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CONTINUITY OF SUPERQUADRATIC SET-VALUED FUNCTIONS

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Abstract

Let $X = (X, +)$ be an arbitrary topological group. The aim of the paper is to prove a regularity theorem for set-valued superquadratic functions, that is solutions of the inclusion

$$2F(s) + 2F(t) \subset F(s+t) + F(s-t), \qquad s, t \in X,$$

with values in a topological vector space.

1. Introduction

In the present paper superquadratic set-valued functions, defined on a topological group X, that is solutions of the inclusion

$$(1) \qquad 2F(s) + 2F(t) \subset F(s+t) + F(s-t), \qquad s, t \in X,$$

with non-empty, compact and convex values in a topological vector space are studied. If the sign of the inclusion in (1) is replaced by "$\supset$" then F is called subquadratic set-valued function and if we have "$=$" instead of "$\subset$" in (1) then we say that F is quadratic set-valued function. A regularity theorem for a subquadratic set-valued function F, which was considered in [7], stating that upper semi-continuity at a point zero with condition $F(0) = \{0\}$ implies the continuity of a subquadratic set-valued function F everywhere in X. Now, we investigate a regularity theorem for superquadratic set-valued functions.

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It is proved here that lower semi-continuity at a point zero implies the continuity of a superquadratic set-valued function, which has the property (O), everywhere in X. In the case of single real valued subquadratic functions the continuity problem is considered in [2]. For functions of this kind properties of subquadratic and superquadratic functions are quite analogous, in view of the fact, if a function f is subquadratic then the function $-f$ is superquadratic and conversely. It is not necessary to investigate functions of these two kinds individually. In the case of set-valued functions the situation is different. Namely, some properties of subquadratic set-valued functions do not have their analogons for superquadratic set-valued functions and conversely. Moreover, even if properties of subquadratic and superquadratic set-valued functions are similar we have to prove them separately. This is the reason why both functions of this kind are considered. Like in the case of subquadratic set-valued functions, the regularity theorem for superquadratic set-valued functions generalizes some earlier results of this type obtained by D. Henney [1], K. Nikodem [3] and W. Smajdor [6] for quadratic set-valued functions. We start our consideration from basic properties for superquadratic set-valued functions, which play an important role in the proof of the main theorem, which is presented in the third part.

Let us start with the notation used in this paper. Throughout this paper $\mathbb{R}$ stands for the set of reals. Let X be a topological group and Y be a topological vector space. Let $n(Y)$ denotes the family of all non-empty subsets of Y, $c(Y)$ — the family of all compact members of $n(Y)$, $cc(Y)$ — the family of all convex members of $c(Y)$ and $B(Y)$ — the family of all subsets of $n(Y)$. The term set-valued function will be abbreviated in the form s.v.f.

Now we present here some definitions for the sake of completeness.

Definition 1.1. (cf. [6]) *A s.v.f. $F\colon X \to n(Y)$ is said to be upper semi-continuous (abbreviated u.s.c.) at $x \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x+t) \subset F(x) + V$$

for every $t \in U$.

Definition 1.2. (cf. [6]) *A s.v.f. $F\colon X \to n(Y)$ is said to be lower semi-continuous (abbreviated l.s.c.) at $x \in X$ iff for every neighbourhood V of zero* *in Y there exists a neighbourhood U of zero in X such that*

$$F(x) \subset F(x+t) + V$$

for every $t \in U$.

Definition 1.3. (cf. [6]) *A s.v.f. $F\colon X \to n(Y)$ is said to be continuous at $x \in X$ iff it is both u.s.c. and l.s.c. at x. It is said to be continuous iff it is continuous at every point $x \in X$.*

Definition 1.4. (cf. [6]) *Let X be a set and Y be a real topological vector space. A s.v.f. $F\colon X \to n(Y)$ is said to be bounded on a set $B \subset X$ iff the set*

$$F(B) := \bigcup\{F(t) | t \in B\}$$

is bounded in Y.

Definition 1.5. (cf. [6]) *Let X be a set and Y be a real topological vector space. A s.v.f. $F\colon X \to n(Y)$ is said to be bounded at a point $x_0 \in X$ iff there exists a neighbourhood U of zero in X such that F is bounded on $x_0 + U$.*

We adopt the following two definitions.

Definition 1.6. *Let X be a topological group. A set $A \subset X$ is bounded in X iff for every neighbourhood U of zero in X there exists an $n \in \mathbb{N} \cup \{0\}$ such that*

$$A \subset 2^n U.$$

Definition 1.7. *Let X be a topological group and Y be a vector space. A s.v.f. $F\colon X \to n(Y)$ has the property (O) iff for every bounded set A in X the set $F(A)$ is bounded in Y.*

We will use frequently the following well known lemma.

