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OPTIMAL REPLACEMENT PERIOD WITH REPAIR COST LIMIT AND CUMULATIVE DAMAGE MODEL

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This paper deals with periodical replacement model with single repair cost limit under cumulative damage process. The system is subject to two types of shocks. Type I shock causes damage to the system. The total damage is additive, and it causes a serious failure eventually if the total additive damage exceeds a failure level K. Type II shock causes the system to a minor failure, which can be maintained by minimal repair if the estimated repair cost is smaller than a predetermined repair-cost limit L_S or by preventive replacement if the estimated repair cost is larger than L_S. The system is also replaced at scheduled time T or at serious failure. The long-term expected cost per unit time is derived using the expected costs as the optimality criterion. The minimum-cost policy is derived, and existence and uniqueness are proved.

Keywords: *periodical replacement model, cumulative damage model, repair cost limit, minimal repair.*

Niniejszy artykuł dotyczy modelu wymiany okresowej z ograniczeniem kosztów pojedynczej naprawy w ramach procesu sumowania uszkodzeń. Układ podlega dwóm rodzajom zaburzeń. Zaburzenie I typu powoduje uszkodzenie systemu. Uszkodzenie całkowite sumuje się, powodując w końcu poważną awarię jeśli łączna wartość uszkodzeń przekroczy poziom awarii K. Zaburzenie II typu powoduje drobną awarię systemu, która może zostać usunięta dzięki minimalnej naprawie jeśli przewidywany koszt naprawy będzie mniejszy niż zakładany limit kosztów naprawy L_S lub na drodze wymiany prewencyjnej, jeżeli przewidywany koszt naprawy będzie większy niż L_S. Układ również podlega wymianie w założonym czasie T lub w przypadku poważnej awarii. Długoterminowe przewidywane koszty na jednostkę czasu obliczono z wykorzystaniem przewidywanych kosztów jako kryterium optymalności. Wyprowadzono strategię minimalnych kosztów, udowadniając istnienie i jedyność.

Słowa kluczowe: *Model wymiany okresowej, model sumowania uszkodzeń, limit kosztów naprawy, naprawa minimalna.*

1. Introduction

Preventing unexpected failure of the system during production process is very important because the production loss from system failure is very expensive, sometimes is very dangerous. In such situation, it is wise to replace the system before failure. Therefore, preventive maintenance (PM) models with regard to deteriorating systems have widely attracted the attention of several researchers and practitioners.

The system suffers external shocks, and then these shocks can incur damage to the system. Lai *et al.* (2006) divided shock models into five categories depending on the effect of shock damage to the system: (1) Cumulative damage model; (2) Instantaneous failure model; (3) Increasing operating cost model; (4) Increasing failure rate model; and (5) δ-shock model. On cumulative damage model, the system is subjected to shocks and suffers some amount of damage such as wear, fatigue, crack growth, creep, and dielectric at each shock. The total damage due to shocks is additive, and the system can fail when the total damage exceeds a failure level. The reliabiity properties and preventive maintennace policies for variuos damage models were summarized sufficiently in Nakagawa (2007).

In cumulative damage model, many researchers have used the optimum control-limit policy where a system is replaced when the total damage exceeds a threshold level (Nagakawa (1976)). On the other hand, the replacement models where a system is replaced at a planned time T were proposed in Taylor (1975), Mizuno (1986), and Perry (2000). Furthermore, the replacement models where a system is replaced at shock N were proposed in Nagakawa (1984).

Nakagawa and Kijima (1989) applied the periodical replacement policy with minimal repair at failure to a cumulative damage model and obtained the optimal values T^*, N^*, and Z^*, individually. Kijima and Nakagawa (1991) considered a cumulative damage shock model with imperfect PM policy. Satow and Nakagawa (1997) presented a modified cumulative damage model and considered a system suffers two kinds of damage. They proposed three replacement policies as following: the system can be replaced before failure at time T, at shock N or at damage level k, where k is less than failure level K. The optimal values T^*, N^*, and k^* which minimize the expected cost rates of three replacement policies are obtained individually. Satow *et al.* (2000) continued the work of Satow and Nakagawa (1997). The system is preventively replaced when the cumulative damage exceeds a threshold k. The optimal value k^* which minimizes the expected cost is obtained.

