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Free Vibrations of Kirchhoff Plates Considering Plate Variable Thickness and Interaction with Water Including Water Boundaries

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Abstract The natural vibrations of thin (Kirchhoff-Love) plates with constant and variable thickness and interaction with water are considered in the paper. The influence of the water free surface on natural frequencies of the coupled water-plate system is analysed too. The Finite Element Method (FEM) and the Finite Difference Method (FDM) are used to describe structural deformation and the Boundary Element Method (BEM) is applied to describe the dynamic interaction of water on a plate surface. The plate inertia forces are expressed by diagonal or consistent mass matrix. The water inertia forces are described by fully-populated mass matrix which is obtained directly from the theory of double layer potential.

Keywords: thin plates, natural vibrations, FEM, FDM, BEM, adjoined mass, fluid-structure interaction

1. Introduction

The dynamic analysis of structures is a very wide spectrum of scientific research and the available literature is very wide. A series of studies on the influence of the adjoined air mass on natural vibration of thin surfaces were done by e.g. Jones and Moore [1], Kwak [2], Sygulski [3], [4], Fu and Price [5] as well as Liang et al. [6] or Lee and Lee [7]. The influence of variable plate stiffness on its dynamic characteristics was analyzed by Nerantzaki and Katsikadelis [8] in terms of the coupled Boundary Element Method (BEM) and the Analog Equation Method (AEM) problem formulation. Rakowski and Guminiak [9] presented non-linear vibrations of thick (Timoshenko) beams using the Finite Element Method (FEM) combined with the Finite Difference Method (FDM). Guminiak and Sygulski [10] presented natural vibrations of thin plates partially and totally submerged in water using the fully BEM approach coupled with the modified formulation of plate boundary conditions. Lenartowicz and Guminiak [11] analyzed plate-water dynamic interaction in terms of FEM-BEM and FDM-BEM. Kamiński [12], [13] presented plate eigenvibrations using stochastic perturbation approach.

Present work includes the free vibration analysis of rectangular thin (Kirchhoff-Love) plates with constant and linearly variable thickness partially and totally submerged in water. In assumption water is treated as inviscid and incompressible medium with resting on the plate and makes no separate flow resulting from this plate vibration. The presence of the fluid is manifested by means of the fully-populated fluid mass matrix which is adjoined to the plate mass matrix..

2. Description of thin plate deformation in terms of the Finite Element Method and the Finite Difference Method

The free vibration problem of the structure can be described by generalized eigenvalue problem, which can be written in matrix notation as follows:

$$(\mathbf{K} - \omega^2 \mathbf{M})\, \mathbf{w} = \mathbf{0} \tag{1}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices of the structure respectively, ω is the natural frequency (eigenvalue) of the structure and $\mathbf{w}$ is the non-zero vector of dynamic degrees of freedom (eigenvector). Bending of a plate with the constant thickness is described by a rectangular four-node finite element with three degrees of freedom at each node. At each element i^{th} node, in the Cartesian coordinate system, there are introduced: deflection w_i and two angles of rotation in mutually perpendicular directions – φ_{ix} and φ_{iy} respectively, wherein the function of deflection can be expressed as the polynomial of the fourth order [14]

$$w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \\ + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} yx^3 + \alpha_{12} xy^3 \tag{2}$$

where $\alpha_{11} \neq 0$ or $\alpha_{12} \neq 0$. The detailed description of the considered finite element is presented by Kuczma [13] and was quoted by Lenartowicz and Guminiak [11].

