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GROUND STATES FOR FRACTIONAL NONLOCAL EQUATIONS WITH LOGARITHMIC NONLINEARITY

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Abstract. In this paper, we study on the fractional nonlocal equation with the logarithmic nonlinearity formed by

$$\begin{cases} \mathcal{L}_K u(x) + u\log|u| + |u|^{q-2}u = 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $2 < q < 2_s^*$, L_K is a non-local operator, Ω is an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. By using the fractional logarithmic Sobolev inequality and the linking theorem, we present the existence theorem of the ground state solutions for this nonlocal problem.

Keywords: linking theorem, ground state, logarithmic nonlinearity, variational methods.

Mathematics Subject Classification: 35J20, 35B33, 58E05.

1. INTRODUCTION

In this paper, our main work is to study the existence of the ground state solutions to the fractional non-local equation with the logarithmic term followed as

$$\begin{cases} \mathcal{L}_K u + u\log|u| + |u|^{q-2}u = 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where $2 < q < 2_s^*$, $2_s^* = \frac{2n}{n-2s}$, $s \in (0,1)$ is fixed with $n > 2s$, $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary, and the integro-differential operator $\mathcal{L}_K$ is defined by

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy \tag{1.2}$$

for any $x \in \mathbb{R}^n$.

Here, the kernel $K : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$ is a function with the properties such that:

(k_1) $\gamma K \in L^1(\mathbb{R}^n)$, where $\gamma(x) = \min\{|x|^2, 1\}$,
(k_2) there exist $\delta > 0$ such that $K(x) \geq \delta |x|^{-(n+2s)}$, for any $x \in \mathbb{R}^n \backslash \{0\}$,
(k_3) $K(x) = K(-x)$, for any $x \in \mathbb{R}^n \backslash \{0\}$.

It is well known that the operator $\mathcal{L}_K$ in (1.2) is a good generalization of the fractional Laplacian operators. For instance, if we take $K(x) = |x|^{-(n+2s)}$, $x \in \mathbb{R}^n \backslash \{0\}$, up to some normalization constant, the nonlocal operator $\mathcal{L}_K$ is equal to the classical operator

$$-(-\Delta)^s u(x) = \frac{1}{2} \int\limits_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{(n+2s)}} dy, \quad x \in \mathbb{R}^n. \tag{1.3}$$

Due to the advantage that they provide a powerful way to describe many complicated physical phenomena, the fractional Laplacian operators $(-\Delta)^s$ play a very important role in many fields of mathematics, especially in harmonic analysis, probability theory and potential theory. As a consequence of studies, there are various definitions about this type of operators. For example, in probability theory (see [2,9,10] for more details), it can be given via a singular integral by

$$(-\Delta)^s u(x) = C(n,s) \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^n \backslash B(x,\varepsilon)} \frac{u(y) - u(x)}{|x-y|^{n+2s}} dy, \quad x \in \mathbb{R}^n, \tag{1.4}$$

where $B(x, \varepsilon)$ is a ball centered at $x \in \mathbb{R}^n$ with radius ε. Also, the fractional Laplacian operator can be defined in an alternative way via the Fourier transform by

$$(-\Delta)^s u(x) = \mathfrak{F}^{-1}\left(|\xi|^{2s} (\mathfrak{F}u)(\xi)\right)(x), \quad \xi \in \mathbb{R}^n, \tag{1.5}$$

where $\mathfrak{F}$ is the Fourier transform. In fact, Nezza *et al.* has proved that (1.4) and (1.5) are equivalent. We refer to [17] for more information. Since the operator $(-\Delta)^s$ and its generalization are both nonlocal operators, the fractional equations are naturally called fractional and nonlocal problems, see [16] for basic results based on variational methods.

In recent years, much attention has been focused on the nonlocal problems. However, the nonlocal problems are more difficult than the local ones. In 2007, Caffarelli and Silvestre established the fundamental characterizations of the fractional Laplacian equations in [3], including the regularity and extremum principle. This is the pioneering work for the later related researches and makes the theory of the nonlocal equations developed rapidly.

For example, more and more researchers have been interested in the nonlocal problems driven by $(-\Delta)^s$ (or its generalization) with the critical nonlinearity. For the following equation

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^\star - 2} u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

Servadei and Valdinoci discussed the non-trivial existence of solutions for the above model by two cases. Precisely, in the case of $2s < n < 4s$, [21] showed that for any $\lambda > \lambda_s$ different from the eigenvalue of the operator $(-\Delta)^s$, there admits a non-trivial solution. Afterward, in the case of $n \geq 4s$, [23] proved that if $\lambda < \lambda_{1,s}$, then there also exists non-trivial solutions, where $\lambda_{1,s}$ is the first eigenvalue of the operator $(-\Delta)^s$. We refer the readers to [11, 13, 14, 18, 29, 31] and references therein for the details of the critical nonlinearity.

Meanwhile, there are many researchers devoted to the nonlocal problems with the subcritical term (see for examples [15, 24, 27, 28]), especially for the existence and multiplicity of solutions for the fractional problems involving different nonlinear term, such as the logarithmic nonlinearity. Precisely, by applying the mountain pass theorem and linking theorem, Servadei and Valdinoci in [22] derived some existence results for the following equation

$$\begin{cases} (-\Delta)^s u - \lambda u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where f satisfies superlinear and sublinear growth condition at zero and infinity. Concerning the discrete case of fractional Laplacian, Ciaurri *et al.* in [5] studied the fractional discrete Laplacian

$$(-\Delta_h)^s u = f,$$

where $u, f : \mathbb{Z}_h \to \mathbb{R}$ and $h > 0$, $0 < s < 1$. $(-\Delta_h)^s$ is the fractional powers of the discrete Laplace operator defined as

$$(-\Delta_h)^s u(j) = \frac{1}{\Gamma(-s)} \int\limits_0^\infty \frac{e^{t\Delta_h} u(j) - u(j)}{t^{1+2s}} dt,$$

where Γ is the Gamma function, see also [30] for more related results obtained by using variational methods. In [7], d'Avenia *et al.* considered the following fractional logarithmic Schrödinger equation

$$(-\Delta)^s u + \omega u = u \log |u|^2, \quad x \in \mathbb{R}^n,$$

where $\omega > 0$, and the existence of infinite many solutions was obtained by using the Sobolev inequality of fractional logarithms. We refer to [1, 4] and references therein for more results in this direction.

