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EXISTENCE AND MULTIPLICITY RESULTS FOR NONLINEAR PROBLEMS INVOLVING THE $p(x)$-LAPLACE OPERATOR

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Abstract. In this paper we study the following nonlinear boundary-value problem

$$-\Delta_{p(x)}u = \lambda f(x,u) \quad \text{in } \Omega,$$
$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = \mu g(x,u) \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, λ, μ are two real numbers such that $\lambda^2 + \mu^2 \neq 0$, p is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Under appropriate assumptions on f and g, we obtain the existence and multiplicity of solutions using the variational method. The positive solution of the problem is also considered.

Keywords: critical points, variational method, $p(x)$-Laplacian, generalized Lebesgue-Sobolev spaces.

Mathematics Subject Classification: 35B38, 35D05, 35J20, 35J60, 35J66.

1. INTRODUCTION

This paper is devoted to finding existence and multiplicity results for the following nonlinear problem

$$\begin{aligned} -\Delta_{p(x)}u &= \lambda f(x,u) \quad \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u &= \mu g(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $\lambda, \mu \in \mathbb{R}$ such that $\lambda^2 + \mu^2 \neq 0$, p is a continuous function on $\overline{\Omega}$

with $p^- := \inf_{x\in\overline{\Omega}} p(x) > 1$ and $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a natural extension of the classical p-Laplace operator $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant. However, such generalizations are not trivial since the $p(x)$-Laplace operator possesses a more complicated structure than p-Laplace operator, for example it is inhomogeneous.

In recent years increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics and the calculus of variations, for information on modelling physical phenomena by equations involving the $p(x)$-growth condition we refer to [1, 8, 10, 17, 19, 20, 24, 29, 31, 32]. In the past decades a vast amount of literature that deal with the existence for problems of the type $-\Delta_{p(x)}u = f(x,u)$ with different boundary conditions (Dirichlet, Neumann, Robin, nonlinear, etc.) have appeared. See, for instance [9, 11, 14, 16, 27, 30] and references therein.

In [16], the authors have studied the problem (1.1) with $g(x,u) \equiv 0$. Using the variational approach based on the nonsmooth critical point theory for locally Lipschitz functions, they obtain the existence of at least two nontrivial solutions. This same problem has been studied in [26]. Under appropriate assumptions on f, and using variational methods, we have obtained important results on existence and multiplicity of solutions. In [3], the authors considered the problem (1.1) with $\lambda f(x,u) \equiv |u|^{p(x)-2}u$ and $\beta(x) \equiv 0$. Using Ricceri's variational principle, they establish the existence of at least three solutions of the problem. If $\beta(x) \equiv 0$ and $\mu g(x,u) \equiv 0$, the problem (1.1) becomes the nonlinear Neumann boundary value problem. It was studied in [27]. Using the three critical point theorem due to Ricceri, under the appropriate assumptions on f, the authors establish the existence of at least three solutions of this problem.

The purpose of this paper is to prove the existence and multiplicity results of solutions to the problem (1.1) under appropriate assumptions on f and g following ideas from [30]. These results extend some of the results in [25] for the p-Laplacian.

Next, we make the following assumptions on f and g:

(f_0) $f : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two constants $C_1 \geq 0$, $C_2 > 0$ such that

$$|f(x,s)| \leq C_1 + C_2|s|^{\alpha(x)-1} \quad \text{for all} \quad (x,s) \in \Omega\times\mathbb{R},$$

where $\alpha(x) \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$, for all $x \in \overline{\Omega}$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N; \end{cases}$$

(f_1) there exist $M_1 > 0$, $\theta_1 > p^+$ such that

$$0 < \theta_1 F(x,s) \leq sf(x,s) \quad \text{for all} \quad |s| \geq M_1, \quad x \in \Omega;$$

(f_2) $f(x,s) = o(|s|^{p^+-1})$, $s \to 0$ for $x \in \Omega$ uniformly;

(f_3) $f(x,-s) = -f(x,s)$ for all $x \in \Omega$, $s \in \mathbb{R}$;

(g_0) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two constants $C_1' \geq 0$, $C_2' > 0$ such that

$$|g(x,s)| \leq C_1' + C_2'|s|^{\gamma(x)-1} \quad \text{for all} \quad (x,s) \in \partial\Omega \times \mathbb{R},$$

where $\gamma(x) \in C_+(\partial\Omega)$ and $\gamma(x) < p^\partial(x)$, for all $x \in \partial\Omega$, where

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N; \end{cases}$$

(g_1) there exist $M_2 > 0$, $\theta_2 > p^+$ such that

$$0 < \theta_2 G(x,s) \leq sg(x,s) \quad \text{for all} \quad |s| \geq M_2, \quad x \in \partial\Omega;$$

(g_2) $g(x,s) = o(|s|^{p^+-1})$, $s \to 0$ for $x \in \partial\Omega$ uniformly;

(g_3) $g(x,-s) = -g(x,s)$ for all $x \in \partial\Omega$, $s \in \mathbb{R}$.

Let H be the energy functional corresponding to problem (1.1).

