ZESZYTY NAUKOWE AKADEMII MARYNARKI WOJENNEJ
SCIENTIFIC JOURNAL OF POLISH NAVAL ACADEMY

2017 (LVIII) 4 (211)

DOI: 10.5604/01.3001.0010.6752

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JACKSON AND HAHN DIFFERENCE MODELS AS DISCRETE BITTNER OPERATIONAL CALCULUS REPRESENTATIONS

ABSTRACT

In the paper, there have been constructed discrete models of the non-classical Bittner operational calculus that are related to the notions of Jackson and Hahn derivatives, both known from the quantum calculus. To achieve it, we have used the Bittner operational calculus representation for the forward difference.

Key words:
operational calculus, derivative, integrals, limit conditions, Jackson difference, Hahn difference.

INTRODUCTION

Let $f(t)$ be a real function determined on the interval $(0, +\infty)$. Moreover, let

$$d_h f(t) := f(t+h) - f(t), \quad h \in \mathbb{R}_{>0}$$
$$\delta_q f(t) := f(qt) - f(t), \quad q \in \mathbb{R}_{>0} \setminus \{1\}.$$

The essence of the *q-calculus* (quantum calculus) [2, 3, 8, 11] is to substitute the shift $t + h$ with a dilatation qt in the difference quotient

$$\frac{f(t+h) - f(t)}{h} = \frac{d_h f(t)}{d_h t} =: D_h f(t).$$

Then, we have

$$D_q f(t) := \frac{\delta_q f(t)}{\delta_q t} = \frac{f(qt) - f(t)}{(q-1)t}.$$

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The operations D_h and D_q are called an *h-derivative* and a *q-derivative* (or *Jackson derivative*) [10, 11], respectively.

Using an increment

$$\delta_{h,q} f(t) := f(qt + h) - f(t),$$

we determine the operation

$$D_{h,q} f(t) := \frac{\delta_{h,q} f(t)}{\delta_{h,q} t} = \frac{f(qt + h) - f(t)}{(q - 1)t + h}$$

called an *h, q-derivative* (or *Hahn derivative*) [7, 9].

It is a generalization of D_h and D_q (which are obtained with $q \to 1$ and $h \to 0$, respectively).

In the finite difference calculus, a *forward difference*

$$\Delta f(k) := f(k + 1) - f(k) \tag{1}$$

corresponds to the increment

$$d_h f(t_k) = f(t_k + h) - f(t_k),$$

where $f(k) \equiv f(t_k) := f(t_0 + kh), k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of naturals.

The *n^{th}-order forward difference*

$$\Delta_n f(k) := f(k + n) - f(k),$$

where n is a given natural number, is a generalization of the operation (1).

A MODEL OF THE OPERATIONAL CALCULUS WITH THE n^{TH}-ORDER FORWARD DIFFERENCE

Let $\mathbb{C}$ be a set of complex numbers, while $C(\mathbb{N}_0, \mathbb{C})$ — a linear space of complex sequences $x = \{x(k)\}_{k \in \mathbb{N}_0}$[1] with a usual sequences addition and multiplication of sequences by complexes.

[1] Formally, in the operational calculus we differentiate a function symbol from a symbol of a function value at a point. In particular, $\{x(k)\}$ signifies a sequence, whereas $x(k)$ – its value for a given $k \in \mathbb{N}_0$. This notation is derived from J. Mikusiński [12]. In what follows, we shall skip the brackets { } whenever it does not cause ambiguity.

Moreover, let

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$$

be n^{th} roots of unity, i.e.

$$\varepsilon_j = \cos\frac{2j\pi}{n} + \mathrm{i}\sin\frac{2j\pi}{n}, \quad j \in \overline{0, n-1},$$

where $\overline{0, n-1} := \{0, 1, \ldots, n-1\}$ and 'i' is the imaginary unit.