On the other hand, we notice that, for the elliptic problem involving the Laplacian operator, by the linking theorem, Liu *et al.* in [12] proved the existence of the ground state solution for a fourth-order nonlinear elliptic problem with logarithmic nonlinearity as form by

$$\begin{cases} \Delta^2 u + c\Delta u = u \log |u|, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 denotes the biharmonic operator, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. See also [25] for Kirchhoff-type fractional diffusion problem with logarithmic nonlinearity. Concerning further applications of the linking theorem, for instance, we refer to [8, 22, 32].

Inspired by the preceding results, in this article we are devoted to studying the ground state solutions of the nonlocal problems with the logarithmic term. More precisely, we investigate the existence of the ground states for the equation (1.1). In order to derive the desired existence theorem, we first introduce a suitable energy functional, and show the continuity and the Gateaux derivative. Then, in order to use the linking theorem, we prove Lemmas 3.1–3.3 and then the existence of the non-trivial solutions is obtained. Finally, we prove the existence of the ground state of the nonlocal problem (1.1).

In order to present the main result, we start with the introduction of the essential function spaces and the basic definitions.

Denote by X the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function u in X belongs to $L^2(\Omega)$ and the map

$$(x,y) \mapsto (u(x)-u(y))\sqrt{K(x-y)} \in L^2(Q, dxdy),$$

with the norm defined as

$$\|u\|_X = \|u\|_{L^2(\Omega)} + \left(\iint\limits_Q |u(x)-u(y)|^2 K(x-y)dxdy \right)^{\frac{1}{2}},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$. Moreover, let

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\},$$

with the norm endowed by

$$\|u\|_{X_0} = \left(\iint\limits_Q |u(x)-u(y)|^2 K(x-y)dxdy \right)^{\frac{1}{2}}.$$

Obviously, X_0 is the Hilbert space. So, for each $u \in X_0$, it can be decomposed as

$$u = \sum_{k=1}^{\infty} a_k \varphi_k,$$

where $\{\varphi_k\}_{k=1}^{\infty}$ is a set of orthogonal bases for X_0. In addition, let

$$E_1 := \mathbb{P}_{k+1} = \{u \in X_0 : \langle u, \varphi_j \rangle_{X_0} = 0, j = 1, \ldots, k\}$$

and

$$E_2 := \operatorname{span}\{\varphi_1, \ldots, \varphi_k\},$$

then we have $X_0 = E_1 \oplus E_2$ (see Proposition 2.4 below).

Now, we give the definition of the weak solution of the problem (1.1) as follows.

Definition 1.1. we say $u \in X_0$ is a weak solution of problem (1.1) if

$$\iint_{\mathbb{R}^{2n}} |u(x)-u(y)||v(x)-v(y)|K(x-y)dxdy = \int_\Omega u \log|u| v dx + \int_\Omega |u|^{q-2} uv dx \quad (1.6)$$

holds for any $v \in X_0$.

Additionally, throughout this paper, we continue to use $\|\cdot\|_p$ as the norm of $L^p(\mathbb{R}^n)(1 \le p < \infty)$ and denote by $H^s(\mathbb{R}^n)$ the fractional Sobolev space with the norm as

$$\|u\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u|^2 dx\right)^{\frac{1}{2}} + \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy\right)^{\frac{1}{2}} \quad (1.7)$$

and the seminorm given by

$$[u]^2_{H^s(\mathbb{R}^n)} = \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy.$$

It should be noted that the constant $c_i > 0$ $(i = 1, 2, \ldots)$ appeared in this paper may differ from one line to another.

2. SOME PRELIMINARY PROPOSITIONS

In this section, we will give some necessary definitions, propositions and some results of the function spaces X and X_0 for the main assertion.

First, according Proposition 4.4 in [17] and Lemma 5 in [20], it is easy to derive Proposition 2.1. For the completeness, we give the proof in details.

Proposition 2.1. *Let X_0' be the dual space of X_0 , then we have that*

$$\|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^n)} \le \frac{2}{\delta}\|u\|_{X_0} \quad (2.1)$$

holds for any $u \in X_0$.

Proof. For every $u \in X_0$, we know that $u \in H^s(\mathbb{R}^n)$, so it follows that

$$[u]_{H^s(\mathbb{R}^n)} = \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)} \quad (2.2)$$

Observe the formula (1.7), according to Lemma 5 in [20], we know that

$$\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \le \frac{1}{\delta} \iint_Q |u(x)-u(y)|^2 K(x-y) dxdy \quad (2.3)$$

where δ is given in (k_2). So, combining (2.2) with (2.3), we obtain that

$$\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)} \le \frac{2}{\delta}\|u\|_{X_0}$$

as desired. □

With the above the fact in mind, we are ready to show a very significant logarithm inequality for our later argument. Actually, the key tool to prove Proposition 2.2 is the fractional logarithm Sobolev inequality in [6].

Proposition 2.2. *For any real number $a > 0$ and $u \in X_0$, we have*

$$\int\limits_{\Omega} u^2 \log\left(\frac{|u|}{\|u\|_2}\right) dx \le \frac{a^2}{\delta\pi^s}\|u\|_{X_0} - \frac{1}{2}\left(n + \frac{n}{s}\log a + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u\|_2^2. \tag{2.4}$$

Proof. According to the fractional logarithm Sobolev inequality

$$\int\limits_{\mathbb{R}^n} u^2 \log\left(\frac{u^2}{\|u\|_2^2}\right) dx + \left(n + \frac{n}{s}\log a + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u\|_2^2 \le \frac{a^2}{\pi^s}\|(-\Delta)^{\frac{s}{2}}u\|_2^2,$$

together with the property of logarithm

$$\int\limits_{\mathbb{R}^n} u^2 \log\left(\frac{u^2}{\|u\|_2^2}\right) dx = 2\int\limits_{\mathbb{R}^n} u^2 \log\left(\frac{|u|}{\|u\|_2}\right) dx,$$

we derive that

$$\int\limits_{\mathbb{R}^n} u^2 \log\left(\frac{|u|}{\|u\|_2}\right) dx \le \frac{a^2}{2\pi^s}\|(-\Delta)^{\frac{s}{2}}u\|_2^2 - \frac{1}{2}\left(n + \frac{n}{s}\log a + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u\|_2^2.$$