The main results of this paper are the following:

Theorem 1.1. *If* $(f_0), (g_0)$ *hold and* $\alpha^+, \gamma^+ < p^-$, *then problem* (1.1) *has a weak solution.*

Theorem 1.2. *If* $(f_0), (f_1), (f_2), (g_0), (g_1), (g_2)$ *hold and* $\alpha^-, \gamma^- > p^+, \lambda, \mu \geq 0$, *then problem* (1.1) *has a nontrivial weak solution.*

Theorem 1.3. *If* $(f_0), (f_1), (f_3), (g_0), (g_1), (g_3)$ *hold and* $\alpha^-, \gamma^- > p^+, \lambda, \mu \geq 0$, *then* H *has a sequence of critical points* $(\pm u_n)$ *such that* $H(\pm u_n) \to \infty$ *as* $n \to \infty$. *Meanwhile, problem* (1.1) *has infinite many pairs of weak solutions.*

Theorem 1.4. *Let* $\alpha(x) \in C_+(\overline{\Omega})$, $\gamma(x) \in C_+(\partial\Omega)$ *and*

$$\alpha(x) < p^*(x) \quad \textit{for all} \quad x \in \overline{\Omega}; \quad \gamma(x) < p^\partial(x) \quad \textit{for all} \quad x \in \partial\Omega.$$

If $f(x,u) = |u|^{\alpha(x)-2}u$, $g(x,u) = |u|^{\gamma(x)-2}u$, $\alpha^- > p^+$, *and* $\gamma^+ < p^-$, *then we have:*

(i) *for all* $\lambda > 0$ *and* $\mu \in \mathbb{R}$, *problem* (1.1) *has a sequence of weak solutions* $(\pm u_k)$ *such that* $H(\pm u_k) \to \infty$ *as* $k \to \infty$;

(ii) *for all* $\mu > 0$ *and* $\lambda \in \mathbb{R}$, *problem* (1.1) *has a sequence of weak solutions* $(\pm v_k)$ *such that* $H(\pm v_k) < 0$, *and* $H(\pm v_k) \to 0$ *as* $k \to \infty$.

Theorem 1.5. *If* $(f_0), (g_0)$ *hold and* $\alpha^+, \gamma^+ < p^-$, *then problem* (1.1) *has a nonnegative weak solution.*

Theorem 1.6. *If* $(f_0), (f_1), (f_2), (g_0), (g_1), (g_2)$ *hold and* $\alpha^-, \gamma^- > p^+, \lambda, \mu \geq 0$, *then problem* (1.1) *has a nonnegative nontrivial weak solution.*

This article is organized as follows. In Section 2, we introduce some necessary preliminary knowledge on variable exponent Lebesgue and Sobolev spaces. In Section 3, we will give the proof of Theorems 1.1–1.4. In Section 4, we will give the proof of Theorems 1.5–1.6.

2. PRELIMINARIES

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \Big\{p \in C(\overline{\Omega})\colon \inf_{x\in\overline{\Omega}} p(x) > 1\Big\}.$$

Denote by $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int\limits_{\Omega} |u|^{p(x)} dx < +\infty\Big\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \Big\{\tau > 0\colon \int\limits_{\Omega} \Big|\frac{u}{\tau}\Big|^{p(x)} dx \le 1\Big\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

with the norm

$$\|u\| = \inf \Big\{\tau > 0\colon \int\limits_{\Omega} \Big(\Big|\frac{\nabla u}{\tau}\Big|^{p(x)} + \Big|\frac{u}{\tau}\Big|^{p(x)}\Big) dx \le 1\Big\},$$

$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$

We refer the reader to [9, 12, 13] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 2.1 ([13])**.** *Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable, reflexive and uniformly convex Banach spaces.*

Lemma 2.2 ([13])**.** *Hölder inequality holds, namely*

$$\int\limits_{\Omega} |uv| dx \le 2|u|_{p(x)}|v|_{p'(x)} \quad \text{for all} \quad u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$

Now, we introduce a norm, which will be used later. For $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_\beta = \inf \left\{ \lambda > 0 : \int\limits_{\Omega} \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx + \int\limits_{\partial\Omega} \beta(x) \left|\frac{u(x)}{\lambda}\right|^{p(x)} d\sigma_x \le 1 \right\}.$$

Then, by Theorem 2.1 in [9], $\|u\|_\beta$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|u\|$.

Lemma 2.3 (see [13,14,30]).

(1) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*
(2) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^\partial(x)$ for any $x \in \partial\Omega$, then the trace imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous.*

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping defined by

$$I_\beta(u) = \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}\, d\sigma_x \quad \text{for all} \quad u \in W^{1,p(x)}(\Omega).$$

Lemma 2.4 ([9]).

(1) $\|u\|_\beta \geq 1 \Rightarrow \|u\|_\beta^{p^-} \leq I_\beta(u) \leq \|u\|_\beta^{p^+}$,
(2) $\|u\|_\beta \leq 1 \Rightarrow \|u\|_\beta^{p^+} \leq I_\beta(u) \leq \|u\|_\beta^{p^-}$,
(3) $\|u\|_\beta \to 0$ *if and only if* $I_\beta(u) \to 0$ *(as $k \to \infty$)*,
(4) $|u(x)|_{p(x)} \to \infty$ *if and only if* $I_\beta(u) \to \infty$ *(as $k \to \infty$)*.

Remark 2.5. From (1) and (2) of the previous lemma, one can easily deduce that

$$\|u\|_\beta < (=;>)1 \Leftrightarrow I_\beta(u) < (=;>)1. \tag{2.1}$$

Theorem 2.6. *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying*

$$|f(x,s)| \leq C(1 + |s|^{\alpha(x)-1}) \quad \textit{for all} \quad (x,s) \in \overline{\Omega} \times \mathbb{R},$$

where $C \geq 0$ is a constant, $\alpha(x) \in C_+(\overline{\Omega})$ such that for all $x \in \overline{\Omega}$, $\alpha(x) < p^*(x)$. Set $X = W^{1,p(x)}(\Omega)$, $F(x,u) = \int_0^u f(x,t)dt$, and $\psi(u) = -\int_\Omega F(x,u(x))\, dx$. Then $\psi(u) \in C^1(X,\mathbb{R})$ and $D\psi(u,\varphi) = \langle \psi'(u), \varphi\rangle = -\int_\Omega f(x,u(x))\varphi\, dx$. Moreover, the operator $\psi' : X \to X^*$ is compact.

