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LIE GROUP ANALYSIS OF HEAT FLUX EFFECT ON MHD SECOND SLIP FLOW FOR A SLIGHTLY RAREFIED GAS PAST A STRETCHING SHEET WITH HEAT GENERATION

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Abstract

The present paper discusses steady MHD second order slip flow and heat transfer for a slightly rarefied gas due to an impermeable stretching sheet with heat flux and internal heat generation. By using the Lie group analysis, new similarity transformations are obtained. Employing these transformations, allows the partial differential equations governing the problem to transform into a system of ordinary differential equations which are later treated numerically using shooting method. Effects of the governing parameters on the dimensionless velocity and dimensionless temperature profiles are outlined graphically. Furthermore, results for the local skin-friction coefficient and the local Nusselt number are presented for some different values of the governing parameters in a tabular form. Also, results show that there is a strong dependency of the dimensionless temperature on the heat flux.

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Introduction

Of late, there has been a significant interest on the study of rarefied gas flows over a stretching sheet to obtain a thorough cognition for their behaviors and their various applications. Motivated by (FANG, AZIZ 2010), we are interested in acquiring the knowledge of heat transfer characteristics for the rarified MHD gas flow over a stretching sheet with heat flux and internal heat generation. The micro-electro-mechanical systems (MEMS) have an immense interest because in these systems, the slip flow regime and its behavior must be treated as a rarefied gas flow (FANG, AZIZ 2010). Also, for problems with low density, the fluid can be also treated as a rarefied gas flow, for example, in outer space applications (SHIDLOVSKIY 1967). The behavior of a rarefied gas flow can be determined by the Knudsen number K_n which defined as ($K_n = \frac{\lambda_e}{D_e}$) the mean free path ($\lambda_e$) divided by a characteristic length (D_e) for the flow. When Knudsen number is very small, no slip is observed between the surface and the fluid. Also, when Knudsen number lies in the range 0.001 to 0.1, slip occurs at the surface fluid interaction and the Navier-Stokes equations with the slip boundary conditions become applicable. But, for large values of Knudsen number, the Navier-stokes equations are not applicable and the kinetic theory of gases must be employed (MAHMOUD 2012). For the accurate prediction of gas flow and heat transfer in many applications second order slip boundary condition is critical. So, a significant amount of research on the fluid flow and heat transfer caused by stretched surfaces with second order slip under different conditions and in the presence of various physical effects has been reported (FANG et al. 2010, KHADER, MEGAHED 2014).

In our study, we will use the Lie-group method to derive the similarity solutions for our proposed problem. Lie group method is one of the most powerful methods in order to determine particular solutions of partial differential equations. It reduces the number of the independent variables of the partial differential equations under consideration and keeps the system of equations and associated initial and boundary condition invariant. The basic concepts of Lie group method can be found in Books (OLVER 1986, IBRAGIMOV 1994). In the field of viscous fluids there are many papers dealing with aspect of Lie group method. In 2001, YÜRÜSOY et al. presented exact solution of boundary layer equations for non-Newtonian fluids by using Lie group method. Lie group method used to study both problems of natural convection heat and mass transfer flow past an inclined plate for various parameters by (SIVASANKARAN et al. 2006). LIKEWISE, in 2010, MEKHEIMER et al. (2010) studied the Lie group analysis and similarity solutions for a couple stress fluids with heat transfer. The similarity reductions for the problem of heat and mass transfer over a moving porous plate with hydrodynamic slip and thermal convective boundary conditions were investigated by (HAMAD et al. 2012). By using the Lie group method (MINA, AMIN 2014) studied the nonlinear inviscid flows with a free surface under gravity. Also (HOSSAN 2015) applied lie group method to study the boundary layer flow and heat transfer of an electrically conducting viscous fluid over a stretching sheet. Thus, the main aim of the present work is to study the effects of heat flux and internal heat generation on the MHD flow and heat transfer for a rarefied gas over a stretching sheet with second order slip velocity.

Mathematical formulation

Let us consider a two-dimensional laminar MHD flow of slightly rarefied gas over a stretching sheet which exposed to a heat flux and internal heat generation in the presence of second order slip effect. The origin is located at a slit, through which the sheet is drawn through the fluid medium. The x-axis is chosen along the plane of the sheet and the y-axis is taken normal to the plane. We assume that the surface starts stretching from rest with the velocity u_w and temperature distribution T_w and the temperature of the fluid at the ambient is T_∞. On the other hand, the gas fluid is assumed to be an electrically conducting in the presence of a uniform magnetic field applied normal to the sheet, and the induced magnetic field is neglected under the approximation of small Reynolds number.