Notice that for any $x \in \mathbb{R}^n \setminus \Omega$, we have $u = 0$. Then, by applying (2.1), it implies that

$$\int\limits_{\Omega} u^2 \log\left(\frac{|u|}{\|u\|_2}\right) dx \le \frac{a^2}{\delta\pi^s}\|u\|_{X_0} - \frac{1}{2}\left(n + \frac{n}{s}\log a + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u\|_2^2.$$

Therefore, we finish the proof of Proposition 2.2. □

Now, before going on, let us go back to the problem (1.1). Since this equation has a variational structure, we can define the energy functional $J : X_0 \to \mathbb{R}$ as the form

$$\begin{aligned} J(u) = \frac{1}{2}\iint\limits_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)dxdy - \frac{1}{2}\int\limits_{\Omega} u^2 \log|u|dx \\ + \frac{1}{4}\int\limits_{\Omega} u^2 dx - \frac{1}{q}\int\limits_{\Omega} |u|^q dx. \end{aligned} \tag{2.5}$$

Thanks to the conditions $(k_1)-(k_3)$, to Proposition 2.2, and to the fact that $L^q \hookrightarrow X_0$ compactly, we can derive easily that J is well defined on X_0.

Next, we need to turn out that $J \in C^1(X_0, \mathbb{R})$, namely, Proposition 2.3, which is equivalent to say that J has a continuous Gâteaux derivative.

Proposition 2.3. *The energy functional J defined as* (2.5) *has the Fréchet derivative and is continuous on X_0. Moreover, for every $u, v \in X_0$, we have that*

$$\langle J^{'}(u), v\rangle = \iint\limits_{\mathbb{R}^n} (u(x) - u(y))(v(x) - v(y))K(x - y)dxdy$$
$$- \int\limits_{\Omega} uv \log |u|dx - \int\limits_{\Omega} |u|^{q-2} uvdx.$$

Proof. We will prove this proposition by two steps, namely, the existence and continuity of Gâteaux derivative. A simple observation tells that there are four terms at the right side of (2.5). Since the proof methods are similar, here, we only show the first and third term.

For the first term, applying the definition of Gâteaux, together with the inner product property, we get that

$$\begin{aligned} \frac{1}{2}\lim_{t\to 0}\frac{\|u+tv\|^2-\|u\|^2}{t} &= \frac{1}{2}\lim_{t\to 0}\frac{\langle u+tv, u+tv\rangle - \langle u, u\rangle}{t} \\ &= \frac{1}{2}\lim_{t\to 0}\frac{2\langle u, tv\rangle + \langle tv, tv\rangle}{t} \\ &= \frac{1}{2}\lim_{t\to 0} 2\langle u, v\rangle + \langle v, tv\rangle = \langle u, v\rangle \end{aligned} \tag{2.6}$$

holds for all $u, v \in X_0$.

Now, let us turn to prove the third term. For the simplicity, define $f(u) = \frac{1}{4}\int_\Omega u^2(x)dx$. According to the definition of Gâteaux derivative, that is, for any $u, v \in X_0$, it is given as

$$\langle f'(u), v\rangle = \frac{1}{4}\lim_{t\to 0}\frac{f(u+tv)-f(u)}{t} = \lim_{t\to 0}\int\limits_{\Omega}\frac{(u+tv)^2-u^2}{4t}dx.$$

So, it is equivalent to show that

$$\langle f'(u), v\rangle = \lim_{t\to 0}\int\limits_{\Omega}\frac{(u+tv)^2-u^2}{4t}dx = \int\limits_{\Omega}\lim_{t\to 0}\frac{(u+tv)^2-u^2}{4t}dx = \frac{1}{2}\int\limits_{\Omega} uvdx.$$

Given $x \in \Omega$ and $0 < |t| < 1$, by the mean value theorem, there exists a parameter $\delta \in (0, 1)$ such that

$$\begin{aligned} \left|\frac{(u(x)+tv(x))^2-u^2(x)}{4t}\right| &= \left|\frac{1}{2}(u(x)+\delta tv(x))v(x)\right| \\ &\le \frac{1}{2}\left(|u(x)|+|v(x)|\right)|v(x)|. \end{aligned}$$

By applying the Hölder inequality and Minkowski inequalities, along with the fact $X_0 \hookrightarrow L^2(\Omega)$, we derive that

$$\begin{aligned}\int_\Omega (|u|+|v|)|v|dx &\le \left(\int_\Omega (|u|+|v|)^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dx\right)^{\frac{1}{2}} \\ &= (\|u\|_{L^2} + \|v\|_2)\cdot \|v\|_2 \\ &\le C(\|u\|_{X_0} + \|v\|_{X_0})\cdot \|v\|_{X_0}\end{aligned}$$

where $C > 0$ is a suitable constant, which implies that $(|u|+|v|)|v| \in L^1(\Omega)$. Thus, using the Lebesgue theorem yields

$$\langle f'(u), v\rangle = \lim_{t\to 0}\int_\Omega \frac{(u+tv)^2 - u^2}{4t}dx = \int_\Omega \lim_{t\to 0}\frac{(u+tv)^2-u^2}{4t}dx = \frac{1}{2}\int_\Omega uvdx. \quad (2.7)$$

Now, let us turn to prove the continuity of the Gâteaux derivative on X_0. Assume that $\{u_n\} \subset X_0$ and $u_n \to u_0$ in X_0, by the embedding $X_0 \hookrightarrow L^\nu$ $(1 \le \nu < 2_s^\star)$ again, which implies

$$\|u_n - u_0\|_\nu \to 0, \quad \text{as} \quad n \to \infty$$

Therefore, due to the definition of the operator norm and the Hölder inequality, we have that

$$\begin{aligned}\|f'(u_n) - f'(u)\| &= \sup_{h\in X_0, \|h\|_{X_0}=1} |\langle f'(u_n) - f'(u), h\rangle| \\ &= \sup_{h\in X_0, \|h\|_{X_0}=1} \int_\Omega |(u_n - u)h|dx \\ &\le \sup_{h\in X_0, \|h\|_{X_0}=1} \left(\int_\Omega |u_n - u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |h|^2 dx\right)^{\frac{1}{2}} \\ &\le C \sup_{h\in X_0, \|h\|_{X_0}=1} \|u_n - u\|_2 \|h\|_{X_0} \to 0.\end{aligned}$$

as $n \to \infty$, where $C > 0$ is a suitable constant. Therefore, we derive that f' is continuous on X_0.

