in\Lambda} \overline{\mathcal{R}(\lambda|T| - T^*T)}$ is a subspace generated by all eigenvectors of $|T|$. Thus $\bigcap_{\lambda\in\Lambda} \overline{\mathcal{R}(\lambda|T| - T^*T)}$ is the maximal subspace reducing T such that $|T|$ has no eigenvalues. □

3. DECOMPOSITIONS OF SOME FAMILIES OF QUASINORMAL OPERATORS

In [4] it has been proved that a family of doubly commuting quasinormal operators has a multiple normal canonical decomposition. The results do not extend to a jointly quasinormal family – Example 1 in [9]. We can obtain the following decomposition.

Theorem 3.1. *Let $\{T_i\}_{i\in Z} \subset L(H)$ be a family of jointly quasinormal operators on a separable Hilbert space H, where $Z \subset \mathbb{Z}$ is finite or infinite. Denote by Λ the set of all sequences $\{\alpha_i\}_{i\in Z}$ such that α_i is an eigenvalue of $|T_i|$ or $\alpha_i = \infty$ for $i \in Z$. There is a decomposition*

$$H = \bigoplus_{\alpha\in\Lambda} H_\alpha$$

into subspaces reducing the family $\{T_i\}_{i\in Z}$, where for every $\alpha \in \Lambda$ and $i \in Z$

(i) $T_i|_{H_\alpha} = 0$ *for* $\alpha_i = 0$,
(ii) $\alpha_i^{-1} T_i|_{H_\alpha}$ *is an isometry for* $\alpha_i \in (0, \infty)$,
(iii) $T_i|_{H_\alpha}$ *is such that* $|T_i|$ *has no eigenvectors for* $\alpha_i = \infty$.

Proof. Denote by $H = H^i_\infty \oplus \bigoplus_{j\in J_i} H^i_{\lambda^i_j}$ the decomposition of the operator T_i given by Theorem 2.4, where $\{\lambda^i_j\}_{j\in J_i}$ are all eigenvalues of $|T_i|$ for $i \in Z$. Precisely, $H^i_0 = \mathcal{N}(T_i)$, $H^i_{\lambda^i_j} = \mathcal{N}((\lambda^i_j)^2 I - T_i^* T_i)$ and $H^i_\infty = \bigcap_{j\in J_i} \overline{\mathcal{R}(\lambda^i_j |T_i| - T_i^* T_i)}$. Note that Λ is the cartesian product of $\{\lambda^i_j\}_{j\in J_i} \cup \{\infty\}$ for $i \in Z$. Since decompositions are $|T_i|$ hyperinvariant for every $i \in Z$ and the family is jointly quasinormal, each subspace in each decomposition reduces the whole family $\{T_i\}_{i\in Z}$. Thus subspaces $H_\alpha = \bigcap_{i\in Z} H^i_{\alpha_i}$ reduce the family, since they are intersections of such subspaces. By the construction above, the subspaces H_α have suitable properties and $H = \bigoplus_{\alpha\in\Lambda} H_\alpha$. □

Beside a normal canonical decomposition of a quasinormal operator there is also a canonical decomposition into an injective operator and a zero operator. Obviously a zero operator is normal. Therefore an injective decomposition can be understand as a partial result compared to the normal decomposition. A jointly quasinormal family need not have a multiple normal canonical decomposition. However, as a corollary of Theorem 3.1, a jointly quasinormal family of operators has a multiple injective canonical decomposition.

Theorem 3.2. *Let $\{T_i\}_{i\in Z} \subset L(H)$ be a family of jointly quasinormal operators on a separable Hilbert space H, where $Z \subset \mathbb{Z}$ is finite or infinite. There is a decomposition*

$$H = \bigoplus_{\alpha\in\{0,1\}^Z} H_\alpha$$

into subspaces reducing the family $\{T_i\}_{i\in Z}$, where for every $\alpha \in \Lambda$ and $i \in Z$

(i) $T_i|_{H_\alpha} = 0$ *for* $\alpha_i = 0$,
(ii) $T_i|_{H_\alpha}$ *is injective for* $\alpha_i = 1$.

There is a natural question of a decomposition with weaker than a joint quasinormality assumption. By the following Remarks 3.3 and 3.5, we can describe an interesting subclass of quasinormal operators.

Remark 3.3. Let H_0 reduces a quasinormal operator T. Let $T = W|T|$ be the polar decomposition, where W is a partial isometry. Obviously H_0 and $H \ominus H_0$ reduce T^*T

which is equivalent to the commutativity of P_{H_0} and $P_{H\ominus H_0}$ with T^*T. Consequently $P_{H_0}, P_{H\ominus H_0}$ commute with $|T|$ and subspaces $H_0,\ H\ominus H_0$ are $|T|$ invariant. Note that $\mathcal{N}(|T|) = \mathcal{N}(T) = \mathcal{N}(W)$. Thus for any $x \in H$ we have $W^*x \perp \mathcal{N}(|T|)$. It follows that $T^*x = |T|W^*x = 0$ if and only if $W^*x = 0$. Thus $\mathcal{N}(W^*) = \mathcal{N}(T^*) \supset \mathcal{N}(T)$. It follows that Wx, W^*x are orthogonal to $\mathcal{N}(T) = \mathcal{N}(|T|)$. Since $\mathcal{N}(|T|)$ is a hyperinvariant subspace of $|T|$, it reduces $P_{H\ominus H_0}$. Thus for any $x \in H_0$ we have

$$0 = P_{H\ominus H_0}Tx = P_{H\ominus H_0}|T|Wx = |T|P_{H\ominus H_0}Wx$$

and consequently

$$P_{H\ominus H_0}Wx = P_{\mathcal{N}(|T|)}P_{H\ominus H_0}Wx = P_{H\ominus H_0}P_{\mathcal{N}(|T|)}Wx = 0.$$

Similarly, $P_{H\ominus H_0}W^*x = 0$. Since x has been taken arbitrary, it follows that H_0 reduces W.

By Remark 3.3 subspaces in any decomposition of a quasinormal operator reduce also a partial isometry in the polar decompositions of the operator. In this sense a decomposition of a quasinormal operator generates the corresponding decomposition of a partial isometry. It can be shown that subspaces in the Wold decomposition of an isometric part of a partial isometry reduce a whole quasinormal operator. In general, a subspace reducing a partial isometry in the polar decomposition of a quasinormal operator does not need to reduce the quasinormal operator.

Example 3.4. Let $H = \bigoplus_{n\geq 0} \mathbb{C}e_n \oplus \mathbb{C}f_n$, where e_n, f_n are orthonormal vectors for $n \geq 0$. Define $Te_n = 2e_{n+1}, Tf_n = f_{n+1}$. Then $We_n = e_{n+1}, Wf_n = f_{n+1}$. A closed subspace generated by $e_n + f_n$ for $n \geq 0$ reduces W but does not reduce T.

Partial isometries which are obtained in the polar decomposition of quasinormal operators are also interesting because they are quasinormal.

Remark 3.5. Let $T \in L(H)$ be a quasinormal operator and $T = W|T|$ denotes the polar decomposition. Then W is quasinormal.