Proof. It is easily adapted from that of [27, Theorem 2.1]. □

Theorem 2.7. *If $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and*

$$|g(x,s)| \leq C"(1 + |s|^{\alpha(x)-1}) \quad \textit{for all} \quad (x,s) \in \partial\Omega \times \mathbb{R},$$

where $C"$ is a positive constant and $\alpha(x) \in C_+(\partial\Omega)$ such that for all $x \in \partial\Omega$, $\alpha(x) < p^\partial(x)$. Set $X = W^{1,p(x)}(\Omega)$, $G(x,u) = \int_0^u g(x,t)dt$, $\psi(u) = -\int_{\partial\Omega} G(x,u(x))d\sigma_x$. Then $\psi(u) \in C^1(X,\mathbb{R})$ and $D\psi(u,\varphi) = \langle \psi'(u), \varphi\rangle = -\int_{\partial\Omega} g(x,u(x))\varphi d\sigma_x$. Moreover, the operator $\psi' : X \to X^*$ is compact.

Proof. It is easily adapted from that of [2, Theorem 2.9]. □

Let $X = W^{1,p(x)}(\Omega)$ and define

$$\phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\, d\sigma_x,$$

$$\psi(u) = -\int_\Omega F(x,u)\, dx, \quad J(u) = -\int_{\partial\Omega} G(x,u)\, d\sigma_x,$$

where $F(x,t) = \int\limits_0^t f(x,s)ds$, and $G(x,t) = \int\limits_0^t g(x,s)ds$. It is easy to see that $\phi \in C^1(X,\mathbb{R})$ and

$$(\phi'(u),v) = \int\limits_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int\limits_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma_x, \quad v \in X.$$

Moreover, we have the following proposition.

Proposition 2.8 ([16, Proposition 2.2]).

(1) $\phi' : X \to X^*$ *is a continuous, bounded and strictly monotone operator.*
(2) $\phi' : X \to X^*$ *is a mapping of type* $(S)^+$, *that is, if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty}(\phi'(u_n) - \phi'(u), u_n - u) \leq 0$, *then* $u_n \to u$ *in* X.
(3) $\phi' : X \to X^*$ *is a homeomorphism.*

Under the conditions (f_0) and (g_0), and from Theorem 2.6 and Theorem 2.7, ψ and J are continuously Gâteaux differentiable functionals whose Gâteaux derivative is compact, and we have

$$\langle \psi'(u), v\rangle = -\int\limits_\Omega f(x,u)v\,dx, \quad \langle J'(u), v\rangle = -\int\limits_{\partial\Omega} g(x,u)v\,d\sigma_x.$$

The energy functional corresponding to problem (1.1) is defined on X as

$$H(u) = \phi(u) + \lambda\psi(u) + \mu J(u).$$

The functional H is of class $C^1(X,\mathbb{R})$, and the weak solution of problem (1.1) corresponds to the critical point of the functional H.

Definition 2.9. *We say that* $u \in W^{1,p(x)}(\Omega)$ *is a weak solution of the problem* (1.1) *if for all* $v \in W^{1,p(x)}(\Omega)$

$$\int\limits_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx + \int\limits_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma_x = \lambda\int\limits_\Omega f(x,u)v\,dx + \mu\int\limits_{\partial\Omega} g(x,u)v\,d\sigma_x,$$

where $d\sigma_x$ *is the measure on the boundary* $\partial\Omega$.

Remark 2.10. In the following sections, the symbols C, D, M denote the generic nonnegative of positive constants, which may not be the same at each occurrence.

3. EXISTENCE AND MULTIPLICITY OF SOLUTIONS

In this section, we shall prove Theorems 1.1–1.4. By using the variational principle, we prove the existence and multiplicity of results for problem (1.1).

Proof of Theorem 1.1. From (f_0) and (g_0), there exist $C > 0$ such that

$$|F(x,t)| \le C(1 + |t|^{\alpha(x)}), \quad (x,t) \in \Omega \times \mathbb{R},$$
$$|G(x,t)| \le C(1 + |t|^{\gamma(x)}), \quad (x,t) \in \partial\Omega \times \mathbb{R}.$$

Obviously, H is weakly lower semicontinuous. It suffices to show that H is coercive. Let $u \in X$ be such that $\|u\|_\beta > 1$. Then

$$\begin{aligned} H(u) = &\int\limits_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \\ &+ \int\limits_{\partial\Omega} \frac{1}{p(x)} \beta(x)|u|^{p(x)} d\sigma_x - \int\limits_\Omega \lambda F(x,u) dx - \int\limits_{\partial\Omega} \mu G(x,u) d\sigma_x \ge \\ &\ge \frac{1}{p^+} \|u\|_\beta^{p^-} - |\lambda| \int\limits_\Omega C(1 + |u|^{\alpha(x)}) dx - |\mu| \int\limits_{\partial\Omega} C(1 + |u|^{\gamma(x)}) d\sigma_x \ge \\ &\ge \frac{1}{p^+} \|u\|_\beta^{p^-} - |\lambda| C \|u\|_\beta^{\alpha^+} - |\mu| C \|u\|_\beta^{\gamma^+} - M. \end{aligned}$$

So $H(u) \to \infty$ as $\|u\|_\beta \to \infty$, since $\alpha^+, \gamma^+ < p^-$. Then H is coercive and H has a minimum point u in X which is a weak solution of problem (1.1). □