Similarly, for the second and last term, we have that

$$\begin{aligned}
&-\frac{1}{2}\lim_{t\to 0}\int_{\Omega}\frac{(u+tv)^2\log(u+tv)^{2\cdot\frac{1}{2}}-u^2\log u^{2\cdot\frac{1}{2}}}{t}dx\\
&=-\frac{1}{2}\int_{\Omega}\lim_{t\to 0}\frac{\frac{1}{2}(u+tv)^2\log(u+tv)^2-\frac{1}{2}u^2\log u^2}{t}dx\\
&=-\frac{1}{2}\int_{\Omega}\lim_{t\to 0}(u+tv)v\log(u+tv)^2+(u+tv)vdx\\
&=-\frac{1}{2}\int_{\Omega}(uv\log u^2+uv)dx\\
&=-\frac{1}{2}\int_{\Omega}uv\log u^2dx-\frac{1}{2}\int_{\Omega}uvdx\\
&=-\int_{\Omega}uv\log udx-\frac{1}{2}\int_{\Omega}uvdx.
\end{aligned}\tag{2.8}$$

and

$$-\frac{1}{q}\lim_{t\to 0}\int_{\Omega}\frac{(u+tv)^{2\cdot\frac{q}{2}}-u^{2\cdot\frac{q}{2}}}{t}dx=-\int_{\Omega}u^{q-2}uvdx. \tag{2.9}$$

So, combining (2.6)–(2.9) together, we derive that

$$\begin{aligned}
\langle J^{'}(u),v\rangle=&\iint_{\mathbb{R}^{2n}}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy\\
&-\int_{\Omega}uv\log|u|dx-\int_{\Omega}|u|^{q-2}uvdx.
\end{aligned}$$

Furthermore, by using the Hölder inequality and definition of continuity for the linear operator, it is simple to see that the Gateaux derivative of J is continuous for every $u\in X_0$. Therefore, the desired result holds. □

Now, we finish this section with the following propositions.

Proposition 2.4 (Proposition 9 in [22])**.** *Let $K:\mathbb{R}^n\backslash\{0\}\to(0,+\infty)$ be the function satisfying assumptions (k_1)–(k_3) and $\{\lambda_k\}$ be the sequence of the eigenvalues of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data and*

$$0<\lambda_1<\lambda_2\leq\ldots\leq\lambda_k\leq\lambda_{k+1}\leq\ldots$$

and

$$\lambda_k\to+\infty$$

as $k\to+\infty$, and let $\{\varphi_k\}$ be the sequence of the eigenfunctions corresponding to λ_k.

Then:

(1) *first eigenvalue* λ_1 *that can be characterized as follows*

$$\lambda_1 = \min_{\substack{u\in X_0\\ \|u\|_2=1}} \iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy$$

or, equivalently

$$\lambda_1 = \min_{u\in X_0} \frac{\displaystyle\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy}{\displaystyle\int_\Omega |u(x)|^2 dx},$$

(2) *for any* $k\in\mathbb{N}$, *the eigenvalues* λ_k *can be characterized as follows*

$$\lambda_{k+1} = \min_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{L^2(\Omega)=1}}} \iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy$$

or, equivalently

$$\lambda_{k+1} = \min_{u\in\mathbb{P}_{k+1}\backslash 0} \frac{\displaystyle\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy}{\displaystyle\int_\Omega |u(x)|^2 dx},$$

where

$$\mathbb{P}_{k+1} := \{u\in X_0 : \langle u,\varphi_j\rangle_{X_0}=0, j=1,\ldots,k\},$$

(3) *the sequence* $\{\varphi_k\}$ *is an orthonormal basis of* $L^2(\Omega)$ *and an orthogonal basis of* X_0.

Proposition 2.5 (Proposition 2.3 in [19]). *Let* $K:\mathbb{R}^n\backslash\{0\}\to(0,+\infty)$ *be the function satisfying assumptions* (k_1)*–*(k_3), $\{\lambda_k\}$ *be the sequence of the eigenvalues of the operator* $-\mathcal{L}_K$ *with homogeneous Dirichlet boundary data and* $\{\varphi_k\}$ *be the sequence of the eigenfunctions corresponding to* $\{\lambda_k\}$. *Then, for any* $k\in\mathbb{N}$, *the eigenvalues* λ_k *can be characterized as follows*

$$\lambda_k = \max_{u\in span\{\varphi_1,\ldots,\varphi_k\}\backslash\{0\}} \frac{\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy}{\int_\Omega |u|^2 dx}.$$

Proposition 2.6 (Lemma 9 in [23]). *Let* $K:\mathbb{R}^n\backslash\{0\}\to(0,+\infty)$ *be the function satisfying assumptions* (k_1)*–*(k_3). *Then the following assertions holds:*

(1) *the embedding* $X_0\hookrightarrow L^\nu(\Omega)$ *is compact for any* $\nu\in[1,2_s^*)$,
(2) *the embedding* $X_0\hookrightarrow L^{2_s^*}(\Omega)$ *is continuous.*

Proposition 2.7 (Linking Theorem [26]). *Let* X_0 *be a real Hilbert space. Suppose that* $J\in C^1(X_0,\mathbb{R})$, $X_0=E_1\bigoplus E_2$, *where* $\dim E_2<\infty$, *and there exist* $R>\rho>0$, $\alpha>0$ *and* $0\neq e_0\in E_1$, *such that*

$$\inf J\left(E_1\bigcap S_\rho\right)\geq\alpha \quad \textit{and} \quad \sup J(\partial T)\leq 0,$$

where

$$S_\rho := \{u \in X_0 : \|u\| = \rho\} \quad \text{and} \quad T := \{u = v + te_0 : v \in E_2, t \geq 0, \|u\| \leq R\}.$$

If all $b \in [\alpha, \sup J(T)]$ *meet the* (PS) *conditions, then* J *has a critical value in* $[\alpha, \sup J(T)]$.

3. MAIN RESULTS AND PROOFS

To facilitate the upcoming main theorem, we will assert the following lemmas.