Proof. For a partial isometry, we have $WW^*W = W$. Thus we need to show that $W^*W^2 = W$. It is trivial for vectors from $\mathcal{N}(W) = \mathcal{N}(|T|) = \mathcal{N}(T)$. Take arbitrary x such that $x = |T|y$ for some $y \in H$. Since $\mathcal{N}(T) \subset \mathcal{N}(T^*)$ for a quasinormal operator, then $\overline{\mathcal{R}(T)} \subset H \ominus \mathcal{N}(T)$ and $W|_{\overline{\mathcal{R}(T)}}$ is an isometry. Consequently, $W^*WT = T$ and

$$W^*W^2x = W^*W^2|T|y = W^*WTy = Ty = W|T|y = Wx.$$

Thus we have $W^*W^2 = W$ on a dense set in $H \ominus \mathcal{N}(W)$. □

Quasinormal partial isometries forms an interesting subclass of quasinormal operators. It is easy to verify that the injective decomposition of a quasinormal operator corresponds to the decomposition of a partial isometry in the polar decomposition to an isometry and a zero operator. We restrict our consideration to pairs of commuting quasinormal partial isometries. Note some properties of such pairs.

Lemma 3.6. *For $T_1, T_2 \in L(H)$ commuting quasinormal partial isometries hold:*

(i) $\mathcal{N}(T_1T_2) = \mathcal{N}(T_i) \oplus T_i^*T_i(\mathcal{N}(T_1T_2))$ *for* $i = 1, 2$,
(ii) $T_1^*T_1(\mathcal{N}(T_1T_2)) \subset \mathcal{N}(T_2^*)$, $T_2^*T_2(\mathcal{N}(T_1T_2)) \subset \mathcal{N}(T_1^*)$,
(iii) $\mathcal{N}(T_1T_2) \subset \mathcal{N}(T_1^*T_2^*)$.

Proof. Since for partial isometries $T_i^*T_i = P_{\mathcal{N}(T_i)^\perp}$, then by inclusions $\mathcal{N}(T_i) \subset \mathcal{N}(T_1T_2)$ for $i = 1, 2$ follows the decomposition in (i).

We will show the first inclusion in (ii). By inclusions $T_1(\mathcal{N}(T_1T_2)) \subset \mathcal{N}(T_2) \subset \mathcal{N}(T_2^*) \subset \mathcal{N}(T_1^*T_2^*)$, it follows that $T_1^*T_1(\mathcal{N}(T_1T_2)) \subset \mathcal{N}(T_2^*)$.

To show (iii) note that by a combination of properties of a quasinormal operator and a partial isometry we have that $T_i^*T_i^2 = T_iT_i^*T_i = T_i$ for $i = 1, 2$. For adjoints we obtain that $T_i^* = T_i^{*2}T_i$. Therefore $T_1^*T_2^* = T_1^*(T_2^*)^2T_2 = (T_2^*)^2T_1^*T_2$. It follows that $\mathcal{N}(T_1T_2) \subset \mathcal{N}(T_1^*T_2) \subset \mathcal{N}((T_2^*)^2T_1^*T_2) = \mathcal{N}(T_1^*T_2^*)$. □

Denote by

$$H_{ni} = \bigcap \{L \subset H : \mathcal{N}(T_1T_2) \subset L \text{ and } L \text{ reduces } T_1, T_2\} \tag{3.1}$$

and

$$H_{Is} = H \ominus H_{ni}. \tag{3.2}$$

Since $\mathcal{N}(T_1T_2) = \mathcal{N}(T_1) \oplus T_2^*T_2(\mathcal{N}(T_1T_2)) \subset \mathcal{N}(T_1) + T_2^*(\mathcal{N}(T_1))$ and $\mathcal{N}(T_i) \subset \mathcal{N}(T_1T_2)$ for $i = 1, 2$, then H_{ni} is the minimal subspace reducing both operators and containing $\mathcal{N}(T_1)$ and $\mathcal{N}(T_2)$. Thus H_{Is} is the maximal subspace reducing the pair T_1, T_2 to isometries. Restrictions $T_1|_{H_{ni}}, T_2|_{H_{ni}}$ form a completely non isometric pair. The commuting isometries are being studied by many authors (see [1, 3, 8]). In the following part we consider a pair of commuting quasinormal partial isometries and describe how one of them acts between the kernel and the isometric part of the other one. Obviously $H_{00} = \mathcal{N}(T_1) \cap \mathcal{N}(T_2)$ reduces both operators. Consider $H_{ni} \ominus H_{00}$. Recall after [6] that a pair of isometries V_1, V_2 is called *compatible* if $P_{\overline{\mathcal{R}(V_1^n)}}$ commutes with $P_{\overline{\mathcal{R}(V_2^m)}}$ for every $n, m \in \mathbb{Z}_+$. It can be shown that isometries are compatible if and only if

$$\mathcal{N}(V_1^{*n}) = (\mathcal{N}(V_1^{*n}) \cap \mathcal{N}(V_2^{*m})) \oplus (\mathcal{N}(V_1^{*n}) \cap \overline{\mathcal{R}(V_2^m)})$$

for every $n, m \in \mathbb{Z}_+$. We will follow the idea of compatibility but decompose kernels of quasinormal operators instead of kernels of theirs adjoints.

Definition 3.7. We call a pair of commuting quasinormal operators $T_1, T_2 \in L(H)$ *q-compatible,* if $\mathcal{N}(T_1) \ominus (\mathcal{N}(T_1) \cap \mathcal{N}(T_2))$ is orthogonal to $\mathcal{N}(T_2) \ominus (\mathcal{N}(T_1) \cap \mathcal{N}(T_2))$.

Since $\mathcal{N}(T)$ reduces a quasinormal operator T, we have $\mathcal{N}(T^n) = \mathcal{N}(T)$ for $n \in \mathbb{Z}_+$. Therefore in the definition of q-compatibility we do not concern powers of the operator. To avoid misunderstanding in case of isometries which are quasinormal we can not call the defined property simply compatibility.

Remark 3.8. If there is the maximal subspace $L \subset H$ reducing the pair T_1, T_2 and such that

$$(\mathcal{N}(T_1) \cap L) \perp (\mathcal{N}(T_2) \cap L),$$

then each subspace L_0 reducing operators T_1, T_2 to isometries or one to an isometry and the other one to a zero operator is a subspace of L. Indeed, in such case $\mathcal{N}(T_1|_{L_0}) \perp \mathcal{N}(T_2|_{L_0})$ because at least one of the kernels is $\{0\}$. Since L_0 reduces both operators $\mathcal{N}(T_i|_{L_0}) = \mathcal{N}(T_i) \cap L_0$ for $i = 1, 2$. Then by the maximality of L follows that $L_0 \subset L$.