Corollary 3.1. *Under the assumptions in Theorem 1.1, if $\lambda, \mu \ne 0$, and there exist two positive constants $d_1, d_2 < p^-$ such that:*

$$\liminf_{t \to 0} \frac{\operatorname{sgn}(\lambda) F(x,t)}{|t|^{d_1}} > 0 \quad \textit{for } x \in \Omega \textit{ uniformly}, \tag{3.1}$$

$$\liminf_{t \to 0} \frac{\operatorname{sgn}(\mu) G(x,t)}{|t|^{d_2}} > 0 \quad \textit{for } x \in \partial\Omega \textit{ uniformly}, \tag{3.2}$$

then the problem (1.1) *has a nontrivial weak solution.*

Proof. From Theorem 1.1 we know that H has a global minimum point u. It suffices to show that u is nontrivial. From (3.1) and (3.2), for $0 < t < 1$ small enough, there exists a positive constant C such that

$$sgn(\lambda) F(x,t) \ge C|t|^{d_1}, \quad \operatorname{sgn}(\mu) G(x,t) \ge C|t|^{d_2}.$$

Choose $u_0 \equiv M > 0$, then $u_0 \in X$. Then we have

$$\begin{aligned} H(tu_0) \le &\frac{t^{p^-}}{p^-} \int\limits_{\partial\Omega} \beta(x)|M|^{p(x)} d\sigma_x - |\lambda| \int\limits_\Omega (sgn(\lambda)) F(x,tM) dx - \\ &- |\mu| \int\limits_{\partial\Omega} (sgn(\mu)) G(x,tM) d\sigma_x \le \\ &\le \frac{t^{p^-}}{p^-} \int\limits_{\partial\Omega} \beta(x)|M|^{p(x)} d\sigma_x - |\lambda| \int\limits_\Omega C|tM|^{d_1} dx - |\mu| \int\limits_{\partial\Omega} C|tM|^{d_2} d\sigma_x \le \\ &\le D_1 t^{p^-} - |\lambda| D_2 t^{d_1} - |\mu| D_3 t^{d_2}. \end{aligned}$$

Since $d_1, d_2 < p^-$, there exists $0 < t_0 < 1$ small enough such that $H(t_0 u_0) < 0$. So the global minimum point u of H is nontrivial. □

Remark 3.2. The conclusion of Corollary 3.1 remains valid if we suppose one of the following conditions:

(i) $\mu = 0, \lambda \neq 0$ and there exist a positive constant $d_1 < p^-$ such that (3.1) holds,
(ii) $\lambda = 0, \mu \neq 0$ and there exist a positive constant $d_2 < p^-$ such that (3.2) holds.

Remark 3.3. If $f(x,u) = sgn(\lambda)|u|^{\alpha(x)-2}u$ and $g(x,u) = sgn(\mu)|u|^{\gamma(x)-2}u$ with $\alpha^+, \gamma^+ < p^-$, then the conditions in Corollary 3.1 can be fulfilled.

To prove Theorem 1.2, we need the following lemma.

Lemma 3.4. *If* $(f_0), (f_1), (g_0), (g_1)$ *hold and* $\lambda, \mu \geq 0$*, then* H *satisfies the* (PS) *condition.*

Proof. Suppose that $(u_n) \subset X$ is a (PS) sequence, i.e.

$$\sup |H(u_n)| \leq M, \quad H'(u_n) \to 0 \text{ as } n \to \infty.$$

Let us show that (u_n) is bounded so as to verify it is precompact in X. By Lemma 2.3, and Theorems 2.6, 2.7, we know that ψ and J are booth weakly continuous and their derivative operators are compact. By Proposition 2.8, we deduce that $H' = \phi' + \lambda\psi' + \mu J'$ is also of type (S^+). For n large enough, we have

$$\begin{aligned}
M + 1 \geq\ & H(u_n) - \frac{1}{\theta}\langle H'(u_n), u_n\rangle + \frac{1}{\theta}\langle H'(u_n), u_n\rangle = \\
=\ & \int\limits_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx + \int\limits_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_n|^{p(x)} d\sigma_x - \\
& - \lambda \int\limits_{\Omega} F(x, u_n) dx - \mu \int\limits_{\partial\Omega} G(x, u_n) d\sigma_x - \\
& - \frac{1}{\theta}\Bigg[\int\limits_{\Omega} |\nabla u_n|^{p(x)} dx + \int\limits_{\partial\Omega} \beta(x)|u_n|^{p(x)} d\sigma_x - \\
& \qquad - \lambda \int\limits_{\Omega} f(x, u_n) u_n dx - \mu \int\limits_{\partial\Omega} g(x, u_n) u_n d\sigma_x \Bigg] + \\
& + \frac{1}{\theta}\langle H'(u_n), u_n\rangle \geq \\
\geq\ & \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|u_n\|_\beta^{p^-} - \frac{1}{\theta}\|H'(u_n)\|_{X^*}\|u_n\|_\beta - C \geq \\
\geq\ & \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|u_n\|_\beta^{p^-} - \frac{1}{\theta}\|u_n\|_\beta - C,
\end{aligned}$$

where $\theta = \min\{\theta_1, \theta_2\}$. From the inequality above, we know that (u_n) is bounded in X since $\theta > p^+$. This completes the proof. □

Proof of Theorem 1.2. We will use the mountain pass theorem (see [4, 28]). By the previous lemma, we know that H satisfies the (PS) condition. So it suffices to verify the geometric conditions in the mountain pass theorem. We have the following compact embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{p^+}(\Omega), \quad W^{1,p(x)}(\Omega) \hookrightarrow L^{p^+}(\partial\Omega),$$

since

$$p^+ < \alpha^- \leq \alpha(x) < p^*(x) \quad \text{for all} \quad x \in \overline{\Omega}; \quad p^+ < \gamma^- \leq \gamma(x) < p^\partial(x) \quad \text{for all} \quad x \in \partial\Omega.$$