Qian *et al.* (1999) presented an extended cumulative damage model with two kinds of shocks: one is failure shock and the other is damage shock at which it suffers only damage. The system is replaced at scheduled time T or at failure. Qian *et al.* (2003) considered an extended cumulative damage model with maintenance at each shock and minimal repair at each failure. The optimal values T^* and N^* which minimize the expected cost are obtained. Qian *et al.* (2005) considered a cumulative damage model, where the system undergoes the PM at a certain time T or the total damage exceeds a managerial level k. The optimal values T^* and k^* are obtained simultaneously.

Ito and Nakagawa (2011) considered three cumulative damage models: (1) a unit is subjected to shocks and suffers some damage due to shocks. (2) The amount of damage due to shocks is measured only

at periodic time. (3) The amount of damage increases linearly with the time. The unit fails when the total damage has exceeded a failure level K. The optimal T^* for Models 1 & 3 and N^* for Model 2 are obtained. Recently, Nakagawa (2007) summarized a large amount of preventive maintenance optimization problems for cumulative damage model.

In PM models with minimal repair, Drinkwater and Hastings (1967) introduced firstly the concept of repair cost limit. When a unit fails, repair cost is estimated and minimal repair is then executed in case the estimated cost is less than a predetermined limit; otherwise, the unit is replaced. PM models with repair-cost limit policy have been discussed in several articles. Several extensions of these policies have been proposed in Kapur *et al.* (1983), Park (1985), Bai and Yun (1986), Kapur and Garg (1989), and Yun and Bai (1987, 1988). Dohi *et al.* (2000) applied the TTT method to determine the optimal repair-time limit, in which it wants to minimize the long-run expected cost per unit time in the steady state case. Continuously, Dohi *et al.* (2003) discussed it to minimize the expected total discounted over an infinite time horizon.

In this paper, we consider a periodical replacement policy incorporating with the concept of repair cost limit under a cumulative damage model. The outline of this paper is as follows. In Section 2, the problem is defined. The long-term expected cost per unit time $A(T, L_s)$ and the conditions characterising the optimal period T^* are derived in Section 3. Finally, a numerical example and conclusions are presented in Sections 4 and 5, respectively.

2. Problem Definition

A system subjected to external shocks is considered and these shocks are supposed to occur randomly at a non-homogeneous Poisson process with an intensity function $\lambda(t)$. These shocks can be divided into two types: type I shock and type II shock. A shock whenever occurs is type I and type II with probailities p $(0 < p \le 1)$ and $q = (1 - p)$, respectively. Thus, we can know that the occurrences of type I and type II shocks are according to two non-homogeneous Poisson processes $\{N_1(t), t \ge 0\}$ and $\{N_2(t), t \ge 0\}$ with intensity rates $p\lambda(t)$ and $q\lambda(t)$, respectively. The effects of two types of shock to the system are described as follows:

Type I shocks cause the damage to the system and these damages are additive. When a type I shock occurs, an amount X_i of damage due to this shock has a probability distribution $H_i(x) = P(X_i \le x)$ and a finite mean μ_x, $i = 1, 2, 3, \dots$. Then the accumulated damage to the j-th type I shock after the installation $W_j = \sum_{i=1}^{j} X_i$ has a distribution function:

$$P(W_j \le w) = H^{(j)}(w) = \begin{cases} 1 & j = 0 \\ H_1 * H_2 * \cdots * H_j(w), & j = 1, 2, 3, \cdots \end{cases} \quad (1)$$

where the "*" mark is denoted the *Stieltjes* convolution, i.e., $a * b(t) \equiv \int_0^t b(t - u) da(u)$ for any function $a(t)$ and $b(t)$. The probabiliity that the number of type I shocks occurred in $[0, t)$ equals to j is given by.

$$P(N_1(t) = j) = \frac{(m_1(t))^j \exp(-m_1(t))}{j!} = P_{1j}(t), \quad (2)$$

where $m_1(t) = \int_0^t p\lambda(x) dx$ denote the mean number of type-I shocks occurred in $[0, t)$.

If the accumulated damage exceeds a failure level K, then a serious failure occurs and the system must be replaced by a new one. The probability that a serious failure occurs at j-th type I shock is $H^{(j-1)}(K) - H^{(j)}(K)$. Let random variable Z denote the occurrence time of the first serious failure, so the survival function of Z is given by

$$\overline{F_z}(t) = P(Z > t) = P(W_{N_1(t)} < K) = \sum_{j=0}^{\infty} P(N_1(t) = j, W_j < K) = \sum_{j=0}^{\infty} P_{1j}(t) H^{(j)}(K) \quad (3)$$

and the density function of Z is $f_z(t) = p\lambda(t) \sum_{j=0}^{\infty} \left(H^{(j)}(t) - H^{(j+1)}(t) \right) P_{1,j}(t)$.