The free vibrations of isotropic thin plate with variable thickness can be described by the following differential equation [8]

$$D\nabla^4 w + 2\frac{\partial D}{\partial x}\nabla^2\frac{\partial w}{\partial x} + 2\frac{\partial D}{\partial y}\nabla^2\frac{\partial w}{\partial y} + \\ -(1-v)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = b(x,y,t) \tag{3}$$

where$D = D(x,y)$ is the thin plate stiffness and $b(x,y,t)$ is the continuous set of inertia forces. It is assumed that the plate thickness varies linearly along one coordinate x and is continuous and smooth function, that is $h = h(x)$. The differential equation (3) can be replaced by the difference equation which have the character of an amplitude equation:

$$D\left(\frac{\Delta^4 w}{\Delta x^4} + 2\frac{\Delta^4 w}{\Delta x^2 \Delta y^2} + \frac{\Delta^4 w}{\Delta y^4}\right) + 2\left(\frac{\Delta D}{\Delta x}\right)\left(\frac{\Delta^2}{\Delta x^2} + \frac{\Delta^2}{\Delta y^2}\right)\frac{\Delta w}{\Delta x} - (1-v)\left(\left(\frac{\Delta^2 D}{\Delta x^2}\right)\frac{\Delta^2 w}{\Delta y^2}\right) = B \tag{4}$$

where the operators $\frac{\Delta^n}{\Delta x^n}(\dots)$ and $\frac{\Delta^n}{\Delta y^n}(\dots)$ indicates the central difference of the n-th order of the function of two variables x and y. The central differences used in formula (4) are approximations of derivatives of functions $w(x,y)$ and $D(x)$ established at selected points belonging to the plate domain. $B = B(x,y)$ states the set of inertia forces acting in nodes and is expressed per unit of sub-domain $\Delta x \Delta y$. The difference operators are created for each internal central point $(i,j) = (x_i, y_j)$ using the set of thirteen points with regular arrangement (see Fig. 1) according to [15].

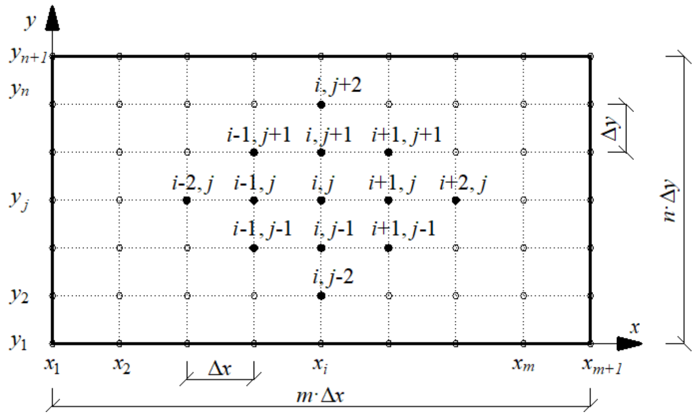

Fig. 1. The set of the finite difference points dividing the plate area [11].

The difference procedures lead to the following matrix equation which has the character of the generalized eigenvalue problem

$$(\mathbf{\Delta} - \omega^2 \mathbf{M})\,\mathbf{w} = \mathbf{0} \tag{5}$$

where $\mathbf{M}$ is a diagonal matrix containing plate masses m_i which are concentrated in subsequent FDM mesh nodes, divided by their surface area.

3. Liquid medium as the source of additional inertia forces

At the beginning, let it be assumed that a plate is fully immersed in a fluid and vibrates at small deflection amplitudes and there is no separation of the liquid from the plate. Starting from the potential theory and building the potential of the double layer the relation between the amplitudes of displacements and the amplitudes of hydrodynamic pressure will be obtained e.g. [10, 11]

$$\omega^2 \tilde{w}(x_m, y_m) = -\frac{1}{4\pi\rho_\mathrm{f}} \sum_{n=1}^{N} \Delta\tilde{p}_n \int_{S_n} \frac{\partial^2}{\partial z_m^2}\left[\frac{1}{r}\right] \underset{z\to 0}{dS_n} \tag{6}$$

where ρ_f is the fluid density. Above relation (6) can be re-written in matrix notation

$$-4\pi\rho_\mathrm{f}\omega^2 \tilde{\mathbf{w}} = \mathbf{H}\,\tilde{\mathbf{p}} \tag{7}$$