So there exists a constant $C > 0$ such that

$$|u|_{L^{p^+}(\Omega)} \leq C\|u\|_\beta, \quad |u|_{L^{p^+}(\partial\Omega)} \leq C\|u\|_\beta \quad \text{for all} \quad u \in X.$$

Conditions $(f_0), (f_2)$ and $(g_0), (g_2)$ assure that there exists an arbitrary constant $0 < \varepsilon < 1$ and two positive constants (both denoted by $C(\varepsilon)$) such that

$$\begin{aligned} |F(x,t)| &\leq \varepsilon|t|^{p^+} + C(\varepsilon)|t|^{\alpha(x)} \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}, \\ |G(x,t)| &\leq \varepsilon|t|^{p^+} + C(\varepsilon)|t|^{\gamma(x)} \quad \text{for all} \quad (x,t) \in \partial\Omega \times \mathbb{R}. \end{aligned}$$

So for $\|u\|_\beta$ small enough ($\|u\|_\beta < 1$). We have

$$\begin{aligned} H(u) &\geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \lambda \int_\Omega F(x,u)dx - \mu \int_{\partial\Omega} G(x,u)d\sigma_x \geq \\ &\geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \lambda \int_\Omega \left(\varepsilon|u|^{p^+} + C(\varepsilon)|u|^{\alpha(x)}\right) dx - \\ &\quad - \mu \int_{\partial\Omega} \left(\varepsilon|u|^{p^+} + C(\varepsilon)|u|^{\gamma(x)}\right) d\sigma_x \geq \\ &\geq \frac{1}{p^+}\|u\|_\beta^{p^+} - (\lambda\varepsilon C + \mu\varepsilon C)\|u\|_\beta^{p^+} - \lambda C(\varepsilon)C\|u\|_\beta^{\alpha^-} - \mu C(\varepsilon)C\|u\|_\beta^{\gamma^-}. \end{aligned}$$

Choose $\varepsilon > 0$ small enough such that $0 < \lambda\varepsilon C + \mu\varepsilon C < \frac{1}{2p^+}$. Then we obtain

$$\begin{aligned} H(u) &\geq \frac{1}{2p^+}\|u\|_\beta^{p^+} - C(\lambda,\mu,\varepsilon)C(\|u\|_\beta^{\alpha^-} + \|u\|_\beta^{\gamma^-}) \geq \\ &\geq \|u\|_\beta^{p^+}\left(\frac{1}{2p^+} - C(\lambda,\mu,\varepsilon)C(\|u\|_\beta^{\alpha^- - p^+} + \|u\|_\beta^{\gamma^- - p^+})\right). \end{aligned}$$

Since $p^+ < \alpha^-, \gamma^-$, the function

$$t \mapsto \frac{1}{2p^+} - C(\lambda,\mu,\varepsilon)C(t^{\alpha^- - p^+} + t^{\gamma^- - p^+})$$

is strictly positive in a neighborhood of zero. It follows that there exist $r > 0$ and $\delta > 0$ such that

$$H(u) \geq \delta \quad \text{for all} \quad u \in X : \|u\|_\beta = r.$$

Now, to apply the mountain pass theorem, we must prove that

$$H(tu) \to -\infty \quad \text{as} \quad t \to +\infty,$$

for a certain $u \in X$. From conditions (f_1) and (g_1) we have for suitable positive constants C, D

$$F(x,s) \geq C|s|^{\theta_1} - D \quad \text{for all} \quad (x,s) \in \Omega \times \mathbb{R},$$
$$G(x,s) \geq C|s|^{\theta_2} - D \quad \text{for all} \quad (x,s) \in \partial\Omega \times \mathbb{R}.$$

Let $u \in X$ and $t > 1$. We have

$$\begin{aligned}
H(tu) = &\int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{t^{p(x)}\beta(x)}{p(x)} |u|^{p(x)} d\sigma_x - \\
&- \lambda \int_\Omega F(x,tu)dx - \mu \int_{\partial\Omega} G(x,tu) d\sigma_x \leq \\
&\leq \frac{t^{p^+}}{p^-} \Big(\int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)} d\sigma_x \Big) - \\
&- \lambda \int_\Omega (C|tu|^{\theta_1} - D) dx - \mu \int_{\partial\Omega} (C|tu|^{\theta_2} - D) d\sigma_x \leq \\
&\leq \frac{t^{p^+}}{p^-} \Big(\int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)} d\sigma_x \Big) - \\
&- t^{\theta_1} \lambda C \int_\Omega |u|^{\theta_1} dx - t^{\theta_2} \mu C \int_{\partial\Omega} |u|^{\theta_2} d\sigma_x + M.
\end{aligned}$$

The fact that $\theta_1, \theta_2 > p^+$ implies

$$H(tu) \to -\infty \quad \text{as} \quad t \to +\infty.$$

It follows that there exists $e \in X$ such that $\|e\|_\beta > r$ and $H(e) < 0$. According to the mountain pass theorem, H admits a critical value $\tau \geq \delta$ which is characterized by

$$\tau = \inf_{h \in \Gamma} \sup_{t \in [0,1]} H(h(t)),$$

where

$$\Gamma = \{h \in C([0,1], X) : h(0) = 0 \text{ and } h(1) = e\}.$$

The proof is complete. □

Proof of Theorem 1.3. Since X is a separable and reflexive Banach space [7,12], there exist $\{e_n\}_{n=1}^\infty \subset X$ and $\{f_n\}_{n=1}^\infty \subset X^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

$$X = \overline{\operatorname{span}}\{e_n : n = 1, 2, \dots\}, \quad X^* = \overline{\operatorname{span}}^{W^*}\{f_n : n = 1, 2, \dots\}.$$

For $n = 1, 2, \dots$ denote by

$$X_n = \operatorname{span}\{e_n\}, \quad Y_n = \oplus_{j=1}^{n} X_j, \quad Z_n = \overline{\oplus_{j=n}^{\infty} X_j}.$$

Then we have the following lemma.