Our aim is to find the maximal subspace L described in Remark 3.8. Decompose

$$\mathcal{N}(T_i) \cap (H_{ni} \ominus H_{00}) = G_i \oplus F_i,$$

where

$$G_1 = \mathcal{N}(T_1) \cap T_2^* T_2(\mathcal{N}(T_1 T_2)), \quad G_2 = \mathcal{N}(T_2) \cap T_1^* T_1(\mathcal{N}(T_1 T_2)) \tag{3.3}$$

and

$$F_i = \mathcal{N}(T_i) \ominus (G_i \oplus H_{00}) \text{ for } i = 1, 2. \tag{3.4}$$

Since T_i is a patrial isometry, then $T_i^* T_i = P_{\mathcal{N}(T_i)^\perp}$ for $i = 1, 2$. Thus $\mathcal{R}(T_i^* T_i)$ and $T_i^* T_i(\mathcal{N}(T_1 T_2)) = \mathcal{N}(T_1 T_2) \ominus \mathcal{N}(T_i)$ are closed subspaces. Note that for $x \in \mathcal{N}(T_1 T_2) \cap \mathcal{R}(T_i^* T_i)$ we have $x = T_1^* T_1 x \in T_1^* T_1(\mathcal{N}(T_1 T_2))$. It follows the inclusion $\mathcal{N}(T_1 T_2) \cap \mathcal{R}(T_i^* T_i) \subset T_i^* T_i(\mathcal{N}(T_1 T_2))$ for $i = 1, 2$. The reverse inclusion holds by Lemma 3.6(i). Consequently, $G_1 = \mathcal{N}(T_1) \cap \mathcal{R}(T_2^* T_2)$ and $G_2 = \mathcal{N}(T_2) \cap \mathcal{R}(T_1^* T_1)$. This means that subspaces G_1, G_2 consist of all vectors which are in the kernel of one operator and are orthogonal to the kernel of the other one.

The result in the following lemma is known. We give it without proof.

Lemma 3.9. *Consider closed subspaces A, B of a Hilbert space K. Then*

$$(A \cap B)^\perp = \overline{A^\perp + B^\perp}.$$

We use the lemma above to describe some properties of the subspaces F_1, F_2.

Proposition 3.10. *Let $T_1, T_2 \in L(H)$ be commuting quasinormal partial isometries and F_1, F_2 be subspaces defined in* (3.4). *Then $F_1, F_2 \subset \mathcal{N}(T_1^*) \cap \mathcal{N}(T_2^*)$.*

Proof. Let $F_{12} = \mathcal{N}(T_1 T_2) \ominus (G_2 \oplus G_1)$, where G_1, G_2 are subspaces defined in (3.3). Denote for convenience: $K = \mathcal{N}(T_1 T_2)$, $A_i = T_i^* T_i(K)$, $B_i = \mathcal{N}(T_i)$ for $i = 1, 2$. Note that $G_1 = B_1 \cap A_2$, $G_2 = B_2 \cap A_1$ and consequently

$$F_{12} = K \ominus ((B_2 \cap A_1) \oplus (B_1 \cap A_2)) = (K \ominus (B_2 \cap A_1)) \cap (K \ominus (B_1 \cap A_2)).$$

If we consider K as a whole space, then by Lemma 3.9 we obtain

$$F_{12} = \overline{(K \ominus B_2) + (K \ominus A_1)} \cap \overline{(K \ominus B_1) + (K \ominus A_2)}.$$

On the other hand, Lemma 3.6(i),(ii) shows that $K = B_i \oplus A_i$ for $i = 1,2$ and $A_1, B_2 \subset \mathcal{N}(T_2^*)$, $A_2, B_1 \subset \mathcal{N}(T_1^*)$. Thus

$$F_{12} = \overline{(A_2 + B_1)} \cap \overline{(A_1 + B_2)} \subset \mathcal{N}(T_1^*) \cap \mathcal{N}(T_2^*).$$

By the definition of F_1, it is a subspace of $\mathcal{N}(T_1) \subset \mathcal{N}(T_1T_2)$ and it is orthogonal to G_1. Since G_2 is orthogonal to $\mathcal{N}(T_1)$, then it is also orthogonal to $F_1 \subset \mathcal{N}(T_1)$. Consequently, $F_1 \subset F_{12}$. Similarly $F_2 \subset F_{12}$. It follows that

$$F_i \subset \mathcal{N}(T_1^*) \cap \mathcal{N}(T_2^*) \quad \text{for} \quad i = 1,2. \qquad \square$$

We will now construct the maximal T_1, T_2 reducing subspace generated by F_1, F_2.

Theorem 3.11. *Let $T_1, T_2 \in L(H)$ be a pair of commuting quasinormal partial isometries and F_1, F_2 be subspaces given in* (3.4)*. The subspace*

$$H_F = \overline{\sum_{n\geq 1} T_1^n(F_2)} \oplus \overline{\sum_{n\geq 1} T_2^n(F_1)} \oplus \overline{\big(T_1^*T_1(F_2) + T_2^*T_2(F_1)\big)}.$$

is the minimal subspace reducing T_1, T_2 and containing F_1, F_2.

Proof. At the beginning we will show that summands in the definition of H_F are indeed orthogonal. By quasinormality $\mathcal{N}(T_i) \subset \mathcal{N}(T_i^*)$. Note that $\overline{\mathcal{R}(T_i^*)} = H \ominus \mathcal{N}(T_i) \supset H \ominus \mathcal{N}(T_i^*) = \overline{\mathcal{R}(T_i)}$ for $i = 1,2$. Thus $\overline{\sum_{n\geq 1} T_2^n(F_1)}$ and $T_2^*T_2(F_1)$ are included in $\overline{\mathcal{R}(T_2^*)}$. On the other hand, $\overline{\sum_{n\geq 1} T_1^n(F_2)} \subset \mathcal{N}(T_2)$. Thus $\overline{\sum_{n\geq 1} T_1^n(F_2)}$ is orthogonal to $\overline{\sum_{n\geq 1} T_2^n(F_1)}$ and $T_2^*T_2(F_1)$. Recall that for a quasinormal partial isometry $T^* = T^{*2}T$ (see proof of Lemma 3.6(iii).) For any $x, y \in F_2$ and $n \geq 1$ by Proposition 3.10 we have that $0 = (T_1^{n-1}x, T_1^*y) = (T_1^{n-1}x, T_1^*T_1^*T_1y) = (T_1^nx, T_1^*T_1y)$. Thus $\overline{\sum_{n\geq 1} T_1^n(F_2)}$ is orthogonal to $T_1^*T_1(F_2)$. We have shown that the subspace $\overline{\sum_{n\geq 1} T_1^n(F_2)}$ is orthogonal to the remaining summands. Similarly, the subspace $\overline{\sum_{n\geq 1} T_2^n(F_1)}$ is orthogonal to $T_1^*T_1(F_2)$ and $T_2^*T_2(F_1)$. This finishes the proof of the orthogonality.