A type II shock whenever occurs makes the system into minor failure, and such a failure can be corrected by minimal repair. Hence, the probability that the number of minor failures occurred in $[0, t)$ equals to j is given by:

$$P(N_2(t) = j) = \frac{(m_2(t))^j \exp(-m_2(t))}{j!} = P_{2j}(t) \quad (4)$$

where $m_2(t) = \int_0^t q\lambda(x) dx$ denote the mean number of minor failures during $[0, t)$.

When a minor failure occurs, the repair cost due to this failure is evaluated. We assume that the repair cost Y_i due to i-th minor failure are nonnegative *i.i.d.* random variables with a probability distribution $G(y) = P(Y_i \le y)$, $i = 1, 2, 3, \dots$. If Y_i is smaller than a predetermined limit L_S, then the system is corrected by minimal repair. Otherwise, the system is replaced. Thus, $\delta = P(Y_i > L_s)$ is the probability of the system's preventive replacement at the minor failure occurement. We let μ_y be the mean of random variable Y_i truncated at L_S. Let random variable U denote the time of replacement due to minor failure has the following distribution:

$$\overline{F_u}(t) = P(U > t) = \sum_{j=0}^{\infty} P(N_2(t) = j) \times P(Y_1 < L_S, Y_2 < L_S, \dots, Y_j < L_S) = \exp(-\delta m_2(t)) \quad (5)$$

and the density function of U is $f_u(t) = \delta q\lambda(t) \exp\left(-\delta m_2(t)\right)$.

Except the system replacement at a serious failure or at one minor failure in case that the corresponding repair cost is larger than L_S, the system is also preventively replaced at a scheduled time T. In summary, the system is replaced at scheduled time T, or at a time of one minor failure where the repair cost exceeds a pre-determined limit L_S, or at serious failure. The probabilities for these three cases will be computed as follows.

First, if all repair cost for minor failures occurred before T are less than L_S and the accumulted damage due to type I shocks up to time T is less than failure level K, i.e., $\min(Z, U, T) = T$, preventive replacement is executed at scheduled time T. Thus, the probability that the system will be replaced preventively at scheduled time T is given by:

$$P\big(\min(Z,U)>T\big)=P(Z>T)\times P(U>T)=e\exp\big(-\delta m_2(T)\big)\sum_{j=0}^{\infty}P_{1j}(T)H^{(j)}(K)\,.\,(6)$$

Second, if $\min(Z,U,T)=U$, the system will be replaced preventively at some one minor failure that the corresponding repair cost is the first time larger than limit L_S. Thus, the probability that the system will be replaced preventively at one minor failure is derived as follows:

$$P\big(U<T,U<Z\big)=\int_0^T P(Z>t)f_u(t)dt=\sum_{j=0}^{\infty}H^{(j)}(K)\int_0^T P_{1j}(t)\exp\big(-\delta m_2(t)\big)\delta q\lambda(t)dt\quad(7)$$

Finally, if $\min(Z,U,T)=Z$, the system will be replaced at a serious failure. Thus, the probability of a failure replacement is given by:

$$P(Z<T,Z<U)=\int_0^T P(U>t)\sum_{j=1}^{\infty}P(W_{j-1}<K\le W_j)P(S_{1j}\in(t,t+\Delta t)$$

$$=\sum_{j=0}^{\infty}\big(H^{(j)}(K)-H^{(j+1)}(K)\big)\int_0^T\exp\big(-\delta m_2(t)\big)P_{1j}(t)p\lambda(t)dt\qquad(8)$$

Moreover, the following assumptions are required:
  (a1) The system is monitored continuously, and all failures are detected immediately.
  (a2) Repairs and replacements are completed instantaneously.
  (a3) The steady state case is considered.

Moreover, the replacement that is executed at scheduled time T or at the occurrence of one minor failure where the repair cost exceeds limit L_S is called preventive replacement of cost C_0. While replacement executed at serious failure is called failure replacement of cost $C_1(C_1>C_0)$. Under a fixed limit L_S, this problem is to find an optimal T^* to minimize the long-term expected cost per unit time $A(T,L_S)$ in the steady state case.