Lemma 1.8. (see [5]) *Let Y be a topological vector space. Let A, B, C be subsets of Y such that $A + C \subset B + C$. If B is closed and convex and C is bounded then $A \subset B$.*

In our proofs we will often use three known lemmas (see Lemma 1.1, Lemma 1.3 and Lemma 1.6 in [3]). The first lemma says that for a convex subset A of an arbitrary real vector space Y the equality $(s+t)A = sA + tA$ holds for every $s, t \geq 0$ (or $s, t \leq 0$). The second lemma says that for two convex subsets $A, B \subset Y$ the set $A + B$ is also convex and the last lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set then the set $A + B$ is closed.

2. Basic properties

Let us start with definition.

Definition 2.1. *A topological group X is said to be locally bounded group iff there exists in it a bounded neighbourhood of zero.*

Lemma 2.2. *Let X be a group and Y be a real topological vector space. If for a superquadratic s.v.f. $F\colon X \to n(Y)$ the set $F(0) \in B(Y)$, then $F(0) = \{0\}$.*

Proof. Putting $t = s = 0$ in (1) and using Lemma 1.1 in [3], we get

$$F(0) + F(0) \supset 2F(0) + 2F(0) = 2\left(F(0) + F(0)\right).$$

Repeating it n-times, we obtain

$$F(0) + F(0) \supset 2^n \left(F(0) + F(0)\right). \tag{2}$$

In spite of the fact, that the sum of two bounded sets is also bounded, the set $F(0) + F(0)$ is bounded. By boundedness of the set $F(0) + F(0)$ and by (2), we get

$$F(0) + F(0) = \{0\}.$$

Hence, $F(0) = \{0\}$. □

Lemma 2.3. *Let X be a group and Y be a real topological vector space. If an s.v.f. $F\colon X \to n(Y)$ with bounded, closed and convex values is superquadratic, then*

$$n^2 F(x) \subset F(nx), \tag{3}$$

for every $x \in X$ and $n \in \mathbb{N}$.

Proof. According to Lemma 2.2 $F(0) = \{0\}$. The proof of the inclusion (3) we can obtain in the same way as the proof of Lemma 2.2 in [7] for subquadratic s.v.f. It is sufficient to replace the sign "$\subset$" in the proof of Lemma 2.2 in [7] by the sign "$\supset$". □

Lemma 2.4. *Let X be a topological group and Y be a real topological vector space. If a s.v.f. $F\colon X \to n(Y)$ is superquadratic and bounded on a neighbourhood of a point $x_0 \in X$, then F is bounded on a neighbourhood of zero in X.*

Proof. Let V_0 be a neighbourhood of zero in Y and let U_0 be a symmetric neighbourhood of zero in X such that the set $F(x_0 + U_0)$ is bounded. Let V be a symmetric neighbourhood of zero in Y such that

$$V + V + V + V \subset V_0. \tag{4}$$

There exists a $c > 0$ such that

$$cF(x_0 + U_0) \subset V. \tag{5}$$

Setting $s = x_0$ in (1) and taking arbitrary $t \in U_0$, we get

$$2F(t) + 2F(x_0) \subset F(x_0 + t) + F(x_0 - t).$$

Fix an $a \in 2F(x_0)$. Then

$$2F(t) \subset F(x_0 + t) + F(x_0 - t) - a \subset F(x_0 + t) + F(x_0 - t) - 2F(x_0), \tag{6}$$

for every $t \in U_0$. By inclusions (4)-(6) and Lemma 1.1 in [3], we get

$$2cF(t) \subset cF(x_0 + t) + cF(x_0 - t) - cF(x_0) - cF(x_0) \subset V + V + V + V \subset V_0$$

for every $t \in U_0$. Thus F is bounded on U_0 and the proof is ended. □

Lemma 2.5. *Let X be a 2-divisible topological group and Y be a real topological vector space. If a s.v.f. $F\colon X \to n(Y)$ is superquadratic with closed, bounded, convex values and bounded on a neighbourhood of a point $x \in X$, then it is u.s.c. at zero in X.*

Proof. The proof of this lemma we can obtain likewise as the proof of Lemma 4.7 in [6] for quadratic set-valued functions. Let V be an arbitrary neighbourhood of zero in Y. By Lemma 2.4 there exists a neighbourhood U of zero in X such that F is bounded on U. Hence, there exists an $n \in \mathbb{N}$ such that

$$\frac{1}{4^n} F(U) \subset V. \tag{7}$$

According to Lemma 2.3, we get

$$F\left(\frac{1}{2^n} U\right) \subset \frac{1}{4^n} F(U). \tag{8}$$

Consequently

$$F\left(\frac{1}{2^n}U\right) \subset V. \tag{9}$$

By Lemma 2.2 $F(0) = \{0\}$. Since

$$F\left(\frac{1}{2^n}U\right) = \bigcup \left\{ F(t) \,\middle|\, t \in \frac{1}{2^n}U \right\},$$

$F(0) = \{0\}$ and the inclusion (9) holds, we get

$$F(t) \subset V + F(0),$$

for every $t \in \frac{1}{2^n}U$, which means that the s.v.f. F is upper semicontinuous at zero. □

The following corollary follows by Lemma 2.5.