Each element of the matrix **H** is defined as follows

$$H_{mn} = \int_{S_n} \frac{\partial^2}{\partial z_m^2}\left[\frac{1}{r}\right] \underset{z\to 0}{dS_n} \tag{8}$$

which can be evaluated analytically (see Fig. 2) according to the following formula:

$$H_{mn} = -\frac{1}{y_p}\left(\frac{x_q}{r_2} - \frac{x_p}{r_1}\right) + \frac{1}{x_q}\left(\frac{y_q}{r_3} - \frac{y_p}{r_2}\right) + \frac{1}{y_q}\left(\frac{x_q}{r_3} - \frac{x_p}{r_4}\right) - \frac{1}{x_p}\left(\frac{y_q}{r_4} - \frac{y_p}{r_1}\right) \tag{9}$$

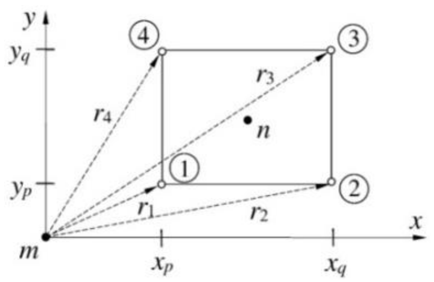

Fig. 2. Designations of rectangular sub-domain for the fluid [10, 11]

Now, let it be assumed, that the same plate is submerged in fluid vertically or horizontally (see Fig. 3).

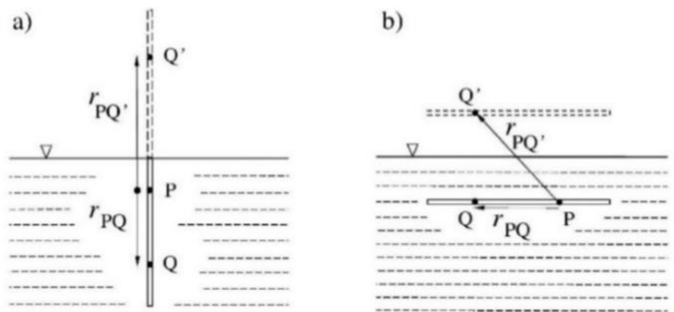

Fig. 3. The fluid mass matrix considering the fluid free surface for plate submerged in water a) vertically and b) horizontally [10].

The liquid free surface can be defined by the liquid hydrodynamic pressure zero vector p_n or the liquid velocity zero vector v_n in normal direction to its surface as Fu and Price [6]. For the first case ($p_n = 0$) the condition of the liquid free surface can be introduced by following replacement in the equation (8)

$$1/r = 1/r(\mathrm{P},\mathrm{Q}) - 1/r(\mathrm{P},\mathrm{Q'}) \tag{10}$$

and for the second case ($v_n = 0$) the replacement is expressed

$$1/r = 1/r(\mathrm{P},\mathrm{Q}) + 1/r(\mathrm{P},\mathrm{Q'}) \tag{11}$$

For a plate horizontally submerged in a fluid, should be taken into account non-zero vertical parameter z occurring in the integral (8) i.e. that $r = \sqrt{x^2 + y^2 + z^2}$. The relation (10) corresponds more closely to the presence of a free surface, and the relation (11) can be assigned when the plate is located near the bottom of the container.

Given the boundary of the fluid medium defined by relations (10) or (11), the matrix **H** will be constructed as follows

$$\mathbf{H} = \mathbf{H}(\mathrm{P}, \mathrm{Q}) - \mathbf{H}(\mathrm{P}, \mathrm{Q}') \quad (12)$$

$$\mathbf{H} = \mathbf{H}(\mathrm{P}, \mathrm{Q}) + \mathbf{H}(\mathrm{P}, \mathrm{Q}') \quad (13)$$

Only matrices defined by relations (12) or (13) can be inverted to obtain a fluid mass matrix considering the barrier of liquid medium

$$\mathbf{M}_{\mathrm{f}} = 4\pi\rho_{\mathrm{f}}\mathbf{S}\mathbf{H}^{-1} \quad (14)$$

wherein the vector of hydrodynamic forces acting on a plate can be specified by the relation

$$\mathbf{P} = -\mathbf{M}_{\mathrm{f}}\omega^2\mathbf{w} \quad (15)$$

where $P_n = \Delta p_n\, S_n$,$\boldsymbol{S} = \mathrm{diag}(S_1 \ldots S_N)$ collects values of areas of the individual sub-domains and N is the number of them.