So, the governing equations of steady boundary layer flow are based on the continuity, momentum and the energy equations taking into account the effect of heat generation and magnetic field, which are given as :

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\mu}{\rho}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \frac{\sigma {\bar{B}_0}^2(\bar{x})}{\rho}\bar{u} \tag{2}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial \bar{y}^2} + \frac{Q(\bar{x})}{\rho c_p}(T - T_\infty) \tag{3}$$

with boundary conditions

$$\bar{y} = 0: \bar{u} = \bar{u}_w(\bar{x}) + a(\bar{x})\frac{\partial \bar{u}}{\partial \bar{y}} + b(\bar{x})\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \bar{v} = 0, -\kappa\frac{\partial T}{\partial \bar{y}} = \bar{q}_s(\bar{x}) \tag{4}$$

$$\bar{y} \to \infty: \bar{u} \to 0, T \to T_\infty \tag{5}$$

Introducing the non-dimensional parameters:

$$x=\frac{\bar{x}}{L},\quad y=\frac{\bar{y}}{L}\sqrt{\mathrm{Re}},\quad u=\frac{\bar{u}}{U_0},\quad v=\frac{\bar{v}}{U_0}\sqrt{\mathrm{Re}},\quad q_s=\frac{L\bar{q}_s}{\kappa\sqrt{\mathrm{Re}}},$$
$$Q(x)=\frac{\bar{Q}(\bar{x})}{Q_0}\sqrt{\mathrm{Re}},\quad \theta=\frac{(T-T_\infty)}{q_s(x)},\quad u_w=\frac{\bar{u}_w(x)}{U_0} \tag{6}$$

where $\mathrm{Re}=\dfrac{LU_0}{\nu}$ is the Reynolds number, U_0, L are the characteristic velocity and the characteristic length.

Use $u=\dfrac{\partial\psi}{\partial y}$, $v=-\dfrac{\partial\psi}{\partial x}$ we get

$$H_1=\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\,\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^3\psi}{\partial y^3}+M^*{B_0}^2(x)\frac{\partial\psi}{\partial y}=0 \tag{7}$$

$$H_2=\frac{1}{q_s(x)}\frac{dq_s}{dx}\frac{\partial\psi}{\partial y}\theta+\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{1}{Pr}\frac{\partial^2\theta}{\partial y^2}+\gamma^*Q(x)\theta=0 \tag{8}$$

The boundary conditions (4) and (5) will be

$$y=0;\quad \frac{\partial\psi}{\partial y}=u_w(x)+a^*(x)\frac{\partial^2\psi}{\partial y^2}+b^*(x)\frac{\partial^3\psi}{\partial y^3},\quad \frac{\partial\psi}{\partial x}=0,\ \frac{\partial\theta}{\partial y}=-1 \tag{9}$$

$$y\to\infty;\quad \frac{\partial\psi}{\partial y}\to 0,\quad \theta\to 0 \tag{10}$$

where $M^*=\dfrac{\sigma L}{\rho U}$, $\gamma^*=\dfrac{L}{\rho c_p U}$, $a^*(x)=\dfrac{\sqrt{\mathrm{Re}}}{L}a(x)$, $b^*(x)=\dfrac{\mathrm{Re}}{L}b(x)$.

Symmetry analysis and infinitesimal generators

In this section, we apply the techniques of Lie group theory to the equations (7) and (8). Details of the theory can be found in (Olver 1986, Ibragimov 1994). To perform this task, we consider the one-parameter (ε) Lie group of infinitesimal transformation in (x, y, ψ, θ) which given by :

$$\bar{x}=x+\varepsilon\xi_1(x,y,\psi,\theta)+o(\varepsilon^2)$$
$$\bar{y}=y+\varepsilon\xi_2(x,y,\psi,\theta)+o(\varepsilon^2)$$
$$\bar{\psi}=\psi+\varepsilon\eta_1(x,y,\psi,\theta)+o(\varepsilon^2)$$
$$\bar{\theta}=\theta+\varepsilon\eta_2(x,y,\psi,\theta)+o(\varepsilon^2) \tag{11}$$

here (ξ_1, ξ_2, η_1, η_2) are the infinitesimal transformation of the variables x, y, ψ, θ.