In [13] it has been shown that the operations $S \equiv \Delta_n, T_{k_0}, s_{k_0}$, where $k_0 \in \mathbb{N}_0$, determined on $C(\mathbb{N}_0, \mathbb{C})$ as follows

$$S\,x := \{x(k+n) - x(k)\}, \tag{2}$$

$$T_{k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=0}^{k-1}\varepsilon_j^{k-i}\,x(i) - \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}\,x(i)\,\Big]\right\}, \tag{3}$$

$$s_{k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0}^{k_0+n-1}\varepsilon_j^{k-i}\,x(i)\right\}, \tag{4}$$

form a *discrete model* (*representation*) of the so-called *Bittner operational calculus* and satisfy two basic formulas of this calculus, i.e.

$$S\,T_{k_0}x = x, \quad T_{k_0}S\,x = x - s_{k_0}x. \tag{5}$$

FUNDAMENTALS OF THE BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* [4–6] is a system

$$CO(L^0, L^1, S, T_\gamma, s_\gamma, \Gamma), \tag{6}$$

in which L^0 and L^1 are linear spaces (over the same scalar field $\mathscr{F}$) such that $L^1 \subset L^0$. The linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in \mathscr{L}(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. Moreover, Γ is a set of indexes γ for the operations $T_\gamma \in \mathscr{L}(L^0, L^1)$ and $s_\gamma \in \mathscr{L}(L^1, L^1)$ such that $S\,T_\gamma f = f, f \in L^0$ and $s_\gamma x = x - T_\gamma S\,x, x \in L^1$. T_γ and s_γ are called *integrals* and *limit conditions*, respectively. The kernel of S, i.e. $\mathrm{Ker}\,S$ is a set of *constants* for the derivative S. The limit conditions s_γ are projections of L^1 on the subspace $\mathrm{Ker}\,S$.

If we define the objects (6), then we speak of a *representation* or a *model* of the operational calculus.

A MODEL OF THE OPERATIONAL CALCULUS WITH THE n^{TH}-ORDER JACKSON DIFFERENCE

A set

$$\mathbb{T}_J \equiv q^{\mathbb{N}_0} := \{k_q : \quad k_q := q^k, k \in \mathbb{N}_0\} = \{1, q, q^2, q^3, \ldots\}$$

will be called the *Jackson time scale* (cf. [1]).

As previously, let $C(\mathbb{T}_J, \mathbb{C})$ be a linear space of complex sequences $x = \{x(k_q)\}_{k_q \in \mathbb{T}_J}$ with a usual sequences addition and sequences multiplication by complexes. Then $C(\mathbb{T}_J, \mathbb{C}) = C(\mathbb{N}_0, \mathbb{C})$. Indeed, each term of a sequence $\{x(k_q)\} \in C(\mathbb{T}_J, \mathbb{C})$ can be presented in the form of

$$x(k_q) \equiv x(q^k) =: \tilde{x}(k),$$

where $\{\tilde{x}(k)\} \in C(\mathbb{N}_0, \mathbb{C})$. For $\{x(k)\} \in C(\mathbb{N}_0, \mathbb{C})$, in turn, we have

$$x(k) = x(\mathrm{lq}(q^k)) =: \hat{x}(k_q),$$

where $\{\hat{x}(k_q)\} \in C(\mathbb{T}_J, \mathbb{C})$ and $\mathrm{lq}(t) := \log_q(t)$.

On the space $C(\mathbb{T}_J, \mathbb{C})$ we define a *forward q-difference*

$$\Delta_{J,q}\{x(k_q)\} := \{x(qk_q) - x(k_q)\}. \tag{7}$$

It corresponds to an increment

$$\delta_q x(t) = x(qt) - x(t),$$

which determines the Jackson q-derivative D_q.

The below *n^{th}-order Jackson difference*

$$\Delta_{J,q,n}\{x(k_q)\} := \{x(q^n k_q) - x(k_q)\}, \tag{8}$$

where n is a given natural number, is a generalization of the operation (7).

Hence, we have

$$\Delta_{J,q,n}\{x(k_q)\} = \{x((k+n)_q) - x(k_q)\} = \{\tilde{x}(k+n) - \tilde{x}(k)\} = \Delta_n\{\tilde{x}(k)\}.$$

Thus, to the operation $\Delta_{J,q,n}$, determined on the Jackson time scale $q^{\mathbb{N}_0}$, there corresponds the difference Δ_n defined on the scale $\mathbb{N}_0$.