We make a construction of the minimal T_1, T_2 reducing subspace containing subspaces F_1, F_2. Note that $\overline{\sum_{n\geq 0} T_2^n(F_1)}$ is T_2 invariant and since it is a subspace of $\mathcal{N}(T_1) \subset \mathcal{N}(T_1^*)$, it is T_1 reducing. Since T_2 is a partial isometry then $T_2^*T_2^n(F_1) = T_2^{n-1}(F_1)$ for $n \geq 2$. In case $n = 0$ Proposition 3.10 shows that $T_2^*(F_1) \subset T_2^*(\mathcal{N}(T_1^*) \cap \mathcal{N}(T_2^*)) = \{0\}$. However, $T_2^*T_2(F_1)$ may not be included in $\overline{\sum_{n\geq 0} T_2^n(F_1)}$. We obtain a similar result for the subspace $\overline{\sum_{n\geq 0} T_1^n(F_2)}$. It follows that the minimal subspace reducing T_1, T_2 and containing F_1, F_2 is not smaller than

$$H_{F_0} := \overline{\sum_{n\geq 0} T_1^n(F_2)} + \overline{\sum_{n\geq 0} T_2^n(F_1)} + \overline{T_1^*T_1(F_2) + T_2^*T_2(F_1)}.$$

We will show that H_{F_0} reduces T_1, T_2. Note that $H_F \subset H_{F_0}$ and by the previous argumentation, the images of $\overline{\sum_{n\geq 0} T_1^n(F_2)}, \overline{\sum_{n\geq 0} T_2^n(F_1)}$ under operators T_1, T_2, T_1^*, T_2^* are subspaces of H_F. We need to show a similar result for $T_1^*T_1(F_2)$

and $T_2^*T_2(F_1)$. Let us prove it for $T_1^*T_1(F_2)$. Note that $T_2(T_2^*T_2(F_1)) = T_2(F_1) \subset H_F$. By $T_2^{*2}T_2 = T_2^*$ and Proposition 3.10, we have

$$T_2^*T_2^*T_2(F_1) = T_2^*(F_1) \subset T_2^*(\mathcal{N}(T_1^*) \cap \mathcal{N}(T_2^*)) = \{0\}.$$

By the definition of F_1 it follows that $T_2(F_1) \subset \mathcal{N}(T_1) \subset \mathcal{N}(T_1^*)$ and consequently $T_1^*(T_2^*T_2(F_1)) = \{0\}$. To show that $T_1T_2^*T_2(F_1) \subset H_F$, take arbitrary $x \in F_1$ and decompose it to $x = x_0 \oplus T_2^*T_2x \in \mathcal{N}(T_2) \oplus \overline{\mathcal{R}(T_2^*)}$. Since $T_1x = 0$, then $T_1T_2^*T_2x = T_1(-x_0)$. For any $z \in G_2$ we have $z = T_1^*T_1z$. Note that that

$$\begin{aligned}(x_0, z) = (x_0, T_1^*T_1z) = (T_1x_0, T_1z) = (-T_1T_2^*T_2x, T_1z) = \\ = -(T_2^*T_2x, T_1^*T_1z) = -(T_2^*T_2x, z) = -(T_2x, T_2z) = -(T_2x, 0) = 0.\end{aligned}$$

Since vector z has been chosen arbitrary, x_0 is orthogonal to the subspace G_2. Thus x_0 being in $\mathcal{N}(T_2)$ belongs to F_2. The equality $T_1T_2^*T_2x = T_1(-x_0)$ shows the inclusion $T_1T_2^*T_2(F_1) \subset T_1(F_2) \subset H_F$. Thus H_{F_0} reduces T_1, T_2.

Note that we have shown that images of H_{F_0} under operators T_1, T_2, T_1^*, T_2^* are not only subspaces of H_{F_0} but also subspaces of H_F. Therefore, since H_F is a subspace of H_{F_0} it also reduces T_1, T_2. By the minimality of H_{F_0} we have that $H_F = H_{F_0}$. □

By Theorem 3.11, we can find the maximal subspace L described in Remark 3.8.

Corollary 3.12. *A subspace*

$$H_{ort} := H_{ni} \ominus (H_{00} \oplus H_F)$$

is the maximal T_1, T_2 *reducing subspace orthogonal to* H_{Is}*, where* $\mathcal{N}(T_1|_{H_{ort}}) \perp \mathcal{N}(T_2|_{H_{ort}})$.

Proof. It is enough to show that any subspace L reducing T_1, T_2 and such that $\mathcal{N}(T_1|_L) \perp \mathcal{N}(T_2|_L)$ is orthogonal to H_{00} and H_F. Note that by assumed orthogonality of kernels it follows that $P_{\mathcal{N}(T_1)}P_{\mathcal{N}(T_2)}L = \{0\}$ and $P_{\mathcal{N}(T_2)}P_{\mathcal{N}(T_1)}L = \{0\}$. On the other hand, $P_{\mathcal{N}(T_i)} = I - T_i^*T_i$. Thus an orthogonal projection on any T_i reducing subspace commutes with $P_{\mathcal{N}(T_i)}$.

The orthogonality of L to H_{00} follows by

$$\{0\} = P_{H_{00}}P_{\mathcal{N}(T_1)}P_{\mathcal{N}(T_2)}L = P_{\mathcal{N}(T_1)}P_{\mathcal{N}(T_2)}P_{H_{00}}L = P_{H_{00}}L.$$

Similarly, $\{0\} = P_{\mathcal{N}(T_1)}P_{\mathcal{N}(T_2)}P_{H_F}L$ and $\{0\} = P_{\mathcal{N}(T_2)}P_{\mathcal{N}(T_1)}P_{H_F}L$. By the definition of F_1, for any $x \in F_1$ holds $P_{\mathcal{N}(T_2)}P_{\mathcal{N}(T_1)}x = P_{\mathcal{N}(T_2)}x \neq 0$. Similarly for F_2. Thus $P_{H_F}L$ is a proper subspace of H_F, reduces T_1, T_2 and does not contain any vector from F_1 nor F_2. By the minimality of H_F, it follows that $P_{H_F}L = \{0\}$. □

As an easy consequence we can find subspaces reducing quasinormal partial isometries to pairs: an isometry – a zero operator and a zero operator – an isometry.

Theorem 3.13. *Let* $T_1, T_2 \in L(H)$ *be a pair of commuting quasinormal partial isometries such that* $\mathcal{N}(T_1) \perp \mathcal{N}(T_2)$*. The subspaces*

$$H_{0,Is} := \bigcap_{n \geq 0} \mathcal{N}(T_1T_2^{*n}), \qquad H_{Is,0} := \bigcap_{n \geq 0} \mathcal{N}(T_2T_1^{*n})$$

are maximal reducing T_1, T_2 *such that* $T_1|_{H_{0,Is}} = 0, T_2|_{H_{Is,0}} = 0$ *and* $T_2|_{H_{0,Is}}, T_1|_{H_{Is,0}}$ *are isometries.*

Proof. By the inclusion $\mathcal{N}(T_i) \subset \mathcal{N}(T_i^*)$ valid for quasinormal operators, every subspace of $\mathcal{N}(T_1)$ reduces T_1. Since $H_{0,Is} \subset \mathcal{N}(T_1) \perp \mathcal{N}(T_2)$, then $H_{0,Is} = T_2^* T_2(H_{0,Is})$. For any $n \geq 1$ holds

$$\{0\} = T_1 T_2^{*n-1}(H_{0,Is}) = T_1 T_2^{*n-1}(T_2^* T_2(H_{0,Is})) = T_1 T_2^{*n}(T_2(H_{0,Is})).$$