The corresponding infinitesimal generator of Lie group is

$$V = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \eta_1 \frac{\partial}{\partial \psi} + \eta_2 \frac{\partial}{\partial \theta}. \tag{12}$$

Now, the infinitesimal transformations ξ_1, ξ_2, η_1 and η_2 can be determined from the following invariance conditions:

$$V^{(3)}\ |H_i|_{H_i=0} = 0, \quad i = 1,2 \tag{13}$$

here: $H_i = 0$, $i = 1, 2$ represent the equations (7) and (8) , $V^{(3)}$ is the third prolongation of infinitesimal generator V and $V^{(3)}$ is given by

$$V^{(3)} = V + \eta_{1x} \frac{\partial}{\partial \psi_x} + \eta_{1y} \frac{\partial}{\partial \psi_y} + \eta_{2x} \frac{\partial}{\partial \theta_x} + \eta_{2y} \frac{\partial}{\partial \theta_y} + \eta_{1xx} \frac{\partial}{\partial \psi_{xx}} + \\ \eta_{1yy} \frac{\partial}{\partial \psi_{yy}} + \eta_{2xx} \frac{\partial}{\partial \theta_{xx}} + \eta_{2yy} \frac{\partial}{\partial \theta_{yy}} + \eta_{1yyy} \frac{\partial}{\partial \psi_{yyy}}, \tag{14}$$

where η_{1x}, η_{1y}, η_{1xx}, η_{1yy}, η_{2x}, η_{2y}, η_{2yy} and η_{1yyy}, can be calculated from the following equations:

$$\eta_{1_x} = D_x(\eta_1) - \psi_x D_x(\xi_1) - \psi_y D_x(\xi_2), \quad \eta_{1_y} = D_y(\eta_1) - \psi_x D_y(\xi_1) - \psi_y D_y(\xi_2),$$

$$\eta_{2_x} = D_x(\eta_2) - \theta_x D_x(\xi_1) - \theta_y D_x(\xi_2), \quad \eta_{2_y} = D_y(\eta_2) - \theta_x D_y(\xi_1) - \theta_y D_y(\xi_2),$$

$$\eta_{1_{xx}} = D_x(\eta_{1_x}) - \psi_{xx} D_x(\xi_1) - \psi_{yx} D_x(\xi_2), \quad \eta_{1_{yy}} = D_y(\eta_{1_y}) - \psi_{xy} D_y(\xi_1) - \psi_{yy} D_y(\xi_2),$$

$$\eta_{2xx} = D_x(\eta_{2x}) - \theta_{xx} D_x(\xi_1) - \theta_{yx} D_x(\xi_2), \quad \eta_{2yy} = D_y(\eta_{2y}) - \theta_{xy} D_y(\xi_1) - \theta_{yy} D_y(\xi_2),$$

$$\eta_{1_{xy}} = D_x(\eta_{1_{xy}}) - \psi_{yx} D_x(\xi_1) - \psi_{yy} D_x(\xi_2),$$

$$\eta_{1_{yyy}} = D_y(\eta_{1_{yy}}) - \psi_{xyy} D_y(\xi_1) - \psi_{yyy} D_y(\xi_2) \tag{15}$$

where D_x, D_y are the operators of total differentiation with respect to x and y, respectively. By applying the third prolongation (14) to the original equations (7) and (8), we get

$$\eta_{1_y}\psi_{xy} + \eta_{1_{xy}}\psi_y - \eta_{1_x}\psi_{yy} - \eta_{1_{yy}}\psi_x - \eta_{1_{yyy}} + M^* B^2(x)\eta_{1_y} + 2M^*\xi_1 \frac{dB(x)}{dx} = 0 \tag{16}$$

$$\left(\frac{d}{dx}\left(\frac{1}{q_w(x)}\frac{dq_w}{dx}\right)\right)\xi_1\psi_y\theta + \frac{1}{q_w(x)}\frac{dq_w}{dx}\eta_2\psi_y + \frac{1}{q_w(x)}\frac{dq_w}{dx}\eta_{1_y}\theta + \eta_{1_y}\theta_x + \\ \eta_{2_x}\psi_y - \eta_{1_x}\theta_y - \eta_{2_y}\psi_x - \frac{1}{\Pr}\eta_{2_{yy}} - \gamma^*\xi_1\frac{dQ(x)}{dx}\theta - \gamma^* Q(x)\eta_2 = 0 \tag{17}$$

Substituting about η_{1x}, η_{1y}, η_{2x}, η_{2y}, η_{1xy}, η_{1yy}, η_{2yy} and η_{1yyy} from equations (15) into equations (16), (17) and setting the coefficients involving ψ_x, ψ_y, ψ_{xx}, ψ_{yy}, ψ_{xy}, ψ_{xyy}, ψ_{yyy}, θ_x, θ_y, θ_{xy}, θ_{xx}, θ_{xy}, θ_{yy} and various products to zero which may be leads to a system of coupled partial differential equations. So, we have:

$$\xi_1 = d_1 + d_2 x, \quad \xi_2 = d_3 y + F(x), \quad \eta_1 = (d_2 - d_3)\psi, \quad \eta_2 = d_3\theta \tag{18}$$

where $\mathbf{d_1}$, $\mathbf{d_2}$ and $\mathbf{d_3}$ are arbitrary constants and $\boldsymbol{F(x)}$ is an arbitrary functions depends on x. The functions $B(x)$, $q_w(x)$ and $Q_w(x)$ are to be determined from the following equations:

$$\frac{dB(x)}{dx} + \left(\frac{d_3}{d_1 - d_2 x}\right) B(x) = 0, \quad \frac{dQ(x)}{dx} + \left(\frac{2d_3}{d_1 - d_2 x}\right) Q(x) = 0,$$
$$\frac{dq_s(x)}{dx} - \left(\frac{d_3}{d_1 + d_2 x}\right) q_w(x) = 0. \tag{19}$$