Lemma 3.1. *Assume that* $u \in E_1$ *satisfies* $\|u\|_{X_0} = \rho$, *then there exist* $\rho > 0$, $\alpha > 0$ *such that* $J(u) \geq \alpha$.

Proof. Let $u \in E_1$, a direct calculation from (2.4) and (2.5) gives that

$$\begin{aligned}
J(u) &= \frac{1}{2} \iint\limits_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx dy - \frac{1}{2} \int\limits_{\Omega} u^2 \log |u| dx \\
&\quad + \frac{1}{4} \int\limits_{\Omega} u^2 dx - \frac{1}{q} \int\limits_{\Omega} |u|^q dx \\
&= \frac{1}{2} \iint\limits_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx dy \\
&\quad - \frac{1}{2} \int\limits_{\Omega} u^2 \left(\log |u| - \log \|u\|_2 + \log \|u\|_2\right) dx \\
&\quad + \frac{1}{4} \int\limits_{\Omega} u^2 dx - \frac{1}{q} \int\limits_{\Omega} |u|^q dx \\
&= \frac{1}{2} \iint\limits_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx dy - \frac{1}{2} \int\limits_{\Omega} u^2 \log \frac{|u|}{\|u\|_2} dx \\
&\quad - \frac{1}{2} \log \|u\|_2 \int\limits_{\Omega} u^2 dx + \frac{1}{4} \int\limits_{\Omega} u^2 dx - \frac{1}{q} \int\limits_{\Omega} |u|^q dx \\
&\geq \frac{1}{2} \|u\|_{X_0}^2 - \frac{a^2}{2\delta\pi^s} \|u\|_{X_0}^2 + \frac{1}{4} \left(n + \frac{n}{s} \log a + \log \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) \|u\|_2^2 \\
&\quad - \frac{1}{2} \|u\|_2^2 \log \|u\|_2 + \frac{1}{4} \|u\|_2^2 - \frac{1}{q} \|u\|_q^q \\
&\geq \left(\frac{1}{2} - \frac{a^2}{2\delta\pi^s}\right) \|u\|_{X_0}^2 + \frac{1}{4} \|u\|_2^2 - \frac{1}{q} \|u\|_q^q \\
&\quad + \left\{\frac{1}{4} \left(n + \frac{n}{s} \log a + \log \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) - \frac{1}{2} \log \|u\|_2\right\} \|u\|_2^2.
\end{aligned} \tag{3.1}$$

Notice that the number $a > 0$ is arbitrary in Proposition 2.2, so taking $a = \sqrt{\frac{\delta\pi^s}{2}}$ leads to

$$\begin{aligned} J(u) \geq \frac{1}{4}\|u\|_{X_0}^2 + \frac{1}{4}\left(n + \frac{n}{s}\log\sqrt{\frac{\delta\pi^s}{2}} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u\|_2^2 \\ - \frac{1}{2}\|u\|_2^2\log\|u\|_2 + \frac{1}{4}\|u\|_2^2 - \frac{1}{q}\|u\|_q^q \end{aligned} \tag{3.2}$$

Meanwhile, when

$$\|u\|_2 \leq \exp\left\{\frac{1}{2}\left(n + \frac{n}{s}\log\sqrt{\frac{\delta\pi^s}{2}} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\right\}$$

due to the monotonicity of the logarithmic function, we get that

$$\log\|u\|_2 \leq \frac{1}{2}\left(n + \frac{n}{s}\log\sqrt{\frac{\delta\pi^s}{2}} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right).$$

Then, we have that

$$\frac{1}{2}\left(n + \frac{n}{s}\log a + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) - \log\|u\|_2 \geq 0.$$

Furthermore, by virtue of $2 < q < 2_s^*$ and Ω is a bounded domain, applying Proposition 2.6 ensures that

$$J(u) \geq \frac{1}{4}\|u\|_{X_0}^2 - \frac{c}{q}\|u\|_{X_0}^q = \frac{1}{4}\|u\|_{X_0}^2\left(1 - \frac{4c}{q}\|u\|_{X_0}^{q-2}\right)$$

holds for some suitable constant c. On the other hand, when $\|u\|_{X_0} \leq \sqrt[q-2]{\frac{q}{8c}}$, it follows that

$$1 - \frac{4c}{q}\|u\|_{X_0}^{q-2} \geq \frac{1}{2}.$$

Thus, let

$$\rho = \min\left\{\exp\left[\frac{c}{2}\left(n + \frac{n}{s}\log\sqrt{\frac{\delta\pi^s}{2}} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\right], \sqrt[q-2]{\frac{q}{8c}}\right\}$$

be such that

$$J(u) \geq \frac{1}{8}\|u\|_{X_0}^2 = \frac{1}{8}\rho^2 =: \alpha$$

holds for all $u \in X_0$ with $\|u\|_{X_0} = \rho$. The proof is thus complete. □

Lemma 3.2. *Suppose that φ_{k+1} is defined as in Proposition 2.5 and*

$$\mathbb{R}_{\varphi_{k+1}} = span\{\varphi_{k+1}\}.$$

Then, there exists $R \geq 0$ such that

$$J(u) \leq 0$$

for all $u \in E_2 \oplus \mathbb{R}_{\varphi_{k+1}}$ with $\|u\|_{X_0} \geq R$.

Proof. First we recall the inequality

$$|t^2 \log t| \leq C_p(|t| + |t|^p), \tag{3.3}$$

where $C_p > 0$, for any $t > 0$, $2 < p < \min\{4, 2_s^*\}$. Then, for any $u \in E_2 \oplus \mathbb{R}_{\varphi_{k+1}}$ with $\|u\|_{X_0} = 1$, we have