Finally the mass matrix of the fluid-plate system $\mathbf{M}_{\mathrm{pf}}$ is the sum of the consistent or lumped (diagonal) plate mass matrix and fully-populated fluid mass matrix [10], [11].

$$\mathbf{M}_{\mathrm{pf}} = \mathbf{M}_{\mathrm{p}} + \mathbf{M}_{\mathrm{f}} \quad (16)$$

The relation (16) informs that the whole problem comes to the construction of the fully-populated fluid mass matrix which is coupled to the mass matrix of the plate.

4. Numerical examples

In the examples presented below, the natural frequencies of selected plate types are determined. Plates are submerged a) vertically or b) horizontally (see Fig. 4). The water free surface is assumed to be defined by the liquid velocity zero vector ($v_n = 0$).

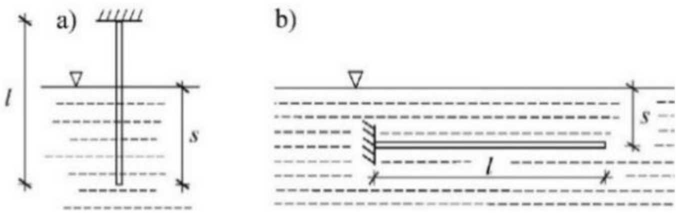

Fig. 4. A square cantilever plate submerged in water a) vertically and b) horizontally.

<u>Example 1.</u> The square cantilever steel plate of dimensions 10.0 m × 10.0 m and thickness 0.238 m is vertically submerged in water. The part of the plate domain immersed in water is divided into the set of regular sub-domains with the number: 400 for the fully immersed plate (*s/l* = 1), 300 for *s/l* = 0.75, 200 for *s/l* = 0.5 and 100 for *s/l* = 0.25. Natural frequencies for the plate obtained using FEM-BEM and FDM-BEM formulation with (I) and without (II) consideration of the free water surface are presented in Table 1 and Table 2 respectively with the use of a 21×21 FEM mesh and 40×40 FDM grid.

Tab. 1. Natural frequencies for the plate vertically immersed in water.

FEM-BEM present solution – natural frequencies ω [rad/s]								
s/l	0.25		0.5		0.75		1.0	
Mode	(I)	(II)	(I)	(II)	(I)	(II)	(I)	(II)
1	9.777	9.591	7.868	7.888	7.451	7.364	7.340	7.257
2	25.587	25.180	21.824	21.644	20.749	20.584	20.530	20.446
3	71.647	77.442	63.742	63.210	52.713	51.939	49.342	48.818
4	85.565	84.679	74.746	74.135	69.768	69.373	68.861	68.700

Tab. 2. Natural frequencies for the plate vertically immersed in water.

FDM-BEM present solution – natural frequencies ω [rad/s]								
s/l	0.25		0.5		0.75		1.0	
Mode	(I)	(II)	(I)	(II)	(I)	(II)	(I)	(II)
1	10.029	9.664	7.792	7.940	7.700	7.410	7.638	7.301
2	28.003	27.070	23.380	23.089	22.911	21.855	22.553	21.692
3	72.058	71.280	64.889	63.019	52.647	51.873	47.489	48.769
4	90.147	87.758	78.029	76.294	74.196	70.852	72.697	70.037

The results for different FDM grids (24×24 and 32×32) are shown below in Tabs. 2a and 2b respectively.