This implies that, the system of nonlinear equations (7) and (8) has the four-parameter Lie group of point symmetries generated by generators

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi}, \quad V_3 = y\frac{\partial}{\partial y} - \psi\frac{\partial}{\partial \psi} + \theta\frac{\partial}{\partial \theta}, \quad V_4 = F(x)\frac{\partial}{\partial y} \tag{20}$$

From the invariance of the boundary conditions, we get $F(x) = 0$ and the functions $u_w(x)$, $a(x)$ and $b(x)$ should satisfy the following equations

$$\frac{du_w(x)}{dx} - \left(\frac{d_2 - 2d_3}{d_1 - d_2 x}\right) u_w(x) = 0, \quad \frac{da^*(x)}{dx} - \left(\frac{d_3}{d_1 - d_2 x}\right) a^*(x) = 0,$$
$$\frac{db^*(x)}{dx} - \left(\frac{2d_3}{d_1 + d_2 x}\right) b^*(x) = 0 \tag{21}$$

From equations (19) and (20) can be obtained on different forms for the functions $B(x)$, $Q(x)$, $u_w(x)$, $a^*(x)$, $q_s(x)$ and $b^*(x)$ via choosing the constants d_1, d_2 and d_3 as follows.

Setting $d_1 = 1$, $d_2 = 1$ and $d_3 \neq 1$ we get:

$$u_w = A_1 e^{-2d_3 x}, \; a^*(x) = \lambda_1 e^{d_3 x}, \quad b^*(x) = \lambda_2 e^{2d_3 x}, \; B(x) = B_0 e^{-d_3 x},$$
$$q_s(x) = A_2 e^{d_3 x}, \quad Q(x) = Q_0 e^{-2d_3 x} \tag{22}$$

Considered the general case in which both constants are involved

$$u_w(x) = A_1(d_1 + d_2 x)^{\frac{d_2 - 2d_3}{d_2}}, a^*(x) = \lambda_1 (d_1 + d_2 x)^{\frac{d_3}{d_2}}, \quad b^*(x) = \lambda_2 (d_1 + d_2 x)^{\frac{2d_3}{d_2}},$$
$$Q_0(x) = Q_0(d_1 + d_2 x)^{\frac{-d_3}{d_2}}, q_w(x) = A_4 (d_1 + d_2 x)^{\frac{-d_3}{d_2}}, \quad B(x) = B_0 (d_1 + d_2 x)^{\frac{-d_3}{d_2}} \tag{23}$$

Choosing d_1, $d_2 \neq 0$ and $d_3 = 0$ we have:

$$u_w = A_1(d_1 + d_2 x), \quad a(x) = \lambda_1, \quad b(x) = \lambda_2, q_w(x) = q_{0s}, \quad B(x) = B_0, \quad Q(x) = Q_0 \tag{24}$$

where A_1, A_2, A_3, q_{0s}, A_4, λ_1, λ_2, B_0 and Q_0 are arbitrary constants.

The solutions of equations (7) and (8) are invariant under infinitesimal transformations (11) if $V(\psi - \psi(x, y)) = 0$ when $\psi = \psi(x, y)$ and $V(\theta - \theta(x, y)) = 0$ when $\theta = \theta(x, y)$.

These conditions can be rewritten as

$$\xi_1 \frac{\partial \psi}{\partial x} + \xi_2 \frac{\partial \psi}{\partial y} = 0, \qquad \xi_1 \frac{\partial \theta}{\partial x} + \xi_2 \frac{\partial \theta}{\partial y} = 0 \tag{25}$$

Equation (25) is called the invariant surface conditions, which are quasi-linear equations. The subsidiary equations can be expressed as:

$$\frac{dx}{\xi_1(x, y, \psi, \theta)} = \frac{dy}{\xi_2(x, y, \psi, \theta)} = \frac{d\psi}{\eta_1(x, y, \psi, \theta)} = \frac{d\theta}{\eta_2(x, y, \psi, \theta)} \tag{26}$$

From the solution of characteristic equation (26), we get on three constants one of them is called similarity variable and other called similarity functions.