$$\|u\|_{X_0}^2 \leq \lambda_{k+1}\|u\|_2^2. \tag{3.4}$$

Combining Proposition 2.5, Proposition 2.6, (3.3), (3.4) with (2.5), we obtain

$$\begin{aligned}
J(tu) = {} & \frac{t^2}{2} \iint\limits_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)dxdy - \frac{t^2}{2} \int\limits_{\Omega} u^2 \log |tu| dx \\
& + \frac{t^2}{4} \int\limits_{\Omega} u^2 dx - \frac{t^q}{q} \int\limits_{\Omega} |u|^q dx \\
\leq {} & \frac{t^2}{2} \iint\limits_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)dxdy - \frac{t^2}{2} \log t \int\limits_{\Omega} u^2 dx \\
& + \frac{t^2}{2} \int\limits_{\Omega} |u^2 \log |u|| dx + \frac{t^2}{4} \int\limits_{\Omega} u^2 dx - \frac{t^q}{q} \int\limits_{\Omega} |u|^q dx \\
\leq {} & \frac{t^2}{2}\|u\|_{X_0}^2 - \frac{t^2 \log t}{2\lambda_{k+1}}\|u\|_{X_0}^2 + \frac{t^2}{2} C(\|u\|_{X_0} + \|u\|_{X_0}^p) \\
& + \frac{t^2}{4}\|u\|_2^2 - \frac{t^q}{q} \int\limits_{\Omega} |u|^q dx \\
\leq {} & t^2 \left[\frac{1}{2}\|u\|_{X_0}^2 + \frac{1}{2} C(\|u\|_{X_0} + \|u\|_{X_0}^p) + \frac{1}{4}\|u\|_{X_0}^2\right] \\
& - \frac{\|u\|_{X_0}^2}{2\lambda_{k+1}} t^2 \log t - \frac{t^q}{q} \int\limits_{\Omega} |u|^q dx,
\end{aligned}$$

where $C > 0$ is a constant. Let

$$c_1 = 1 + C > 0, \quad c_2 = \frac{1}{2\lambda_{k+1}} > 0, \quad c_3 = \frac{1}{q} > 0.$$

Since all norms are equivalent in a subspace of the finite dimensional space, it is readily to derive that

$$J(tu) \leq c_1 t^2 - c_2 t^2 \log t - c_3 t^q \to -\infty \quad \text{as} \quad t \to +\infty. \tag{3.5}$$

Therefore, there exists $t_1 > 0$ large enough such that for all $u \in \partial T$ with

$$R = \|t_1 u\|_{X_0} = t_1,$$

it results that

$$J(u) < 0,$$

where $T = \{u = v + te_0 : v \in E_2, t \geq 0, \|u\|_{X_0} \leq R\}$. □

Now, with the above lemmas in hand, we are ready to assert the following result.

Lemma 3.3. *Let $u \in X_0$ and $J(u)$ given as (2.5). Then, we have that $J(u)$ satisfies Palais-Smale (PS) condition, that is, for any (PS) sequence $\{u_j\} \subset X_0$, there admits a subsequence strongly convergent in X_0.*

Proof. Similarly to the proof of Lemma 3.3 in [12], we proceed by two steps.

Step 1. The sequence $\{u_j\}$ is bounded in X_0. Assume that $\{u_j\} \subset X_0$ is a sequence satisfying that

$$|J(u_j)| \le k \tag{3.6}$$

and

$$\sup\{|\langle J^{'}(u_j), \varphi\rangle| : \varphi \in X_0, \|\varphi\|_{X_0} = 1\} \to 0 \quad \text{as} \quad n \to \infty, \tag{3.7}$$

where k is some positive constant. For any $j \in \mathbb{N}$, according to (3.6) and (3.7), there exists a constant $b > 0$ such that

$$|J(u_j)| \le b \tag{3.8}$$

and

$$\frac{\langle J^{'}(u_j), u_j\rangle}{\|u_j\|_{X_0}} \le 2b. \tag{3.9}$$

From Proposition 2.3, (3.8), (3.9) and (2.5), it follows that

$$\begin{aligned}
4b(1 + \|u_j\|_{X_0}) &\ge 4[J(u_j) - \frac{1}{2}\langle J^{'}(u_j), u_j\rangle] \\
&= 2\iint\limits_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 K(x-y)dxdy - 2\int\limits_{\Omega} u_j^2 \log|u_j|dx \\
&\quad + \int\limits_{\Omega} u_j^2 dx - \frac{4}{q}\int\limits_{\Omega} |u|^q dx \\
&\quad - 2\iint\limits_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 K(x-y)dxdy + 2\int\limits_{\Omega} u_j^2 \log|u_j|dx \\
&\quad + 2\int\limits_{\Omega} |u_j|^{q-2} u_j^2 dx \\
&= \int\limits_{\Omega} u_j^2 dx - \frac{4}{q}\int\limits_{\Omega} |u_j|^q dx + 2\int\limits_{\Omega} |u_j|^{q-2} u_j^2 dx \\
&= \|u_j\|_2^2 - \frac{4}{q}\|u_j\|_q^q + 2\|u_j\|_q^q \\
&= \|u_j\|_2^2 + \frac{2q-4}{q}\|u_j\|_q^q.
\end{aligned}$$

Then, it is easy to derive that

$$\|u_j\|_2^2 \le 4b(1+\|u_j\|_{X_0}) \quad \text{and} \quad \|u_j\|_q^q \le \frac{2qb}{q-2}(1+\|u_j\|_{X_0}). \tag{3.10}$$

According to inequality (3.2) and (3.3), we have

$$\begin{aligned}
b &\ge |J(u_j)| \\
&\ge \frac{1}{4}\|u_j\|_{X_0}^2 + \frac{1}{4}\left(n + \frac{n}{s}\log\sqrt{\frac{\delta\pi^s}{2}} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u_j\|_2^2 - \frac{1}{2}\|u_j\|_2^2\log\|u_j\|_2 \\
&\quad + \frac{1}{4}\|u_j\|_2^2 - \frac{1}{q}\|u_j\|_q^q,
\end{aligned}$$

which yields

$$\begin{aligned}
\|u_j\|_{X_0}^2 &\le 4b - \left(n + \frac{n}{s}\log\sqrt{\frac{\delta\pi^s}{2}} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u_j\|_2^2 + 2\|u_j\|_2^2\log\|u_j\|_2 \\
&\quad - \|u_j\|_2^2 + \frac{4}{q}\|u_j\|_q^q \\
&\le 4b - \left(n + \frac{n}{s}\log\sqrt{\frac{\delta\pi^s}{2}} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})} + 1\right)\|u_j\|_2^2 \\
&\quad + 2C_p(\|u_j\|_2 + \|u_j\|_2^p) + \frac{4}{q}\|u_j\|_q^q \\
&\le 4b - 4b\left(n + \frac{n}{s}\log a + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})} + 1\right)(1+\|u_j\|_{X_0}) \\
&\quad + \frac{8b}{q-2}(1+\|u_j\|_{X_0}) \\
&\quad + 2C_p\left[2b^{1/2}(1+\|u_j\|_{X_0})^{1/2} + 2^p b^{p/2}(1+\|u_j\|_{X_0})^{p/2}\right] \\
&\le c_3\|u_j\|_{X_0} + c_4,
\end{aligned}$$

where $c_3 > 0$, $c_4 > 0$ are some suitable constants, independent of j, and $2 < p < \min\{4, 2_s^*\}$. Hence, the proof of Step 1 is complete.