Lemma 3.5 ([15, Proposition 3.5])**.** *If $\alpha(x) \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, and $\gamma(x) \in C_+(\partial\Omega)$, $\gamma(x) < p^\partial(x)$ for all $x \in \partial\Omega$, denote*

$$\alpha_k = \sup\left\{|u|_{L^{\alpha(x)}(\Omega)} : \|u\|_\beta = 1,\ u \in Z_k\right\},$$
$$\gamma_k = \sup\left\{|u|_{L^{\gamma(x)}(\partial\Omega)} : \|u\|_\beta = 1,\ u \in Z_k\right\}.$$

Then $\lim_{k\to\infty} \alpha_k = 0$ and $\lim_{k\to\infty} \gamma_k = 0$.

Now, we return to the proof of Theorem 1.3. To do that, we will use the Fountain theorem (see [28]). Obviously, H is an even functional and satisfies the (PS) condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that

(A1) $b_k := \inf\{H(u) : u \in Z_k, \|u\|_\beta = r_k\} \to +\infty$ as $k \to +\infty$,
(A2) $a_k := \max\{H(u) : u \in Y_k, \|u\|_\beta = \rho_k\} \le 0$ as $k \to +\infty$.

(A1) For $u \in Z_k$ such that $\|u\|_\beta = r_k > 1$, by conditions (f_0) and (g_0), we have

$$\begin{aligned}
H(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}d\sigma_x - \lambda\int_\Omega F(x,u)dx - \mu\int_{\partial\Omega} G(x,u)d\sigma_x \ge \\
&\ge \frac{1}{p^+}\|u\|_\beta^{p^-} - \lambda\int_\Omega C(1+|u|^{\alpha(x)})dx - \mu\int_{\partial\Omega} C(1+|u|^{\gamma(x)})d\sigma_x \ge \\
&\ge \frac{1}{p^+}\|u\|_\beta^{p^-} - \lambda C\max\left\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\right\} - \\
&\quad - \mu C\max\left\{|u|^{\gamma^+}_{L^{\gamma(x)}(\partial\Omega)}, |u|^{\gamma^-}_{L^{\gamma(x)}(\partial\Omega)}\right\} - M \ge \\
&\ge \frac{1}{p^+}\|u\|_\beta^{p^-} - \\
&\quad - C(\lambda,\mu)\max\left\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\gamma^+}_{L^{\gamma(x)}(\partial\Omega)}, |u|^{\gamma^-}_{L^{\gamma(x)}(\partial\Omega)}\right\} - M.
\end{aligned}$$

If $\max\left\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\gamma^+}_{L^{\gamma(x)}(\partial\Omega)}, |u|^{\gamma^-}_{L^{\gamma(x)}(\partial\Omega)}\right\} = |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}$, then we have

$$H(u) \ge \frac{1}{p^+}\|u\|_\beta^{p^-} - C(\lambda,\mu)\alpha_k^{\alpha^+}\|u\|_\beta^{\alpha^+} - M.$$

If we choose $r_k = (\alpha^+ C(\lambda,\mu)\alpha_k^{\alpha^+})^{\frac{1}{p^- - \alpha^+}}$, we obtain

$$H(u) \ge r_k^{p^-}\left(\frac{1}{p^+} - \frac{1}{\alpha^+}\right) - M.$$

Since $\alpha_k \to 0$, $r_k \to +\infty$ and $p^+ < \alpha^- \leq \alpha^+$, we have $H(u) \to +\infty$ as $k \to +\infty$. In the other three cases, we can deduce in the same way that

$$H(u) \to \infty, \quad \text{since } \alpha_k \to 0,\ \gamma_k \to 0 \text{ as } k \to +\infty.$$

So (A1) holds.

(A2) Conditions (f_1) and g_1 implies that there exist positive constants C, D such that

$$F(x,s) \geq C|s|^{\theta_1} - D \quad \text{for all} \quad (x,s) \in \Omega \times \mathbb{R},$$

$$G(x,s) \geq C|s|^{\theta_2} - D \quad \text{for all} \quad (x,s) \in \partial\Omega \times \mathbb{R}.$$

Let $u \in Y_k$ be such that$\|u\|_\beta = \rho_k > r_k > 1$. Then

$$\begin{aligned} H(u) &\leq \frac{1}{p^-}\|u\|_\beta^{p^+} - \lambda \int_\Omega (C|u|^{\theta_1} - D)\, dx - \mu \int_{\partial\Omega} (C|u|^{\theta_2} - D)\, d\sigma_x \leq \\ &\leq \frac{1}{p^-}\|u\|_\beta^{p^+} - \lambda C \int_\Omega |u|^{\theta_1}\, dx - \mu C \int_{\partial\Omega} |u|^{\theta_1}\, d\sigma_x + M. \end{aligned}$$

Since the space Y_k has finite dimension, then all norms are equivalents and we obtain

$$H(u) \leq \frac{1}{p^-}\|u\|_\beta^{p^+} - \lambda C\|u\|_\beta^{\theta_1} - \mu C\|u\|_\beta^{\theta_2} + M.$$