Reduction to ordinary differential equations

Here, we will consider the various reductions of the partial differential equations (7) and (8).

1. We consider the combination $V_1 + d_3 V_3$, then the characteristic equations would be

$$\frac{dx}{1} = \frac{dy}{d_3 y} = \frac{d\psi}{-d_3 \psi} = \frac{d\theta}{d_3 \theta} \tag{27}$$

Solving equation (27), we have the following similarity variable and functions

$$\eta = y e^{-d_3 x}, \qquad \psi = e^{-d_3 x} f(\eta), \quad \theta = e^{d_3 x} G(\eta). \tag{28}$$

Inserting the similarity variable $\boldsymbol{\eta}$ and functions into equations (7) and (8) yields the following ordinary differential system:

$$f''' - d_3 f f'' + 2 d_3 f'^2 - M f' = 0, \tag{29}$$

$$G'' - \Pr(d_3 f G' + 2 d_3 f' G - \gamma G) = 0 \tag{30}$$

and boundary conditions are transformed to

$$f'(0) = 1 + \lambda_1 f''(0) + \lambda_2 f'''(0), \quad f(0) = 0, \quad G'(0) = -1, \text{ at } \eta = 0 \tag{31}$$

$$f'(\infty) = 0, \quad G(\infty) = 0, \text{ at } \eta \to 0 \tag{32}$$

where the functions $\underline{u_w}(x)$, $a(x)$, $b(x)$, $B(x)$, $Q(x)$ are defined in equation (22) and $A_1 = 1$, $M = M^* B_0^2$, $\gamma = \gamma^* Q_0$.

2. Consider the general case $d_1V_1 + d_2V_2 + d_3V_3$, then equation (26) take the form

$$\frac{dx}{d_1 + d_2 x} = \frac{dy}{d_3 y} = \frac{d\psi}{(d_2 - d_3)\psi} = \frac{d\theta}{d_3\theta} \tag{33}$$

from equation (33), we get

$$\eta = y(d_1 + d_2 x)^{-\frac{d_3}{d_2}}, \quad \psi = (d_1 + d_2 x)^{\frac{d_2 - d_3}{d_2}} f(\eta), \quad \theta = (d_1 + d_2 x)^{\frac{d_3}{d_2}} G(\eta) \tag{34}$$

Consequently, substituting equation (34) into equations (7) and (8) and boundary conditions (9) and (10), we obtain the following similarity equations

$$f''' + (d_2 - d_3)ff'' - (d_2 - 2d_3)f'^2 - Mf' = 0 \tag{35}$$

$$G'' + \Pr((d_2 - d_3)fG' - (A_4 + 1)d_3 f'G + \gamma G) = 0 \tag{36}$$

and boundary conditions take the form

$$f'(0) = 1 + \lambda_1 f''(0) + \lambda_2 f'''(0), \quad f(0) = 0, \quad G'(0) = -1, \text{ at } \eta = 0 \tag{37}$$

$$f'(\infty) = 0, \quad G(\infty) = 0, \text{ at } \eta \to 0 \tag{38}$$

where the functions $\underline{u_w}(x)$, $a(x)$, $b(x)$, $B(x)$, $Q(x)$ are defined in equation (23) and $A_1 = 1$, $M = M^* B_0^2$, $\gamma = \gamma^* Q_0$.

3. Consider $d_1V_1 + d_2V_2$, from equation (26) we have

$$\eta = y, \qquad \psi = (d_1 + d_2 x) f(\eta), \quad \theta = G(\eta) \tag{39}$$

Substituting from equation (39) into equations (7) to (10) we get the following system of ordinary differential equations:

$$f''' + d_2 ff'' - d_2 f'^2 - Mf' = 0 \tag{40}$$

$$G'' + \Pr(d_2 fG' + \gamma G) = 0 \tag{41}$$

and boundary conditions will be

$$f'(0) = 1 + \lambda_1 f''(0) + \lambda_2 f'''(0), \quad f(0) = 0, \quad G'(0) = -1, \text{ at } \eta = 0 \tag{42}$$

$$f'(\infty) = 0, \quad G(\infty) = 0, \text{ at } \eta \to 0 \tag{43}$$

where the functions $\underline{u_w}(x)$, $a(x)$, $b(x)$, $B(x)$, $Q(x)$ and $\underline{q_w}(x)$ are defined in equation (24) $A_1 = 1$, $M = M^* B_0^2$, $\gamma = \gamma^* Q_0$.