Step 2. The sequence $\{u_j\}$ (up to a subsequence) converges strongly to u_0, that is, the following relation holds

$$\|u_j - u_0\|_{X_0} \to 0 \quad \text{as} \quad j \to \infty.$$

Recall that the sequence $\{u_j\}$ is bounded in X_0 and X_0 is reflexive Hilbert space. Then, there exists a subsequence of $\{u_j\}$ weakly convergent to u_0 in X_0. Without loss of generality, this subsequence is still denoted by $\{u_j\}$, that is

$$u_j \rightharpoonup u_0 \quad \text{in} \quad X_0. \tag{3.11}$$

First, making use of Proposition 2.3 again yields

$$
\begin{aligned}
&\langle J^{'}(u_j), u_j - u_0\rangle \\
&= \iint\limits_{\mathbb{R}^{2n}} [u_j(x) - u_j(y)][(u_j(x) - u_0(x)) - (u_j(y) - u_0(y))]K(x-y)dxdy \\
&\quad - \int\limits_{\Omega} u_j(u_j - u_0)\log|u_j|dx - \int\limits_{\Omega} |u_j|^{q-2}u_j(u_j - u_0)dx \\
&= \iint\limits_{\mathbb{R}^{2n}} [(u_j(x) - u_j(y))^2 - (u_j(x) - u_j(y))(u_0(x) - u_0(y))]K(x-y)dxdy \\
&\quad - \int\limits_{\Omega} u_j(u_j - u_0)\log|u_j|dx - \int\limits_{\Omega} |u_j|^{q-2}u_j(u_j - u_0)dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle J^{'}(u_0), u_j - u_0\rangle \\
&= \iint\limits_{\mathbb{R}^{2n}} [u_0(x) - u_0(y)][(u_j(x) - u_0(x)) - (u_j(y) - u_0(y))]K(x-y)dxdy \\
&\quad - \int\limits_{\Omega} u_0(u_j - u_0)\log|u_0|dx - \int\limits_{\Omega} |u_0|^{q-2}u_0(u_j - u_0)dx \\
&= \iint\limits_{\mathbb{R}^{2n}} [(u_j(x) - u_j(y))(u_0(x) - u_0(y)) - (u_0(x) - u_0(y))^2]K(x-y)dxdy \\
&\quad - \int\limits_{\Omega} u_0(u_j - u_0)\log|u_0|dx - \int\limits_{\Omega} |u_0|^{q-2}u_0(u_j - u_0)dx.
\end{aligned}
$$

Then, by the definition of the norm for space X_0, we have

$$
\begin{aligned}
\|u_j - u_0\|_{X_0}^2 &= \iint\limits_{\mathbb{R}^{2n}} [(u_j(x) - u_0(x)) - (u_j(y) - u_0(y))]^2 K(x-y)dxdy \\
&= \langle J^{'}(u_j), u_j - u_0\rangle - \langle J^{'}(u_0), u_j - u_0\rangle \\
&\quad + \int\limits_{\Omega} (u_j\log|u_j| - u_0\log|u_0|)(u_j - u_0)dx \\
&\quad + \int\limits_{\Omega} (|u_j|^{q-2}u_j - |u_0|^{q-2}u_0)(u_j - u_0)dx.
\end{aligned}
$$

Also, by using the embedding properties repeatedly, along with (3.11), we know that

$$u_j \to u_0 \quad \text{in} \quad L^{\nu}(\Omega) \tag{3.12}$$

and

$$u_j \to u_0 \quad \text{a.e. in} \quad \Omega, \tag{3.13}$$

as $j \to \infty$ for any $1 \leq \nu < 2_s^*$. Moreover, by the Hölder inequality, (3.12) and Step 1, we have that

$$\int_\Omega |u_j|^{q-2} u_j (u_j - u_0) dx \quad \text{as} \quad j \to \infty \tag{3.14}$$

and

$$\int_\Omega |u_0|^{q-2} u_0 (u_j - u_0) dx \quad \text{as} \quad j \to \infty. \tag{3.15}$$

According to the inequality (3.3), we get that

$$\begin{aligned}
&\int_\Omega (u_j \log |u_j| - u_0 \log |u_0|)(u_j - u_0) dx \\
&\leq 2 \int_\Omega |u_j \log |u_j|^{\frac{1}{2}}||u_j - u_0| dx - 2 \int_\Omega |u_0 \log |u_0|^{\frac{1}{2}}||u_j - u_0| dx \\
&\leq 2C_p \int_\Omega ||u_j|^{\frac{1}{2}} + |u_j|^{\frac{p}{2}}||u_j - u_0| dx - 2C_p \int_\Omega ||u_0|^{\frac{1}{2}} + |u_0|^{\frac{p}{2}}||u_j - u_0| dx.
\end{aligned}$$

Then, the Hölder inequality, (3.12) and the boundedness of $\{u_j\}$ ensure that

$$\int_\Omega ||u_j|^{\frac{1}{2}} + |u_j|^{\frac{p}{2}}||u_j - u_0| dx \to 0 \tag{3.16}$$

and

$$\int_\Omega ||u_0|^{\frac{1}{2}} + |u_0|^{\frac{p}{2}}||u_j - u_0| dx \to 0 \tag{3.17}$$

as $j \to \infty$. Meanwhile, by (3.7), (3.9) and the boundedness of $\{u_j\}$, it is easy to derive that

$$\langle J^{'}(u_j), u_j - u_0 \rangle \to 0 \quad \text{as} \quad j \to \infty. \tag{3.18}$$

and

$$\langle J^{'}(u_0), u_j - u_0 \rangle \to 0 \quad \text{as} \quad j \to \infty. \tag{3.19}$$

Therefore, the assertion of Step 2 comes for (3.17) and (3.18). □

Lemma 3.4. *Assume that* $2 < q < 2_s^*$ *and the operator* K *satisfies the conditions* (k_1)*–*(k_3)*. Then, we have that equation* (1.1) *has a nontrivial solution.*

Proof. Lemmas 3.1–3.3 imply that all the conditions for the proposition are satisfied. Thus, the functional $J(u)$ has a nontrivial critical point in X_0, that is to say, the equation (1.1) has a nontrivial solution. □

It should be pointed out that all these basic results given in this section will be used to obtain Theorem 3.5, which is our main result.