Therefore, on the basis of (2)–(4), to the derivative S_J understood as the forward q-difference (8), there correspond integrals

$$T_{J,k_{0q}}\{x(k_q)\} := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=0}^{\mathrm{lq}(k_q)-1}\varepsilon_j^{\mathrm{lq}(k_q)-i}\,x(i_q) - \sum_{i=0}^{\mathrm{lq}(k_{0q})-1}\varepsilon_j^{\mathrm{lq}(k_q)-i}\,x(i_q)\Big]\right\} \tag{9}$$

and limit conditions

$$s_{J,k_{0q}}\{x(k_q)\} := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=\mathrm{lq}(k_{0q})}^{\mathrm{lq}(k_{0q})+n-1}\varepsilon_j^{\mathrm{lq}(k_q)-i}\,x(i_q)\right\}, \tag{10}$$

where $k_{0q} = q^{k_0} \equiv \gamma \in \Gamma := \mathbb{T}_J$, $\mathrm{lq}(k_q) = k$ and $\{x(k_q)\} \in L^0 = L^1 := C(\mathbb{T}_J, \mathbb{C})$.

For the above operations (9) and (10) the formulas (5) are satisfied.

So, we get the

Corollary 1. *The operations* (8)–(10) *form a discrete Jackson model of the Bittner operational calculus*

$$CO(C(\mathbb{T}_J,\mathbb{C}), C(\mathbb{T}_J,\mathbb{C}), S_J, T_{J,k_{0q}}, s_{J,k_{0q}}, \mathbb{T}_J).$$

Using *Mathematica*, we can easily determine the consecutive terms of the sequences (8)–(10). For example, for $n = 3$ and $k_0 = 0$ we obtain

$$S_J\{x(k_q)\} \equiv \Delta_{J,q,3}\{x(q^k)\} = \{x(q^{k+3}) - x(q^k)\},$$

$$T_{J,0q}\{x(k_q)\} \equiv T_{J,0q}\{x(q^k)\} = \{0, 0, 0, x(q^0), x(q), x(q^2), x(q^0) + x(q^3), x(q) + x(q^4), x(q^2) + x(q^5), x(q^0) + x(q^3) + x(q^6), x(q) + x(q^4) + x(q^7), x(q^2) + x(q^5) + x(q^8), x(q^0) + x(q^3) + x(q^6) + x(q^9), x(q) + x(q^4) + x(q^7) + x(q^{10}), \ldots\},$$

$$s_{J,0q}\{x(k_q)\} \equiv s_{J,0q}\{x(q^k)\} = \{x(q^0), x(q), x(q^2), x(q^0), x(q), x(q^2), x(q^0), x(q), x(q^2), \ldots\}.$$

Example 1. Using the described model, we will solve the q-difference equation

$$x(4^3 k_4) - x(k_4) = 4\,\mathrm{lq}(k_4), \quad k_4 \in 4^{\mathbb{N}_0} \tag{11}$$

with initial conditions

$$x(2_4) \equiv x(16) = 2, x(3_4) \equiv x(64) = 1, x(4_4) \equiv x(256) = 1. \tag{12}$$

Hence we have $q = 4, n = 3, k_0 = 2$.

The equation (11) can be shown in the form of

$$S_J x = f, \tag{13}$$

where $S_J \equiv \Delta_{J,4,3}$, $x = \{x(k_4)\}$, $f = \{4\,\mathrm{lq}(k_4)\} = \{4k\}$.

Using *Mathematica*, on the basis of the formula (10), we determine the limit condition corresponding to the initial conditions (12). Finally, we get

$$s_{J,2_4}\{x(k_4)\} = \left\{\frac{1}{3}\left(4 - \sqrt{3}\sin\frac{2\pi\,\mathrm{lq}(k_4)}{3} - \cos\frac{2\pi\,\mathrm{lq}(k_4)}{3}\right)\right\} =: c. \tag{14}$$

The problem (13), (14) has exactly one solution (Th. 3 [6])

$$x = c + T_{J,2_4} f.$$

Hence, using *Mathematica* and basing on (9), we obtain a solution to the Cauchy problem (11), (12):

$$x(k_4) = \frac{1}{9}\left(8 + 6\,(\mathrm{lq}(k_4) - 3)\,\mathrm{lq}(k_4) - 15\sqrt{3}\sin\frac{2\pi\,\mathrm{lq}(k_4)}{3} + \cos\frac{2\pi\,\mathrm{lq}(k_4)}{3}\right), \quad k_4 \in 4^{\mathbb{N}_0}.$$

A MODEL OF THE OPERATIONAL CALCULUS WITH THE n^{TH}-ORDER HAHN DIFFERENCE

In the quantum calculus, there are determined the so-called *q-numbers* (*q*-analogs of non-negative integers)

$$[k]_q := \frac{1 - q^k}{1 - q}, \quad k \in \mathbb{N}_0.$$

The set

$$\mathbb{T}_H \equiv \mathbb{G}_q := \{0, 1, 1 + q, 1 + q + q^2, \ldots, 1 + q + q^2 + \ldots + q^{k-1}, \ldots\},$$

i.e. $\mathbb{T}_H = \{[k]_q : k \in \mathbb{N}_0\}$, will be called the *Hahn time scale* (cf. [1]).