The physical quantities of interest are the local skin-friction coefficient C_f and the local Nusselt number Nu_x which are defined as:

$$C_f = \frac{2\tau_w}{\rho u_w{}^2}, \quad Nu_x = \frac{xq_s}{\kappa(T_w - T_\infty)} \tag{44}$$

Further, τ_w is the shear stress which given by :

$$\tau_w = -\mu(\frac{\partial u}{\partial y})_{y=0} \tag{45}$$

Using Eqs. (39) (The third case), we obtain:

$$\frac{1}{2}\mathrm{Re}_x^{\frac{1}{2}} C_f = -f''(0), \qquad Nu_x \mathrm{Re}_x^{-\frac{1}{2}} = \frac{1}{\theta(0)} \tag{46}$$

where $\mathrm{Re}_x = \frac{xu_w(x)}{\nu}$ is the local Reynolds number.

Results and discussion

In this section, to get clear insight of the physical problem, numerical results for the second order slip flow and heat transfer for a slightly rarefied gas within a boundary layer in the presence of heat generation/absorption, magnetic field and heat flux effects was performed with the purpose of identifying the characteristics of the gas flow over a stretching sheet (third case). Figures 1-7 depict the graphical illustrations of the various controlling parameters on the velocity and temperature profiles. The dimensionless velocity profiles for some selected values of magnetic parameter M are plotted in Figure 1. It is apparent that the velocity decreases along the surface with an increase in the magnetic parameter. Physically, it is well known that the magnetic field presents a damping effect on the velocity field by creating linear magnetic drag force in the form of ($-Mf'$) which appear in the non-dimensional momentum boundary layer equation (40), is directly proportional to M. Therefore greater retarding effect is generated in the flow with greater M values which causes a decrease for the velocity distribution inside the momentum boundary layer.

Figure 2, shows the effect of magnetic parameter M on the temperature profiles above the surface. It is noticed that, an increase in the parameter M has the effect of increasing the temperature distribution, the wall temperature $\theta(0)$ and the thermal boundary layer thickness.

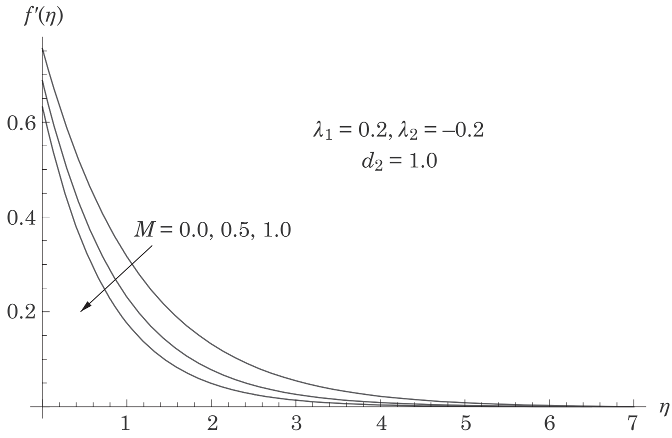

Fig. 1. The velocity distribution for various values of M

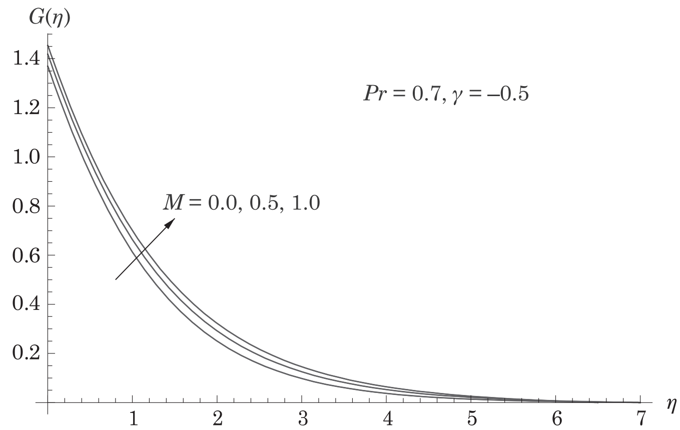

Fig. 2. The temperature distribution for various values of M

The effects of the first order velocity slip parameter on the dimensionless velocity and temperature profiles are depicted in Figures 3 and 4, respectively. It is clear from these figures that the velocity decreases with the increase of the first order velocity slip parameter, while the temperature distribution and the wall temperature $\theta(0)$ are increased with the increase of the same parameter. Physically, as the slip parameter increases in magnitude, causes a creation for the friction force which permit more fluid to slip past the sheet, the flow slows down for distances close to the sheet and the temperature rise due to the presence for this force.