Corollary 2.6. *Let X be a 2-divisible locally bounded topological group and Y be a real topological vector space. If a s.v.f. $F\colon X \to n(Y)$ is superquadratic with closed, bounded, convex values and has the property (O), then it is u.s.c. at zero.*

Proof. Let $x_0 \in X$ and let U be a bounded neighbourhood of zero in X. The set $F(x_0 + U)$ is bounded since F has property (O).

Then s.v.f. F is u.s.c. at zero according to Lemma 2.5. □

3. The main result

The proof of the next lemma is similar to the proof of Theorem 3.2 in [7], which is a regularity theorem for subquadratic set-valued functions.

Lemma 3.1. *Let X be a 2-divisible locally bounded topological group and Y be a locally convex real topological space. If a superquadratic s.v.f. $F\colon X \to cc(Y)$ has property (O), then it is upper semicontinuous everywhere in X.*

Proof. Similar as in the proof of Theorem 3.2 in [7], the idea of the proof of this lemma is due to W. Smajdor (Lemma 4.8 in [6]). Suppose that s.v.f. is not u.s.c. at $z \in X$. Then there exists a neighbourhood V of zero in Y such that for every neighbourhood U of zero in X there exists $x_u \in U$ such that

$$F(z + x_u) \not\subseteq F(z) + V.$$

Take a convex balanced neighbourhood W of zero in Y such that $\overline{W} \subset V$. Then

$$F(z + x_u) \not\subseteq F(z) + \overline{W}. \tag{10}$$

By induction we shall show that

$$F\left(z + 2^k x_u\right) \not\subseteq F(z) + 2^k\left(2^k - 1\right)F(x_u) + 2^k\overline{W} \tag{11}$$

for $k = 1, 2, ...$ For $k = 0$ (11) holds by (10). Now, assume that (11) holds for some positive integer $k \geq 0$. By (1), we have

$$2F(z + 2^k x_u) + 2F(2^k x_u) \subset F(z + 2^k x_u + 2^k x_u) + F(z + 2^k x_u - 2^k x_u) =$$

$$= F(z + 2^{k+1} x_u) + F(z).$$

By Lemma 2.3

$$2F(z + 2^k x_u) + 2^{2k+1} F(x_u) \subset 2F(z + 2^k x_u) + 2F(2^k x_u).$$

Consequently

$$2F(z + 2^k x_u) + 2^{2k+1} F(x_u) \subset F(z + 2^{k+1} x_u) + F(z). \tag{12}$$

According to (11) and Lemma 1.8, we obtain

$$2F(z+2^k x_u)+2^{2k+1}F(x_u) \not\subseteq 2F(z)+2^{k+1}\left(2^k - 1\right)F(x_u)+2^{k+1}\overline{W}+2^{2k+1}F(x_u). \tag{13}$$

By (12) i (13), we have

$$F(z + 2^{k+1} x_u) + F(z) \not\subseteq 2F(z) + 2^{k+1}\left(2^k - 1\right)F(x_u) + 2^{k+1}\overline{W} + 2^{2k+1}F(x_u)$$

By convexity of the set $F(x_u)$ and by Lemma 1.1 in [6], we get

$$2F(z) + 2^{k+1}\left(2^k - 1\right)F(x_u) + 2^{k+1}\overline{W} + 2^{2k+1}F(x_u) =$$

$$= 2F(z) + [2^{k+1}\left(2^k - 1\right) + 2^{2k+1}]F(x_u) + 2^{k+1}\overline{W}.$$

Applying the equality

$$2^{k+1}\left(2^k - 1\right) + 2^{2k+1} = 2^{k+1}\left(2^{k+1} - 1\right),$$

we obtain

$$F(z + 2^{k+1} x_u) + F(z) \not\subseteq 2F(z) + 2^{k+1}\left(2^{k+1} - 1\right)F(x_u) + 2^{k+1}\overline{W}.$$

By convexity of the set $F(z)$ and by Lemma 1.1 in [6], we get finally

$$F(z+2^{k+1}x_u) \not\subseteq F(z) + 2^{k+1}\left(2^{k+1}-1\right)F(x_u) + 2^{k+1}\overline{W}.$$

Thus (11) is generally valid for all integer $k \geq 0$.