Now, with the aid of compactly embedding properties of the function space and the above propositions and lemmas, we shall assert the existence of the ground state solution for the nonlocal problem (1.1).

Theorem 3.5. *Assume that* $2 < q < 2_s^*$ *and the operator* K *satisfies the conditions* (k_1)*–*(k_3)*. Then, the problem* (1.1) *has a ground state solution.*

Proof. Let

$$\mathcal{N} = \{u \in X_0 \setminus \{0\}\} : J^{'}(u) = 0\}.$$

By the above proof, we know that $\mathcal{N}$ is nonempty. So for any $u \in \mathcal{N}$, we get Proposition 2.3. For any solution of (1.1) $u \in X_0$, due to that $J^{'}(u) = 0$, it follows that $\langle J^{'}(u), u\rangle = 0$. By Propositions 2.2 and 2.3, we can get

$$\begin{aligned}
0 &= \langle J^{'}(u), u\rangle \\
&= \iint\limits_{Q} |u(x) - u(y)|^2 K(x-y)dxdy - \int\limits_{\Omega} u^2 \log|u|dx - \int\limits_{\Omega} |u|^{q-2}u^2 dx \\
&= \iint\limits_{Q} |u(x) - u(y)|^2 K(x-y)dxdy - \int\limits_{\Omega} u^2(\log|u| - \log\|u\|_2 + \log\|u\|_2)dx \\
&\quad - \int\limits_{\Omega} |u|^{q-2}u^2 dx \\
&= \iint\limits_{Q} |u(x) - u(y)|^2 K(x-y)dxdy - \int\limits_{\Omega} u^2 \log\frac{|u|}{\|u\|_2}dx - \log\|u\|_2 \int\limits_{\Omega} u^2 dx \\
&\quad - \int\limits_{\Omega} |u|^{q-2}u^2 dx \\
&\geq \iint\limits_{Q} |u(x) - u(y)|^2 K(x-y)dxdy - \frac{a^2}{\delta\pi^s}\|u\|_{X_0} \\
&\quad + \frac{1}{2}\left(n + \frac{n}{s}\log a + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u\|_2^2 - \|u\|_2^2\log\|u\|_2 - \|u\|_q^q \\
&= \left(1 - \frac{a^2}{\delta\pi^s}\right)\|u\|_{X_0} + \frac{1}{2}\left(n + \frac{n}{s}\log a + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u\|_2^2 \\
&\quad - \|u\|_2^2\log\|u\|_2 - \|u\|_q^q.
\end{aligned}$$

Take $a = \sqrt{\frac{\delta\pi^s}{2}}$ into the above inequality and let

$$\kappa = \frac{1}{2}\left(n + \frac{n}{s}\log\sqrt{\frac{\delta\pi^s}{2}} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right).$$

Then, we obtain that

$$0 \geq \frac{1}{2}\|u\|_{X_0} + (\kappa - \log\|u\|_2)\,|u\|_2^2 - \|u\|_q^q. \tag{3.20}$$

For any $u \in \mathcal{N}$, if $\kappa - \log\|u\|_2 \leq 0$, that is $\|u\|_2 \geq e^{\kappa}$, by a simple calculation and $q > 2$, we have that

$$J(u) - \frac{1}{2}\langle J^{'}(u), u\rangle = \frac{1}{4}\|u\|_2^2 + \frac{q-2}{2q}\|u\|_q^q \geq \frac{1}{4}\|u\|_2^2 \geq \frac{1}{4}e^{\kappa}. \tag{3.21}$$

On the other side, if $\kappa - \log\|u\|_2 > 0$, by the expression of (3.20) and $X_0 \hookrightarrow L^q(\Omega)$, there exists a constant $c_{1,q} > 0$, independent of u, such that

$$\frac{1}{2}\|u\|_{X_0} \leq \|u\|_q^q \leq c_{1,q}\|u\|_{X_0}^q, \tag{3.22}$$

which yields that

$$\|u\|_{X_0} \geq (2c_{1,q})^{1/(q-1)} =: c_{2,q}. \tag{3.23}$$

Moreover, by adapting the process of (3.21), along with (3.22) and (3.23), there exists a constant $c_{3,q} > 0$, such that

$$\begin{aligned} J(u) - \frac{1}{2}\langle J^{'}(u), u\rangle &= \frac{1}{4}\|u\|_2^2 + \frac{q-2}{2q}\|u\|_q^q \geq \frac{q-2}{2q}\|u\|_q^q \\ &\geq \frac{c_{1,q}(q-2)}{2q}\|u\|_{X_0} \geq c_{3,q}. \end{aligned} \tag{3.24}$$

Notice that $\frac{1}{2}\langle J^{'}(u), u\rangle = 0$ holds for each $u \in \mathcal{N}$. So, combining (3.21) with (3.24), we get that

$$\inf_{u\in\mathcal{N}} J(u) \geq \bar{c} > 0,$$

where $\bar{c} = \min\left\{\frac{1}{4}e^{\kappa}, c_{3,q}\right\}$. This indicates that any limit points of the sequence in $\mathcal{N}$ are different from zero. Let the sequence $\{u_n\} \subset \mathcal{N}$ satisfy $\lim_{n\to\infty} J(u_n) = \inf_{u\in\mathcal{N}} J(u)$. Similarly to the process of Lemmas 3.3 and 3.4, we can assert that $\{u_n\}$ is bounded in X_0 and there is a subsequence converging strongly to $\bar{u}_0 \in X_0 \setminus \{0\}$. Hence, by $J^{'}(u_n) = 0$ and $J \in C^1(X_0, \mathbb{R})$, we can achieve readily that $J(\bar{u}_0) = 0$ and $J^{'}(\bar{u}_0) = 0$. Therefore, $u \in X_0$ is a ground state solution of the problem (1.1) as desired. □

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