The below operation

$$\Delta_{H,q,n}\{x([k]_q)\} := \{x(q^n[k]_q + [n]_q) - x([k]_q)\}, \tag{15}$$

which is determined on the space $C(\mathbb{T}_H, \mathbb{C})$ and where $q \in \mathbb{R}_{>0} \setminus \{1\}$ and $n \in \mathbb{N}$ are given, will be called the n^{th}*-order Hahn difference*.

Notice that

$$q^n[k]_q + [n]_q = \frac{q^n(1-q^k)+(1-q^n)}{1-q} = \frac{1-q^{k+n}}{1-q} = [k+n]_q.$$

Thus

$$\Delta_{H,q,n}\{x([k]_q)\} = \{x([k+n]_q) - x([k]_q)\} = \{\breve{x}(k+n) - \breve{x}(k)\} = \Delta_n\{\breve{x}(k)\},$$

which means that to the operation $\Delta_{H,q,n}$ determined on the Hahn time scale $\mathbb{T}_H$, there corresponds the difference Δ_n defined on the scale $\mathbb{N}_0$.

Hence, on the basis of (2)–(4), we infer that to the derivative S_H understood as the Hahn difference (15), there correspond the below integrals

$$T_{H,[k_0]_q}\{x([k]_q)\} := \left\{ \frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=0}^{\mathrm{lq}(1-(1-q)[k]_q)-1} \varepsilon_j^{\mathrm{lq}(1-(1-q)[k]_q)-i}\, x([i]_q) - \sum_{i=0}^{\mathrm{lq}(1-(1-q)[k_0]_q)-1} \varepsilon_j^{\mathrm{lq}(1-(1-q)[k]_q)-i}\, x([i]_q)\Big] \right\} \tag{16}$$

and limit conditions

$$s_{H,[k_0]_q}\{x([k]_q)\} := \left\{ \frac{1}{n}\sum_{j=0}^{n-1} \sum_{i=\mathrm{lq}(1-(1-q)[k_0]_q)}^{\mathrm{lq}(1-(1-q)[k_0]_q)+n-1} \varepsilon_j^{\mathrm{lq}(1-(1-q)[k]_q)-i}\, x([i]_q) \right\}, \tag{17}$$

where $[k_0]_q = (1-q^{k_0})/(1-q) \equiv \gamma \in \Gamma := \mathbb{T}_H, \mathrm{lq}(1-(1-q)[k]_q) = k$ and $\{x([k]_q)\} \in L^0 = L^1 := C(\mathbb{T}_H, \mathbb{C})$

Thus, we come to the

Corollary 2. *The operations* (15)–(17) *form a discrete Hahn model of the Bittner operational calculus*

$$CO(C(\mathbb{T}_H,\mathbb{C}), C(\mathbb{T}_H,\mathbb{C}), S_H, T_{H,[k_0]_q}, s_{H,[k_0]_q}, \mathbb{T}_H).$$

Example 2. In the considered model, using *Mathematica*, we will determine such a solution to the equation

$$x(0.5^4[k]_{0.5} + [4]_{0.5}) - x([k]_{0.5}) = [k]_{0.5}, \quad [k]_{0.5} \in \mathbb{G}_{0.5} \tag{18}$$

that satisfies the initial conditions

$$x([0]_{0.5}) \equiv x(0) = 0, x([1]_{0.5}) \equiv x(1) = 1, x([2]_{0.5}) \equiv x(1.5) = 0, x([3]_{0.5}) \equiv x(1.75) = 2.$$

We have here $q = 0.5, n = 4, k_0 = 0$ as well as the equation (18) in the form of $S_H x = f$, where $S_H \equiv \Delta_{H,0.5,4}, x = \{x([k]_{0.5})\}, f = \{[k]_{0.5}\} = \{2(1-0.5^k)\}$.