Also for $n = 0$ it is obvious that $T_2(H_{0,Is}) \subset T_2(\mathcal{N}(T_1)) \subset \mathcal{N}(T_1)$. Consequently,

$$T_2(H_{0,Is}) \subset \bigcap_{n\geq 0} \mathcal{N}(T_1 T_2^{*n}) = H_{0,Is}.$$

Note that $H_{0,Is} = \bigcap_{n\geq 0} \mathcal{N}(T_1 T_2^{*n})$ is the maximal subspace of $\mathcal{N}(T_1)$ invariant for T_2^*. Since we have shown that $H_{0,Is}$ is invariant also for T_2, it is the maximal subspace of $\mathcal{N}(T_1)$ that reduce T_2. Since $\mathcal{N}(T_1)$ is orthogonal to $\mathcal{N}(T_2)$ it follows that $H_{0,Is}$ reduces T_2 to an isometry. □

Denote

$$H_G = H \ominus (H_{Is} \oplus H_{Is,0} \oplus H_{0,Is} \oplus H_{00} \oplus H_F). \tag{3.5}$$

By Lemma 3.6(iii), we have $\mathcal{N}(T_1T_2) \subset \mathcal{N}(T_1^* T_2^*)$. Thus the product of quasinormal partial isometries is a quasinormal partial isometry if it is a partial isometry.

Remark 3.14. Subspaces H_{00}, $H_{0,Is}$, $H_{Is,0}$ reduce the product T_1T_2 to a zero operator while H_{Is} defined in (3.2) reduce it to an isometry. Recall that

$$H_F = \overline{\sum_{n\geq 1} T_1^n(F_2)} \oplus \overline{\sum_{n\geq 1} T_2^n(F_1)} \oplus \overline{\big(T_1^* T_1(F_2) + T_2^* T_2(F_1)\big)}.$$

Note that $\overline{\sum_{n\geq 1} T_1^n(F_2)} \subset \mathcal{N}(T_2)$, $\overline{\sum_{n\geq 1} T_2^n(F_1)} \subset \mathcal{N}(T_1)$. Since

$$T_1 T_2 T_2^* T_2(F_1) = T_1 T_2(F_1) \subset T_2 T_1(\mathcal{N}(T_1)) = \{0\},$$

then $T_2^* T_2(F_1) \subset \mathcal{N}(T_1T_2)$. Similarly, $T_1^* T_1(F_2) \subset \mathcal{N}(T_1T_2)$. Thus $H_F \subset \mathcal{N}(T_1T_2)$. Consequently, a product of quasinormal partial isometries restricted to the subspace $H \ominus H_G$ is a quasinormal partial isometry.

In the next paragraph it will be shown that if $H_G \neq \{0\}$, then the product of quasinormal partial isometries can be, but do not need to be, a quasinormal partial isometry (Examples 4.2 and 4.3). Recall after [5] the following lemma.

Lemma 3.15. *Let* T_1, T_2 *be partial isometries* (*possibly not commuting*). *Product* T_1T_2 *is a partial isometry if and only if* $T_1^* T_1 T_2 T_2^* = T_2 T_2^* T_1^* T_1$.

The subspace $H_{Is} \oplus H_{Is,0} \oplus H_{0,Is} \oplus H_{00} \oplus H_F$ reduce T_1, T_2 such that the product T_1T_2 is a partial isometry. However, it is not the maximal such subspace.

Proposition 3.16. *Let T_1, T_2 be a pair of commuting quasinormal partial isometries such that $H = H_G$, where H_G is given in* (3.5)*. Let H_p be the maximal subspace of $\mathcal{N}(T_1T_2)$ reducing T_1, T_2. Then H_p is the maximal subspace reducing T_1, T_2 where the product $T_1T_2|_{H_p}$ is a quasinormal partial isometry.*

Proof. Let H_p be the maximal subspace of $\mathcal{N}(T_1T_2)$ reducing T_1, T_2. Obviously $(T_1T_2)|_{H_p}$ being a zero operator is a quasinormal partial isometry. Let $L \subset H \ominus H_p$ be any non zero subspace reducing T_1, T_2. Since we have assumed $H = H_G$, then L can not reduce T_1, T_2 to a pair of isometries. Consequently, $\mathcal{N}(T_1T_2) \cap L \neq \{0\}$. Since L is orthogonal to H_p, we can choose $x \in \mathcal{N}(T_1T_2) \cap L$ such that T_1^*x or T_2^*x is not in $\mathcal{N}(T_1T_2)$. Assume that $T_1T_2T_2^*x \neq 0$. It follows that $T_1^*T_1T_2T_2^*x \neq 0$. On the other hand, $x \in \mathcal{N}(T_2T_1) \subset \mathcal{N}(T_2^*T_1)$. Hence $T_2T_2^*T_1^*T_1x = T_2T_1^*T_2^*T_1x = 0$. By Lemma 3.15 the product is not a partial isometry. □

We can formulate the decomposition theorem for pairs of commuting quasinormal partial isometries.

Theorem 3.17. *Let H be a Hilbert space and $T_1, T_2 \in L(H)$ be a pair of commuting quasinormal partial isometries. There is a decomposition*

$$H = H_J \oplus H_F \oplus H_p \oplus H_n,$$

where H_J, H_F, H_p, H_n are the maximal subspaces reducing operators T_1, T_2 such that:

(i) *$T_1|_{H_J}, T_2|_{H_J}$ are jointly quasinormal,*
(ii) *$T_1|_{H_F}, T_2|_{H_F}$ are completely non q-compatible,*
(iii) *$T_1|_{H_p}, T_2|_{H_p}$ are q-compatible, completely non jointly quasinormal and the product T_1T_2 is a partial isometry,*
(iv) *$(\mathcal{N}(T_1) \cap H_n) \perp (\mathcal{N}(T_2) \cap H_n)$ and there is no non trivial T_1, T_2 reducing subspace of H_n, where the product T_1T_2 is a partial isometry.*

Proof. Define $H_J = H_{Is} \oplus H_{0,Is} \oplus H_{Is,0} \oplus H_{00}$, where H_{Is} defined in (3.2) reduces T_1, T_2 to a pair of isometries, $H_{Is,0}, H_{0,Is}$ are given by Theorem 3.13 and $H_{00} = \mathcal{N}(T_1) \cap \mathcal{N}(T_2)$. By Theorem 3.2, every jointly quasinormal pair has a multiple injective canonical decomposition. Note that an injective partial isometry is just an isometry. On the other hand, $H_{Is}, H_{0,Is}, H_{Is,0}, H_{00}$ are the maximal subspaces reducing T_1, T_2 to suitably: a pair of isometries, T_1 to a zero operator and T_2 to an isometry, T_1 to an isometry and T_2 to a zero operator, a pair of zero operators. Consequently, the maximality of H_J follows from the maximality of their summands.

Define H_F, H_p, H_G suitably by Theorem 3.11, Proposition 3.16, formula (3.5) and $H_n = H_G \ominus H_p$. From these results follows also that restrictions of T_1, T_2 to the subspaces H_F, H_p, H_n have suitable properties and $H \ominus H_J = H_F \oplus H_p \oplus H_n$. □

It may be surprising that in the subspace where the product of quasinormal partial isometries is not a partial isometry we have the orthogonality of kernels.