Finally,

$$H(u) \to -\infty \quad \text{as} \quad \|u\|_\beta \to +\infty, u \in Y_k$$

since $\theta_1, \theta_2 > p^+$. So the assertion (A2) is then satisfied and the proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. (i) As in the proof of Theorem 1.3, we will use in a similar way, the Fountain theorem. So, it suffices to verify the (PS) condition. Assume

$$(u_n) \subset X, \quad \sup H(u_n) \leq M, \quad H'(u_n) \to 0 \quad \text{as } n \to +\infty.$$

For n large enough, we have

$$M+1 \geq H(u_n) - \frac{1}{\alpha^-}\langle H'(u_n), u_n\rangle + \frac{1}{\alpha^-}\langle H'(u_n), u_n\rangle =$$
$$= \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}d\sigma_x -$$
$$- \lambda \int_\Omega \frac{1}{\alpha(x)}|u_n|^{\alpha(x)}dx - \mu \int_{\partial\Omega} \frac{1}{\gamma(x)}|u_n|^{\gamma(x)}d\sigma_x -$$
$$- \frac{1}{\alpha^-}\Bigg[\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)}d\sigma_x -$$
$$- \lambda \int_\Omega |u_n|^{\alpha(x)}dx - \mu \int_{\partial\Omega} |u_n|^{\gamma(x)}d\sigma_x\Bigg] +$$
$$+ \frac{1}{\alpha^-}\langle H'(u_n), u_n\rangle \geq$$
$$\geq \left(\frac{1}{p^+} - \frac{1}{\alpha^-}\right)\|u_n\|_\beta^{p^-} - C\|u_n\|_\beta^{\gamma^+} - \frac{1}{\alpha^-}\|H'(u_n)\|_{X^*}\|u_n\|_\beta \geq$$
$$\geq \left(\frac{1}{p^+} - \frac{1}{\alpha^-}\right)\|u_n\|_\beta^{p^-} - C\|u_n\|_\beta^{\gamma^+} - \frac{1}{\alpha^-}\|u_n\|_\beta.$$

Since $\alpha^- > p^+, \gamma^+ < p^-$, we know that (u_n) is bounded in X. This completes the proof.

(ii) We will use the dual of the Fountain theorem. We need to prove that H satisfies the $(PS)_c^*$ condition (see [28]) and there exist $\rho_k > r_k > 0$ such that for k large enough we have

(B1) $a_k := \max\{H(u) : u \in Y_k, \|u\|_\beta = r_k\} < 0$,
(B2) $b_k := \inf\{H(u) : u \in Z_k, \|u\|_\beta = \rho_k\} \geq 0$,
(B3) $d_k := \max\{H(u) : u \in Y_k, \|u\|_\beta \leq \rho_k\} \to 0$ as $k \to +\infty$.

Let us show that (B1) holds. We assume $\|u\|_\beta < 1$ for convenience. For $u \in Y_k$, we have

$$H(u) = \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}d\sigma_x -$$
$$- \lambda \int_\Omega \frac{1}{\alpha(x)}|u_n|^{\alpha(x)}dx - \mu \int_{\partial\Omega} \frac{1}{\gamma(x)}|u_n|^{\gamma(x)}d\sigma_x \leq$$
$$\leq \frac{1}{p^-}\|u\|_\beta^{p^-} + \frac{|\lambda|}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx - \frac{\mu}{\gamma^+}\int_{\partial\Omega} |u|^{\gamma(x)}d\sigma_x.$$

If we choose $r_k > 0$ small enough, we get $a_k := \max\{H(u) : u \in Y_k, \|u\|_\beta = r_k\} < 0$, since $\dim Y_k < \infty$ and $p^- > \gamma^+, \alpha^- > p^+$. So (B1) holds.

(B2) Let $u \in Z_k$, then

$$H(u) \geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \frac{|\lambda|}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx - \frac{\mu}{\gamma^-}\int_{\partial\Omega} |u|^{\gamma(x)}d\sigma_x \geq$$

$$\geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \frac{C|\lambda|}{\alpha^-}\|u\|_\beta^{\alpha^-} - \frac{\mu}{\gamma^-}\int_{\partial\Omega} |u|^{\gamma(x)}d\sigma_x \geq$$

$$\geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \frac{C|\lambda|}{\alpha^-}\|u\|_\beta^{\alpha^-} - \frac{\mu}{\gamma^-}\max\{|u|^{\gamma^+}_{L^{\gamma(x)}(\partial\Omega)}, |u|^{\gamma^-}_{L^{\gamma(x)}(\partial\Omega)}\}.$$

There exists $\rho_0 > 0$ small enough such that $\frac{C|\lambda|}{\alpha^-}\|u\|_\beta^{\alpha^-} \leq \frac{1}{2p^+}\|u\|_\beta^{p^+}$ as $0 < \rho = \|u\|_\beta \leq \rho_0$, since $\alpha^- > p^+$. Then we have

$$H(u) \geq \frac{1}{2p^+}\|u\|_\beta^{p^+} - \frac{\mu}{\gamma^-}\max\{|u|^{\gamma^+}_{L^{\gamma(x)}(\partial\Omega)}, |u|^{\gamma^-}_{L^{\gamma(x)}(\partial\Omega)}\}.$$

If $\max\{|u|^{\gamma^+}_{L^{\gamma(x)}(\partial\Omega)}, |u|^{\gamma^-}_{L^{\gamma(x)}(\partial\Omega)}\} = |u|^{\gamma^+}_{L^{\gamma(x)}(\partial\Omega)}$, then

$$H(u) \geq \frac{1}{2p^+}\|u\|_\beta^{p^+} - \frac{\mu}{\gamma^-}\gamma_k^{\gamma^+}\|u\|_\beta^{\gamma^+}.$$

Choose $\rho_k = \left(\frac{2p^+\mu\gamma_k}{\gamma^-}^{\gamma^+}\right)^{\frac{1}{p^+-\gamma^+}}$, then $H(u) \geq 0$. Since $p^- > \gamma^+$, $\gamma_k \to 0$, we get $\rho_k \to 0$ as $k \to \infty$. The case $\max\{|u|^{\gamma^+}_{L^{\gamma(x)}(\partial\Omega)}, |u|^{\gamma^-}_{L^{\gamma(x)}(\partial\Omega)}\} = |u|^{\gamma^-}_{L^{\gamma(x)}(\partial\Omega)}$ is similar, so (B2) holds.