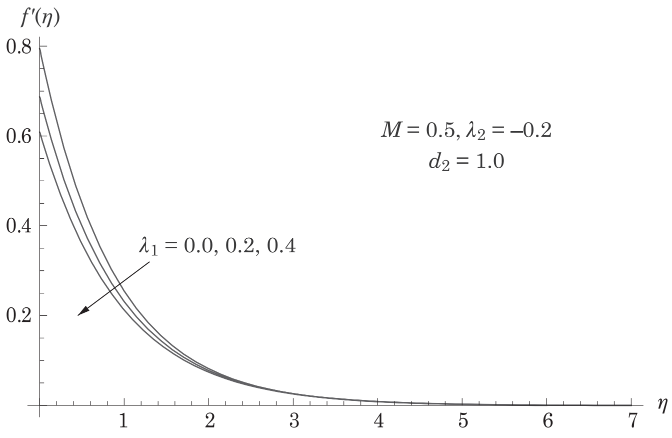

Fig. 3. The velocity distribution for various values of λ_1

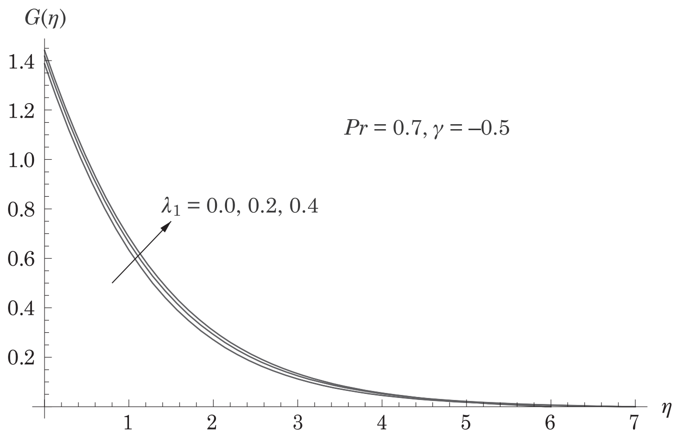

Fig. 4. The temperature distribution for various values of λ_1

Figures 5 and 6, illustrate the effects of the second velocity slip parameter on the dimensionless velocity and temperature profiles, respectively. It can be seen that the dimensionless velocity is gradually reduced with increasing the amount of the absolute value of second velocity slip parameter but the reverse is true for the temperature distribution along the thermal boundary layer and the wall temperature $\theta(0)$.

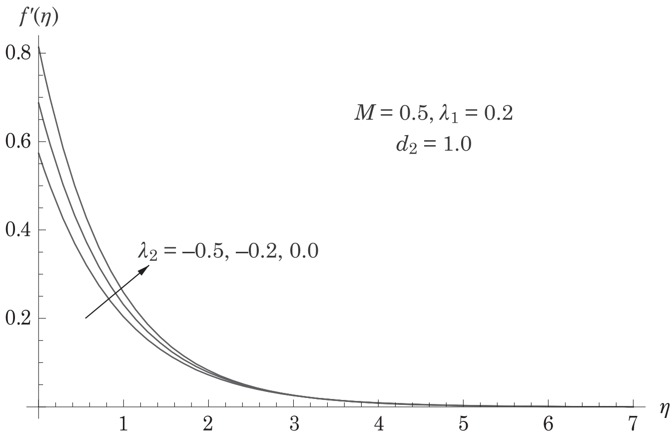

Fig. 5. The velocity distribution for various values of λ_2

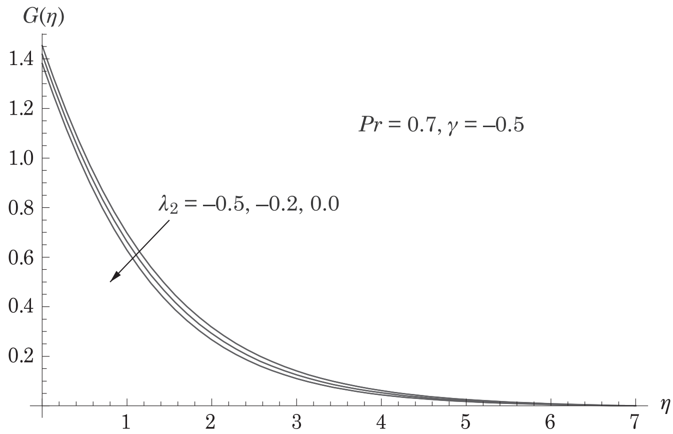

Fig. 6. The temperature distribution for various values of λ_2

Figure 7 shows the temperature profile against the similarity variable η for various values of heat generation/absorption parameter γ. This figure shows that the heat generation or absorption has a profound effect on the thermal boundary layer thickness in which the absorption parameter $\gamma < 0$ reduces the thermal boundary layer thickness and the wall temperature $\theta(0)$, whereas heat generation parameter $\gamma > 0$ thickens the thermal boundary layer and increases the wall temperature $\theta(0)$. However, the net effect for the absorption parameter is to slow down the temperature distribution but the reverse is true for the heat generation parameter.