4. EXAMPLES

Each subspace in the decomposition in Theorem 3.17 can be non trivial. In this paragraph we give examples of non jointly quasinormal pairs. We use the fact that in case of partial isometries the inclusion $\mathcal{N}(T) \subset \mathcal{N}(T^*)$ is equivalent to quasinormality. The first example concerns the non q-compatible case.

Example 4.1. Let $H = \bigoplus_{n\geq 1} H_n$, where $H_n = \mathbb{C}e \oplus \mathbb{C}f$ for every $n = 1, 2, \dots$ and e, f are orthonormal vectors. Denote the canonical basis in H by $e_i = (0, 0, \dots 0, e, 0, \dots)$ and $f_i = (0, 0, \dots 0, f, 0, \dots)$ with non zero value on the i-th coordinate. Define:

$$\begin{aligned} T_1(e_i) &= e_{i+1}, & T_1(f_i) &= 0, \quad \text{for } i = 1, 2, \dots, \\ T_2(\sqrt{2}/2e_1 + \sqrt{2}/2f_1) &= 0, & T_2(\sqrt{2}/2e_1 - \sqrt{2}/2f_1) &= f_2, \\ T_2(e_i) &= 0, & T_2(f_i) &= f_{i+1}, \quad \text{for } i = 2, 3, \dots \end{aligned}$$

Obviously, T_1, T_2 are partial isometries. We leave to the reader to check that $T_1T_2 = 0 = T_2T_1$ and $\mathcal{N}(T_j) \subset \mathcal{N}(T_j^*)$, for $j = 1, 2$. It follows that T_1, T_2 are commuting quasinormal operators. From (3.3) and (3.4) follows that:

$$G_1 = \{f_i : i = 2, 3, \dots\},$$

$$G_2 = \{e_i : i = 2, 3, \dots\},$$

$$F_1 = \{f_1\}, \quad F_2 = \{\sqrt{2}/2e_1 + \sqrt{2}/2f_1\}.$$

Thus $H = H_F$.

The next example concerns the q-compatible case, where the product is quasinormal – the case of the H_p subspace in the decomposition.

Example 4.2. Let $H = \bigoplus_{i=-\infty}^{\infty} \mathbb{C}e_i$ be a Hilbert space generated by orthonormal vectors $\{e_i\}_{i=-\infty}^{\infty}$. Define operators:

$$T_1(e_i) = 0 \text{ for } i \leq -1, \quad T_1(e_i) = e_{i+1} \text{ for } i \geq 0,$$

$$T_2(e_i) = e_{i-1} \text{ for } i \leq 0, \quad T_2(e_i) = 0 \text{ for } i \geq 1.$$

The operators are partial isometries. By a simple calculation we can check that $\mathcal{N}(T_j) \subset \mathcal{N}(T_j^*)$ for $j = 1, 2$. Since $T_1T_2 = 0 = T_2T_1$, the operators commute and their product is a partial isometry. Moreover $G_j = \mathcal{N}(T_j)$ and consequently $F_j = \{0\}$ for $j = 1, 2$. By formulas in Theorem 3.13, also $H_{0,Is} = H_{Is,0} = \{0\}$. Eventually, $H = H_p$.

The last example concerns the q-compatible pair, where the product is not quasinormal – the case of the H_n subspace in the decomposition.

Example 4.3. Consider a Hilbert space generated by an orthonormal basis $\{e_i\}_{i=0}^{\infty}$, $\{f_i, g_i\}_{i=1}^{\infty}$. Define operators:

$$T_1(g_i) = 0, \quad T_1(e_i) = e_{i+1}, \quad T_1(f_i) = f_{i+1} \text{ for } i \geq 1, \quad T_1(e_0) = 1/\sqrt{2}e_1 + 1/\sqrt{2}f_1,$$
$$T_2(f_i) = 0, \quad T_2(e_i) = e_{i+1}, \quad T_2(g_i) = g_{i+1} \text{ for } i \geq 1, \quad T_2(e_0) = 1/\sqrt{2}e_1 + 1/\sqrt{2}g_1.$$

One can check that the operators commute and

$$\mathcal{N}(T_2) = \bigoplus_{i=1}^{\infty} \mathbb{C}f_i, \quad \mathcal{N}(T_1) = \bigoplus_{i=1}^{\infty} \mathbb{C}g_i.$$

Thus the kernels are orthogonal. On the other hand,

$$(T_2^{*n} g_n, e_0) = (T_2^* g_1, e_0) = (g_1, T_2 e_0) = (g_1, 1/\sqrt{2}e_1 + 1/\sqrt{2}g_1) = 1/\sqrt{2}.$$

Note that for every $x \in \mathcal{N}(T_1)$, there is n such that $T_2^{*n}x$ is not orthogonal to e_0. Since e_0 is orthogonal to $\mathcal{N}(T_1)$ then $T_2^{*n}x \notin \mathcal{N}(T_1)$. It follows that $H_{0,Is} = \bigcap_{n\geq 0} \mathcal{N}(T_1 T_2^{*n}) = \{0\}$. Similarly, $H_{Is,0} = \{0\}$. Thus $H = H_G$.

Check that $\|T_1T_2e_0\| = \|1/\sqrt{2}e_2\| = 1/\sqrt{2}$. We will show that the product T_1T_2 is not a partial isometry, if we check that $e_0 \perp \mathcal{N}(T_1T_2)$. Decompose $e_0 = x \oplus y \in \overline{\mathcal{R}(T_1^*T_2^*)} \oplus \mathcal{N}(T_1T_2)$. Note that e_0 and x are orthogonal to both kernels $\mathcal{N}(T_1), \mathcal{N}(T_2)$. Thus $y = e_0 - x$ is orthogonal to both kernels, precisely y is orthogonal to f_i and g_i for $i \geq 1$. Since $\|T_1T_2e_k\| = \|e_{k+2}\| = \|e_k\|$ and the product T_1T_2 is a contraction, then $e_k \perp \mathcal{N}(T_1T_2)$ for $k \geq 1$. Consequently, y is orthogonal to every vector in the basis except e_0. On the other hand, $y \in \mathcal{N}(T_1T_2)$, while e_0 is not in $\mathcal{N}(T_1T_2)$. Therefore $y = 0$.

5. APPLICATION TO PAIRS OF QUASINORMAL OPERATORS

The subspaces in the decomposition Theorem 3.17 have been described by geometrical properties of kernels. The kernel of any operator is equal to the kernel of a partial isometry in the polar decomposition of this operator. Thus the decomposition of a pair of quasinormal partial isometries may be used to find the decomposition of a pair of arbitrary quasinormal operators. We will generalize Theorem 3.17 to a pair of quasinormal operators.