Tab. 2a. Natural frequencies for the plate vertically immersed in water.

FDM-BEM present solution – natural frequencies ω [rad/s] for 24×24 FDM grid								
s/l	0.25		0.5		0.75		1.0	
Mode	(I)	(II)	(I)	(II)	(I)	(II)	(I)	(II)
1	10.121	9.467	7.958	7.799	7.577	7.295	7.415	7.192
2	29.215	27.535	24.163	23.360	23.159	22.073	22.638	21.909
3	71.009	70.492	64.456	61.699	50.881	50.812	47.794	47.945
4	92.077	88.025	79.034	76.049	73.705	70.386	71.805	69.544

Tab. 2b. Natural frequencies for the plate vertically immersed in water.

FDM-BEM present solution – natural frequencies ω [rad/s] for 32×32 FDM grid								
s/l	0.25		0.5		0.75		1.0	
Mode	(I)	(II)	(I)	(II)	(I)	(II)	(I)	(II)
1	9.809	9.588	8.474	7.886	7.711	7.366	7.187	7.259
2	28.043	27.244	24.842	23.192	23.127	21.938	21.967	21.775
3	70.791	70.984	62.152	62.516	51.370	51.474	51.472	48.461
4	89.9611	87.841	80.311	76.191	74.291	70.665	70.826	69.839

The numerical tests results are compared with the analytical [6] and BEM [10] results taking into account the effect of the free water surface, which are listed in the Table 3. Four first modes of vibrations for both plates fully submerged in water are shown in the Fig. 5.

Tab. 3. Natural frequencies for the plate vertically immersed in water [6], [10]

Analytical [6] and BEM [10] solution – natural frequencies ω [rad/s]								
s/l	0.25		0.5		0.75		1.0	
Mode	[6]	[10]	[6]	[10]	[6]	[10]	[6]	[10]
1	10.25	9.99	8.22	8.16	7.57	7.50	7.35	7.31
2	26.40	25.61	21.85	21.83	20.50	20.55	20.20	20.30
3	73.26	71.84	64.40	64.36	54.65	53.53	50.45	49.44
4	87.10	85.82	76.68	75.39	71.52	69.81	70.41	68.80

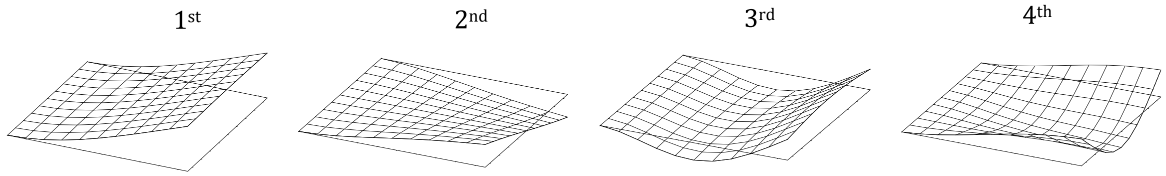

Fig. 5. Modes of vibration of the clamped isotropic plate fully immersed in fluid [11].

Example 2. The square cantilever isotropic plate of the constant thickness is horizontally submerged in water. The plate material properties and the geometry are the same as in the Example 1. Water sub-domains with the number 400 are used while FEM mesh and FDM grid are the same as in the Example 1. The results of calculations are presented in Table 4 and compared to analytical solutions [6].

Tab. 4. Natural frequencies for the plate submerged horizontally in water

Mode	Solution – natural frequencies ω [rad/s]								
	FEM-BEM – present			FDM-BEM – present			Analytical [6]		
s/l	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
1	7.961	7.467	7.337	8.014	7.514	7.382	8.04	7.51	7.35
2	21.586	20.616	20.482	22.929	21.877	21.731	21.54	20.34	20.19
3	52.307	49.564	49.053	52.239	49.508	49.001	53.23	50.83	50.11
4	71.411	68.971	68.751	72.896	70.329	70.093	72.20	70.81	69.50

The results for different FDM grids (24×24 and 32×32) are shown below in Tab. 4a.