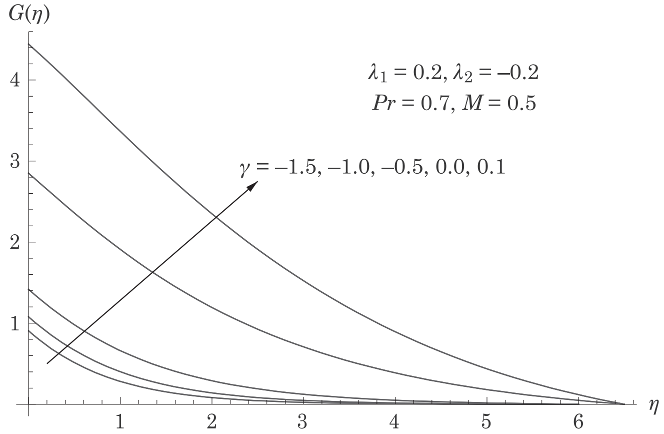

Fig. 7. The temperature distribution for various values of γ

Finally, to show the behavior of the quantities of relevant physical interest like the local skin-friction coefficient $\frac{1}{2}\mathrm{Re}_x^{1/2}C_f$ and the local Nusselt number $Nu_x\mathrm{Re}_x^{-1/2}$ with changes in the first order slip velocity parameter λ_1, the second order slip velocity parameter λ_2, the magnetic parameter M and the heat generation/absorption parameter γ. One can then see from Table 1 that, increases in the first order slip velocity parameter or the absolute value of the second order slip velocity parameter leads to a decrease in both the local skin-friction coefficient and the local Nusselt number. Physically, when the slip velocity on the surface rises, friction between the fluid and the surface is reduced. With decreasing friction, the heat generated on the surface which transferred to the flow is reduced. This leads to a decrease in the rate of heat transfer. Also, it is observed from the same table that, an increase in the magnetic parameter causes an increase in the local skin-friction coefficient but the reverse trend is noted for the local Nusselt number. On the other hand, increases in the heat absorption parameter leads to an enhancement in the local Nusselt number. Likewise, an increase in the heat generation parameter causes a decrease in the local Nusselt number.

Table 1

Values of the local skin-friction coefficient $\frac{1}{2}\mathrm{Re}_x^{1/2}C_f$ and the local Nusselt number $Nu_x\mathrm{Re}_x^{-1/2}$ for various values of M, λ_1, λ_2 and γ with Pr = 0.7

$Nu_x\mathrm{Re}_x^{-1/2}$	$\frac{1}{2}\mathrm{Re}_x^{1/2}C_f$	γ	λ_2	λ_1	M
0.730256	0.656093	-0.5	-0.2	0.2	0.0
0.705121	0.748668	-0.5	-0.2	0.2	0.5
0.687691	0.807587	-0.5	-0.2	0.2	1.0
0.720946	0.903753	-0.5	-0.2	0.0	0.5
0.705121	0.748668	-0.5	-0.2	0.2	0.5
0.693715	0.640927	-0.5	-0.2	0.4	0.5
0.687897	0.594106	-0.5	-0.5	0.2	0.5
0.705121	0.748668	-0.5	-0.2	0.2	0.5
0.723733	0.93238	-0.5	0.0	0.2	0.5
1.103830	0.748668	-1.5	-0.2	0.2	0.5
0.927601	0.748668	-1.0	-0.2	0.2	0.5
0.705121	0.748668	-0.5	-0.2	0.2	0.5
0.350826	0.748668	0.0	-0.2	0.2	0.5
0.225079	0.748668	0.1	-0.2	0.2	0.5

Conclusion

This article deals with the effects of MHD second order slip on a slightly rarefied gas flow and heat transfer over a stretching surface in the presence of internal heat generation and heat flux. The governing partial differential equations for the flow and temperature fields are reduced to a system of coupled nonlinear ordinary differential equations by deducing suitable similarity transformations via Lie group analysis. These nonlinear differential equations are then solved numerically by the shooting method coupled with the fourth-order Runge Kutta scheme. As here it clearly brings out, the rate of heat transfer decreases with an increase in the heat generation parameter, magnetic parameter, the first order slip parameter and the absolute value of the second order slip velocity parameter. Also, it was observed that the local Nusselt number increases as the heat absorption parameter increases. Thus fast cooling of the stretching sheet can be achieved by implementing this effect. Finally, a large value of the first order slip velocity parameter and the absolute value of the second order slip velocity parameter lead to a decrease in the value of the skin-friction coefficient.

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