Theorem 5.1. *Let H be a Hilbert space and $T_1, T_2 \in L(H)$ be a pair of commuting quasinormal operators. There is a decomposition*

$$H = H_J \oplus H_0 \oplus H_n,$$

where H_J, H_0, H_n are the maximal subspaces reducing T_1, T_2 such that:

(i) *$T_1|_{H_J}, T_2|_{H_J}$ are jointly quasinormal,*
(ii) *$H_0 \subset \mathcal{N}(T_1T_2)$ and $T_1|_{H_0}, T_2|_{H_0}$ are completely non jointly quasinormal,*
(iii) *H_n reduces T_1, T_2 to a completely non jointly quasinormal pair and the product T_1T_2 can not be a zero operator on any nontrivial subspace of H_n reducing T_1, T_2.*

Proof. First note some property we will use in the proof. Let $K \subset H$ be any subspace. The maximal subspace of K reducing T_1, T_2 is an orthogonal complement of the minimal T_1, T_2 reducing subspace containing $K^\perp$. Thus the maximal subspace of K reducing T_1, T_2 is the following

$$\left(\bigcap\{L \text{ reducing } T_1, T_2 : K^\perp \subset L\}\right)^\perp.$$

We will construct the subspace H_J. The commutants of $T_i^*T_i$ and $|T_i|$ are equal. Thus T_1, T_2 are jointly quasinormal, when T_i commutes with $|T_j|$ for $i, j = 1, 2$. Consequently, the subspace reduces T_1, T_2 to a jointly quasinormal pair if and only if it is a T_1, T_2 reducing subspace of $\mathcal{N}(|T_1|T_2 - T_2|T_1|) \cap \mathcal{N}(|T_2|T_1 - T_1|T_2|)$. By the previous argumentation, the subspace

$$H_J = \left(\bigcap \left\{L \text{ reducing } T_1, T_2 : (\mathcal{N}(|T_1|T_2 - T_2|T_1|) \cap \mathcal{N}(|T_2|T_1 - T_1|T_2|))^\perp \subset L\right\}\right)^\perp$$

is the maximal T_1, T_2 reducing subspace of $\mathcal{N}(|T_1|T_2 - T_2|T_1|) \cap \mathcal{N}(|T_2|T_1 - T_1|T_2|)$. Thus H_J is the maximal subspace reducing T_1, T_2 to a jointly quasinormal pair.

We will construct the H_0 subspace. The subspace

$$L_{max} = \left(\bigcap \left\{L \text{ reduce } T_1, T_2 : \mathcal{N}(T_1T_2)^\perp \subset L\right\}\right)^\perp$$

is the maximal subspace of $\mathcal{N}(T_1T_2)$ reducing T_1, T_2. By Theorem 3.2 applied to the pair $T_1|_{H_J}, T_2|_{H_J}$ we obtain $H_J = H_{11} \oplus H_{10} \oplus H_{01} \oplus H_{00}$. Note that H_{11} is orthogonal to $\mathcal{N}(T_1T_2)$ and $H_{01}, H_{10}, H_{00} \subset \mathcal{N}(T_1T_2)$. Since H_{01}, H_{10}, H_{00} reduce T_1, T_2, it follows they are also subspaces of L_{max}. Consequently, $H_0 = L_{max} \ominus (H_{10} \oplus H_{01} \oplus H_{00})$ is the maximal subspace of $\mathcal{N}(T_1T_2)$ reducing T_1, T_2 to a completely non jointly quasinormal pair.

The subspace $H_n := H \ominus (H_J \oplus H_0)$ have required properties since it is the orthogonal complement of $H_J \oplus H_0$. □

Theorem 5.1 has been proved independently to Theorem 3.17. However, for further decompositions we will use some of the previous results on quasinormal partial isometries.

Remark 5.2. Let $T_1, T_2 \in L(H)$ be commuting quasinormal operators and $L \subset H$ be a subspace reducing T_1, T_2 such that $T_1|_L = 0$ or $T_2|_L = 0$. Restrictions $T_1|_L, T_2|_L$ are jointly quasinormal since suitable products are equal to 0. Therefore, to find a subspace reducing T_1, T_2 to a pair where at least one of operators is a zero operator, it is enough to check only those subspaces where they are jointly quasinormal.

The next result generalize some formulas to pairs of quasinormal operators.

Theorem 5.3. *Let $T_1, T_2 \in L(H)$ be a pair of commuting quasinormal operators, where $T_1 = W_1|T_1|, T_2 = W_2|T_2|$ are the polar decompositions. Denote by H_J the maximal subspace reducing T_1, T_2 to a jointly quasinormal pair and by $H_{0,Is}, H_{Is,0}$ the maximal subspaces reducing W_1, W_2 to pairs: a zero operator – an isometry, an isometry – a zero operator. Then $H_{0,Is} \cap H_J, H_{Is,0} \cap H_J$ are the maximal subspaces reducing*

T_1, T_2 to pairs: a zero operator – an injective operator, an injective operator – a zero operator.

Proof. Let L be any subspace reducing T_1 to a zero operator and T_2 to an injective operator. By Remark 3.3 and properties of the polar decomposition, the subspace L reduces W_1 to a zero operator and W_2 to an isometry. Thus $L \subset H_{0,Is}$. By Remark 5.2, we have the inclusion $L \subset H_J$. Thus every subspace reducing T_1 to a zero operator and T_2 to an injective operator is a subspace of $H_{0,Is} \cap H_J$.

By Remark 3.3, the subspace H_J reduces W_1, W_2 and consequently $H_{0,Is} \cap H_J$ reduces W_1, W_2, since it is an intersection of such subspaces. Denote $K = Span\{|T_2|^n(H_{0,Is} \cap H_J) : n \geq 0\}$. Since $K \subset H_J$, it follows that $T_1|T_2|^n(H_{0,Is} \cap H_J) = |T_2|^n T_1(H_{0,Is} \cap H_J) = \{0\}$. Therefore K reduces T_1 to a zero operator. Since $H_{0,Is} \cap H_J$ reduces W_2 it follows that

$$T_2|T_2|^n(H_{0,Is} \cap H_J) = |T_2|^{n+1} W_2(H_{0,Is} \cap H_J) \subset |T_2|^{n+1}(H_{0,Is} \cap H_J).$$

Similarly, $T_2^*|T_2|^n(H_{0,Is} \cap H_J) \subset |T_2|^{n+1}(H_{0,Is} \cap H_J)$. Therefore K reduces T_2 to an injective operator. By the first part of the proof $K \subset H_{0,Is} \cap H_J$. The reverse inclusion follows by the definition of K. Thus $K = H_{0,Is} \cap H_J$ reduces T_1, T_2. □

It can be shown that commutativity of jointly quasinormal operators is inherited on partial isometries in their polar decompositions. Unfortunately, it is not true for arbitrary pairs of quasinormal operators. Another problem is that a subspace reducing a partial isometry in the polar decomposition of T does not need to reduce T. Note the following.