Take a bounded set U_0 of zero in X. Since F has property (O), there exists $\lambda > 0$ such that

$$\lambda F(z+x) \subset W, \qquad x \in U_0. \tag{14}$$

Now we choose an $k \in \mathbb{N}$ so large that the inequality

$$2^k > \frac{3}{\lambda} \tag{15}$$

holds. By Corollary 2.6 F is u.s.c at zero and $F(0) = \{0\}$ according to Lemma 2.2. Since F is u.s.c. at zero and $F(0) = \{0\}$ there exists a neighbourhood U of zero in X such that

$$F(t) \subset \frac{1}{\lambda 2^k(2^k-1)}W, \qquad t \in U$$

and

$$U \subset \frac{1}{2^k}U_0.$$

Let $x_u \in U$ satisfies condition (11). Moreover

$$2^k x_u \in U_0 \tag{16}$$

and

$$2^k(2^k-1)F(x_u) \subset \frac{1}{\lambda}W. \tag{17}$$

Let $a \in F\left(z+2^k x_u\right)$, $b \in F(z)$ and $c \in F(x_u)$. Then

$$a = b + \left(a - b - 2^k(2^k-1)c\right) + 2^k(2^k-1)c.$$

By inclusions (14), (16) , (17) and (15), we obtain

$$a - b - 2^k(2^k-1)c \in \frac{1}{\lambda}W + \frac{1}{\lambda}W + \frac{1}{\lambda}W = \frac{3}{\lambda}W \subset 2^k W.$$

Therefore

$$a \in F(z) + 2^k W + 2^k(2^k-1)F(x_u)$$

and

$$F\left(z+2^k x_u\right) \subset F(z) + 2^k W + 2^k(2^k-1)F(x_u),$$

in spite of (11), which ends the proof □

The following theorem is the main result of this paper.

Theorem 3.2. *Let X be a $2-$divisible locally bounded topological group and let Y be a locally convex real topological space. If a superquadratic s.v.f. $F\colon X \to cc(Y)$ is l.s.c. at zero and has property (O), then it is continuous everywhere in X.*

Proof. By Lemma 3.1 F is u.s.c. everywhere in X. Now, we show that F is l.s.c. in X. Let $x_0 \in X$ and let V be a neighbourhood of zero in Y. We choose convex neighbourhood V_0 of zero in Y such that $3\overline{V_0} \subset V$. Since F is u.s.c. at x_0 there exists a symmetric neighbourhood U of zero in X such that

$$F(x_0 + t) \subset F(x_0) + V_0, \tag{18}$$

$$F(x_0 - t) \subset F(x_0) + V_0 \tag{19}$$

if $t \in U$.
According to lemma 2.2 $F(0) = \{0\}$. Since F is l.s.c. at zero and $F(0) = \{0\}$, there exists a neighbourhood U_0 of zero in X such that

$$0 \in F(t) + V_0 \qquad t \in U_0. \tag{20}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in X, such that $\widetilde{U} \subset U \cap U_0$.

Now, let $t \in \widetilde{U}$. By (1), (19), (20) and by the fact $F(0) = \{0\}$, we obtain

$$F(x_0) + \{0\} \subset F(x_0) + F(t) + V_0 \subset \frac{1}{2}F(x_0 + t) + \frac{1}{2}F(x_0 - t) + V_0 \subset$$
$$\subset \frac{1}{2}F(x_0 + t) + \frac{1}{2}F(x_0) + \frac{3}{2}V_0.$$

By convexity of the set $F(x_0)$ and by Lemma 1.1 in [3], we get

$$\frac{1}{2}F(x_0) + \frac{1}{2}F(x_0) \subset \frac{1}{2}F(x_0) + \frac{1}{2}F(x_0 + t) + \frac{3}{2}\overline{V_0}.$$

Since the set $\frac{1}{2}F(x_0)$ is bounded and the set $\frac{1}{2}F(x_0 + t) + \frac{3}{2}\overline{V_0}$ is convex (see Lemma 1.3 in [3]) and closed (Lemma 1.6 in [3]), then according to Lemma 1.8 we have proved that

$$\frac{1}{2}F(x_0) \subset \frac{1}{2}F(x_0 + t) + \frac{3}{2}\overline{V_0}.$$

Therefore,

$$F(x_0) \subset F(x_0 + t) + 3\overline{V_0} \subset F(x_0 + t) + V.$$

Thus F is l.s.c in X. The proof is completed. □

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