Tab. 4a. Natural frequencies for the plate submerged horizontally in water

Mode	Solution – natural frequencies ω [rad/s]								
	FDM-BEM – present (24×24 FDM grid)			FDM-BEM – present (32×32 FDM grid)			Analytical [6]		
s/l	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
1	7.922	7.411	7.276	7.978	7.474	7.342	8.04	7.51	7.35
2	23.247	22.112	21.952	23.049	21.966	21.815	21.54	20.34	20.19
3	51.452	48.688	48.179	51.944	49.202	48.694	53.23	50.83	50.11
4	72.592	69.857	69.603	72.769	70.139	69.896	72.20	70.81	69.50

Example 3. The square cantilever isotropic plate of the linearly variable thickness is considered (see Fig. 6) and vertically submerged in water. The manner of submerging the plate is shown in the Fig. 4. The immersion effect on natural frequencies for h = 0.238 m and $H = 2h$ is presented in Table 5. The FDM grid of dimensions 40×40 is used.

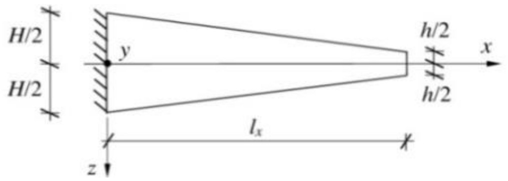

Fig. 6. The square clamped plate with linearly variable thickness.

Obtained natural frequencies for two different plate thickness values at its fixed edge and for different FDM grid dimensions respectively are presented in Table 6 taking into account the effect of the free water surface. The modes of vibrations are similar to those presented in Example 2.

Tab. 5. Natural frequencies for the plate vertically immersed in water.

FDM-BEM present solution – natural frequencies ω [rad/s]								
s/l	0.25		0.5		0.75		1.0	
Mode	(I)	(II)	(I)	(II)	(I)	(II)	(I)	(II)
1	20.373	19.700	16.444	16.747	16.384	15.897	16.296	15.727
2	38.969	37.755	33.687	33.457	33.387	32.408	33.027	32.295
3	104.989	102.493	95.285	94.064	90.369	89.018	86.029	87.186
4	134.328	132.568	120.357	117.340	106.189	103.781	102.425	101.403

Tab. 6. Natural frequencies for the plate obtained using different grid dimensions.

Mode	Solution FDM-BEM present solution – natural frequencies ω [rad/s]							
	$H = 2h$ and $s/l = 0.75$				$H = 3h$ and $s/l = 0.75$			
	16 × 16	24 × 24	32 × 32	40 × 40	16 × 16	24 × 24	32 × 32	40 × 40
1	15.502	16.101	16.377	16.384	23.639	24.486	24.888	24.935
2	43.907	34.183	33.796	33.387	47.579	45.780	44.979	44.351
3	88.324	88.031	89.077	90.369	119.378	119.039	119.448	119.751
4	108.914	106.252	105.984	106.189	152.708	148.030	148.136	149.398

5. Conclusions

The linear theory of natural vibrations of thin rectangular isotropic plates considering interaction with water has been presented in the paper. To describe the plate deformation the Finite Element Method (FEM) and the Finite Difference Method (FDM) were applied for plates with constant and variable thickness respectively. The numerical tests carried out in the paper showed that the deeper the plate is immersed in the water, the lower its natural frequencies become. This proves the significant influence of the adjoined mass on the dynamics of the plate. The calculations for plates with constant thickness presented in the paper showed that both the FEM and FDM methods used allow to obtain similar results. In addition, doubling the thickness of the plate at its fixed edge approximately doubled its first natural frequency. The combined BEM-FDM approach results are closer to the analytical solutions than the FEM-BEM combined methodology results. Numerical convergence studies carried out for a plate with variable thickness showed that starting from a 24×24 grid, the results obtained began to approach each other.

Acknowledgments

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