Similarly as before, from the formula

$$x = s_{H,[0]_{0.5}} x + T_{H,[0]_{0.5}} f,$$

we eventually get

$$x([k]_{0.5}) = \frac{5}{6}\cos(\pi\,\mathrm{lb}(2-[k]_{0.5})) - \frac{1}{10}\left(3\sin\frac{\pi\,\mathrm{lb}(2-[k]_{0.5})}{2} + 4\cos\frac{\pi\,\mathrm{lb}(2-[k]_{0.5})}{2}\right)$$
$$+\frac{1}{30}(49-32[k]_{0.5}) - \frac{1}{2}\mathrm{lb}(2-[k]_{0.5}), \quad [k]_{0.5} \in \mathbb{G}_{0.5},$$

where $\mathrm{lb}(t)$ denotes the binary logarithm $\log_2(t)$.

SOME GENERALIZATIONS

In [13], there was also considered a *forward difference* S_b *with the basis* $b \in \mathbb{C} \setminus \{0\}$

$$S_b\{x(k)\} := \{x(k+n) - bx(k)\},$$

which is a generalization of the operation (2). It was shown that to the derivative S_b there correspond the integrals

$$T_{b,k_0}\{x(k)\} := \{e(k)\}T_{k_0}\left\{\frac{x(k)}{e(k+n)}\right\}$$

and the limit conditions

$$s_{b,k_0}\{x(k)\} := \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\},$$

where $\{e(k)\} := \{b^{k/n}\} \in \operatorname{Ker} S_b$, while T_{k_0} and s_{k_0} are the operations (3) and (4), respectively.

Therefore, on the basis of the foregoing considerations we obtain the two following corollaries:

Corollary 3. *The operations*

$$S_{J,b}\{x(k_q)\} := \{x(q^n k_q) - bx(k_q)\},$$

$$T_{J,b,k_{0q}}\{x(k_q)\} := \{\hat{e}(k_q)\}T_{J,k_{0q}}\left\{\frac{x(k_q)}{\hat{e}((k+n)_q)}\right\},$$

$$s_{J,b,k_{0q}}\{x(k_q)\} := \{\hat{e}(k_q)\}s_{J,k_{0q}}\left\{\frac{x(k_q)}{\hat{e}(k_q)}\right\},$$

where $\hat{e}(k_q) := b^{\frac{1}{n}\mathrm{lq}(k_q)} = e(k)$*, form a discrete Jackson model of the Bittner operational calculus*

$$CO(C(\mathbb{T}_J, \mathbb{C}), C(\mathbb{T}_J, \mathbb{C}), S_{J,b}, T_{J,b,k_{0q}}, s_{J,b,k_{0q}}, \mathbb{T}_J).$$

Corollary 4. *The operations*

$$S_{H,b}\{x([k]_q)\} := \{x(q^n[k]_q + [n]_q) - bx([k]_q)\},$$

$$T_{H,b,[k_0]_q}\{x([k]_q)\} := \{\check{e}([k]_q)\}T_{H,[k_0]_q}\left\{\frac{x([k]_q)}{\check{e}([k+n]_q)}\right\},$$

$$s_{H,b,[k_0]_q}\{x([k]_q)\} := \{\check{e}([k]_q)\}s_{H,[k_0]_q}\left\{\frac{x([k]_q)}{\check{e}([k]_q)}\right\},$$

where $\check{e}([k]_q) := b^{\frac{1}{n}\mathrm{lq}(1-(1-q)[k]_q)} = e(k)$*, form a discrete Hahn model of the Bittner operational calculus*

$$CO(C(\mathbb{T}_H, \mathbb{C}), C(\mathbb{T}_H, \mathbb{C}), S_{H,b}, T_{H,b,[k_0]_q}, s_{H,b,[k_0]_q}, \mathbb{T}_H).$$

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RÓŻNICOWE MODELE JACKSONA I HAHNA JAKO DYSKRETNE REPREZENTACJE RACHUNKU OPERATORÓW BITTNERA

STRESZCZENIE

W artykule skonstruowano dyskretne modele nieklasycznego rachunku operatorów Bittnera związane z pojęciami pochodnej Jacksona i pochodnej Hahna znanymi z rachunku kwantowego. Do tego celu wykorzystano reprezentację rachunku operatorów Bittnera dla różnicy progresywnej.

Słowa kluczowe:
rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica Jacksona, różnica Hahna.