(B3) From the proof above and the fact that $Y_k \cap Z_k \neq \emptyset$, we know that for $u \in Z_k$, $\|u_k\|_\beta \leq \rho_k$ small enough

$$H(u) \geq -\frac{\mu}{\gamma^-}\gamma_k^{\gamma^+}\|u\|_\beta^{\gamma^+} \quad \text{or} \quad -\frac{\mu}{\gamma^-}\gamma_k^{\gamma^-}\|u\|_\beta^{\gamma^-}.$$

Since $\gamma_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$, (B3) holds and obviously we can choose $\rho_k > r_k > 0$.

Now, to verify the $(PS)_c^*$ condition, we consider a sequence $(u_{n_j}) \subset X$ such that

$$n_j \to \infty, \quad u_{n_j} \in Y_{n_j}, \quad H(u_{n_j}) \to C, \quad (H|_{Y_{n_j}})'(u_{n_j}) \to 0.$$

Assume $\|u\|_\beta > 1$, then for n large enough and $\lambda \geq 0$ we have

$$C + 1 \geq H(u_{n_j}) - \frac{1}{\alpha^-}\langle H'(u_{n_j}), u_{n_j}\rangle + \frac{1}{\alpha^-}\langle H'(u_{n_j}), u_{n_j}\rangle \geq$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{\alpha^-}\right)\|u_{n_j}\|_\beta^{p^-} - D\|u_{n_j}\|_\beta^{\gamma^+} - \frac{1}{\alpha^-}\|u_{n_j}\|_\beta.$$

Since $\alpha^- > p^+$ and $p^- > \gamma^+$, we deduce that (u_{n_j}) is bounded in X.
If $\lambda < 0$, then for n large enough, we have

$$C + 1 \geq H(u_{n_j}) - \frac{1}{\alpha^+}\langle H'(u_{n_j}), u_{n_j}\rangle + \frac{1}{\alpha^+}\langle H'(u_{n_j}), u_{n_j}\rangle.$$

Going if necessary to a subsequence, we can assume that $u_{n_j} \rightharpoonup u$ weakly in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned}\lim_{n_j\to\infty} H'(u_{n_j})(u_{n_j}-u) &= \lim_{n_j\to\infty} H'(u_{n_j})(u_{n_j}-v_{n_j}) + \lim_{n_j\to\infty} H'(u_{n_j})(v_{n_j}-u) = \\ &= \lim_{n_j\to\infty} (H|_{Y_{n_j}})'(u_{n_j})(u_{n_j}-v_{n_j}) = 0.\end{aligned}$$

Then we can conclude that $u_{n_j} \to u$ since H' is of type (S^+). Moreover, we have $H'(u_{n_j}) \to H'(u)$. Now, it only remains to prove that $H'(u) = 0$. For an arbitrary $w_k \in Y_k$, we have for $n_j \geq k$

$$\begin{aligned}H'(u)w_k &= (H'(u) - H'(u_{n_j}))w_k + H'(u_{n_j})w_k = \\ &= (H'(u) - H'(u_{n_j}))w_k + (H|_{Y_{n_j}})'(u_{n_j})w_k.\end{aligned}$$

Going to the limit on the right side of the above equation, one get

$$H'(u)w_k = 0 \quad \text{for all} \quad w_k \in Y_k,$$

so $H'(u) = 0$, this shows that the functional H satisfies the $(PS)^*_c$ condition for every $c \in \mathbb{R}$. The proof of Theorem 1.4 is complete. □

4. EXISTENCE OF NONNEGATIVE SOLUTION AND POSITIVE SOLUTION

In this section, we will assume that f and g satisfy the following condition:

$$f(x,0) = 0 \quad \text{for all} \quad x \in \Omega, \quad \text{and} \quad g(x,0) = 0 \quad \text{for all} \quad x \in \partial\Omega.$$

Define

$$f_+(x,t) = \begin{cases} f(x,t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

$$g_+(x,t) = \begin{cases} g(x,t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Let $F_+(x,t) = \int_0^t f_+(x,s)ds$ and $G_+(x,t) = \int_0^t g_+(x,s)ds$. Consider the following problem:

$$\begin{aligned} -\Delta_{p(x)} u &= \lambda f_+(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u &= \mu g_+(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

The energy functional associated with problem (4.1) is

$$\begin{aligned} H_+(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{1}{p(x)}\beta(x)|u|^{p(x)}d\sigma_x - \\ - \int_\Omega \lambda F_+(x,u)dx - \int_{\partial\Omega} \mu G_+(x,u)d\sigma_x. \end{aligned}$$

Proof of Theorem 1.5. By Theorem 1.1, we know that problem (4.1) has a weak solution u. Multiplying the equation in (4.1) by $u^- := \max\{-u, 0\}$ and integrating over Ω, in view of the boundary condition, we get

$$\int_{\Omega} |\nabla u^-|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u^-|^{p(x)} d\sigma_x = 0,$$

which implies that $\|u^-\|_\beta = 0$ and then $u^- = 0$ in X. So we conclude that u is a nonnegative solution of the problem (4.1). □

By the same arguments, and using Theorem 1.2, we prove Theorem 1.6.

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