Lemma 5.4. *Let $T_1 = W_1|T_1|, T_2 = W_2|T_2|$ be the polar decompositions of commuting quasinormal operators. Denote by L a subspace of $\mathcal{N}(T_1T_2)$.*

(i) *If L reduces T_1, T_2, then $L \subset \mathcal{N}(W_1W_2)$ and L reduces W_1, W_2.*
(ii) *If L reduces W_1, W_2 and $\mathcal{N}(T_i) \subset L$ for $i = 1, 2$, then L reduces T_1, T_2.*

Proof. To show (i) take arbitrary $x \in L$ such that $x = x_0 \oplus |T_2|y$, where $x_0 \in \mathcal{N}(W_2) = \mathcal{N}(|T_2|)$ and y is orthogonal to $\mathcal{N}(|T_2|)$. Since L reduces T_1, T_2, by Remark 3.3 it reduces also W_1, W_2. It follows that $W_2x = T_2y \in L$ and consequently $y = T_2^*W_2x \in L \subset \mathcal{N}(T_1T_2) = \mathcal{N}(W_1T_2)$. On the other hand, $W_1W_2x = W_1T_2y = 0$. Since x has been taken arbitrary in a dense set in L, we have the thesis.

To show (ii) it is enough to prove that L is $|T_1|, |T_2|$ invariant. By $\mathcal{N}(W_1) = \mathcal{N}(|T_1|)$, it follows that W_1 is an isometry on $\overline{\mathcal{R}(|T_1|)}$. Thus $|T_1| = W_1^*W_1|T_1| = W_1^*T_1$ and consequently

$$|T_1|(L) = W_1^*T_1(L) \subset W_1^*T_1(\mathcal{N}(T_1T_2)) \subset W_1^*(\mathcal{N}(T_2)) \subset W_1^*(L) \subset L.$$

It follows that L is $|T_1|$ invariant. Similarly, L is $|T_2|$ invariant which finishes the proof. □

As a consequence of Lemma 5.2, we obtain the following corollary.

Corollary 5.5. *Denote by* $T_1 = W_1|T_1|, T_2 = W_2|T_2|$ *the polar decompositions of commuting quasinormal operators. If* $L \subset \mathcal{N}(T_1T_2)$ *reduces* T_1, T_2*, then* $L \subset \mathcal{N}(W_1W_2)$ *and* W_1, W_2 *commute on* L*.*

Proof. By the first part of the Lemma 5.4 and commutativity of T_1, T_2, we have $W_1W_2|_L = 0 = W_2W_1|_L$. □

As it was shown above, with appropriate assumptions subspaces reducing W_1, W_2 reduce also T_1, T_2. If necessary, some of subspaces in the decomposition of W_1, W_2 can be treated as generators of subspaces reducing T_1, T_2. By Corollary 5.5, commutativity of T_1, T_2 implies commutativity of W_1, W_2 on the subspace H_0 in Theorem 5.1. By Theorem 3.17 applied to restrictions of operators to the subspace H_0, we obtain the decomposition $H_0 = H_F \oplus H_p$. Since H_0 is a subspace of $\mathcal{N}(W_1W_2)$ it follows that $H_n = \{0\}$. If necessary, we can extend H_F to the minimal subspace reducing T_1, T_2. In this way we obtain the maximal subspace reducing T_1, T_2 to a completely non q-compatible pair.

For any operator $T = W|T|$ we have $\mathcal{N}(T) = \mathcal{N}(|T|)$. For the product we do not have the similar property, not always $\mathcal{N}(T_1T_2) = \mathcal{N}(|T_1||T_2|)$. On some of the subspaces in Theorem 3.17 the above equality never holds.

Remark 5.6. Consider a pair of commuting quasinormal operators $T_1 = W_1|T_1|$, $T_2 = W_2|T_2|$, where W_1, W_2 are partial isometries. Assume that $H_F \neq \{0\}$, where H_F is the subspace defined in Theorem 3.11. Since T_1, T_2 are not jointly quasinormal on H_F, there is $x \in H_F$ such that $T_2|T_1|x \neq |T_1|T_2x$ or $T_1|T_2|x \neq |T_2|T_1x$. On the other hand, by $\mathcal{N}(T_1) = \mathcal{N}(|T_1|)$, it follows that $\mathcal{N}(T_1T_2) = \mathcal{N}(|T_1|T_2)$. Since $H_F \subset \mathcal{N}(T_1T_2)$, then $|T_1|T_2x = 0$ and $0 \neq T_2|T_1|x = W_2|T_2||T_1|x$. Consequently, $|T_2||T_1|x \neq 0$ and $\mathcal{N}(T_1T_2) \neq \mathcal{N}(|T_2||T_1|)$. The similar result can be obtained if $H_p \neq \{0\}$.

As a corollary we obtain the following proposition.

Proposition 5.7. *Let* T_1, T_2 *be commuting quasinormal operators and their partial isometries in the polar decompositions* W_1, W_2 *also commute. Then the following conditions are equivalent:*

(i) T_1, T_2 *have a multiple injective canonical decomposition,*
(ii) W_1, W_2 *have a multiple isometric canonical decomposition,*
(iii) $\mathcal{N}(T_1T_2) = \mathcal{N}(|T_1||T_2|)$ *and the product* W_1W_2 *is a partial isometry.*

Proof. We have (i) $\Rightarrow$ (ii) by Remark 3.3.

We will show implication (iii) $\Rightarrow$ (i). Using the notation of Theorem 3.17 applied to W_1, W_2 we have $H_n = \{0\}$ and by Remark 5.6 also $H_F = H_p = \{0\}$. We need to show that subspaces $H_{00}, H_{0,Is}, H_{Is,0}, H_{Is}$ reduce T_1, T_2. By Lemma 5.4(ii), the subspace $H_{0,Is} \oplus H_{Is,0} \oplus H_{00}$ reduces T_1, T_2. Consequently, H_{Is} reduces T_1, T_2. By the inclusion $\mathcal{N}(T_i) \subset \mathcal{N}(T_i^*)$ every subspace of $\mathcal{N}(T_i)$ reduces T_i for $i = 1, 2$. Since each of the operators T_1, T_2 is zero on two of the subspaces $H_{00}, H_{0,Is}, H_{Is,0}$, then it is reduced by these two subspaces and consequently by all three subspaces. The equality $\mathcal{N}(T_i) = \mathcal{N}(W_i)$ for $i = 1, 2$ shows that the obtained decomposition of T_1, T_2 is a multiple injective canonical decomposition.

We will show implication (ii) $\Rightarrow$ (iii). Consider the decomposition of W_1, W_2 given by Theorem 3.17. It follows immediately that product $W_1 W_2$ is a partial isometry. We need to show that $\mathcal{N}(T_1 T_2) = \mathcal{N}(|T_1||T_2|)$. Similarly like in the proof of the implication (iii) $\Rightarrow$ (i) the subspaces in the decomposition reduce also T_1, T_2. Since (iii) trivially holds on H_{00} and H_{Is}, we can assume for convenience that $H = H_{0,Is} \oplus H_{Is,0}$ and consequently that $H = \mathcal{N}(T_1 T_2)$. An arbitrary x can be decomposed to $x = x_1 + x_2 \in H_{0,Is} \oplus H_{Is,0}$. Since

$$|T_2|x = |T_2|W_2^* W_2 x_1 + |T_2|x_2 = T_2^* W_2 x_1 \in H_{0,Is} \subset \mathcal{N}(T_1) = \mathcal{N}(|T_1|),$$

then $|T_1||T_2|x = 0$. □

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