3. Long-term expected cost per unit time

A replacement cycle is actually a time interval between the installation of the system and the first replacement or a time interval between consecutive replacements. Therefore, the successive replacement cycles will constitute a renewal process. Let $\overline{R}(T,L_S)$ and $\overline{Z}(T,L_S)$ denote the mean length of a replacement cycle and the expected total cost incurred during a replacement cycle, respectively. By using renewal-reward theorem, we can see that the long-term expected cost per unit time in the steady state case is given by (Ross (1983)):

$$A(T,L_S)=\frac{\overline{R}(T,L_S)}{\overline{Z}(T,L_S)}\,.\qquad(9)$$

Under the defined preventive maintenance policy, the expected length of a replacement cycle $\overline{Z}(T,L_S)$ is given by:

$$\overline{Z}(T,L_S)=\int_0^{\infty}P(\min(T,Z,U)>t)dt=\sum_{j=0}^{\infty}H^{(j)}(K)\int_0^T P_{1j}(t)\exp\big(-\delta m_2(t)\big)dt\quad(10)$$

If the system is replaced at scheduled time T, then the number of minor failures during $[0,T]$ will be $N_2(T)$. If the system is replaced at time U, then the number of minor failures during $[0,U]$ will be $N_2(U)$. And, if the system is replaced at the first serious failure, then the number of minor failures during $[0,Z]$ will be $N_2(Z)$. Therefore, the expected number of minor failures until replacement is:

$$E\big(N_2(\min(T,Z,U))\big)=H^{(j)}(K)P_{1j}(T)e\exp\big(-\delta m_2(T)\big)E(N_2(T))+\sum_{j=0}^{\infty}H^{(j)}(K)\int_0^T P_{1j}(t)f_u(t)E(N_2(t))dt$$

$$+\sum_{j=0}^{\infty}\big(H^{(j)}(K)-H^{(j+1)}(K)\big)\int_0^T P_{1,j}(t)p\lambda(t)\exp\big(-\delta m_2(t)\big)E(N_2(t))dt$$

$$=\sum_{j=0}^{\infty}H^{(j)}(K)\int_0^T P_{1j}(t)\exp\big(-\delta m_2(t)\big)q\lambda(t)dt$$

Furthermore, the expected total cost $\overline{R}(T,L_S)$ during a replacement cycle can be dervied dervied as follows:

$$\overline{R}(T,L_S)=C_0\sum_{j=0}^{\infty}H^{(j)}(K)P_{1j}(T)e\exp\big(-\delta m_2(T)\big)+C_0\sum_{j=0}^{\infty}H^{(j)}(K)\int_0^T P_{1j}(t)\exp\big(-\delta m_2(t)\big)\delta q\lambda(t)dt$$

$$+C_1\sum_{j=0}^{\infty}\big(H^{(j)}(K)-H^{(j+1)}(K)\big)\int_0^T\exp\big(-\delta m_2(t)\big)P_{1j}(t)p\lambda(t)dt+\mu_y E\big(N_2(\min(T,Z,U))\big)$$

$$=C_0+(C_1-C_0)\sum_{j=0}^{\infty}\big(H^{(j)}(K)-H^{(j+1)}(K)\big)\int_0^T\exp\big(-\delta m_2(t)\big)P_{1j}(t)p\lambda(t)dt$$

$$+\mu_y\sum_{j=0}^{\infty}H^{(j)}(K)\int_0^T P_{1j}(t)\exp\big(-\delta m_2(t)\big)q\lambda(t)dt$$

(11)

Combining (10) and (11), the long-term expected cost per unit time $A(T,L_S)$ is obtained as follows:

$$A(T,L_s) = \frac{\left[\begin{array}{l} C_0 + (C_1 - C_0)\sum_{j=0}^{\infty}\left(H^{(j)}(K) - H^{(j+1)}(K)\right)\int_0^T \exp\left(-\delta m_2(t)\right)P_{1j}(t)\,p\lambda(t)dt \\ \qquad\qquad + \mu_y \sum_{j=0}^{\infty} H^{(j)}(K)\int_0^T P_{1j}(t)\exp\left(-\delta m_2(t)\right)q\lambda(t)dt \end{array}\right]}{\left(\sum_{j=0}^{\infty} H^{(j)}(K)\int_0^T P_{1j}(t)\exp(-\delta m_2(t))dt\right)} \tag{12}$$

A necessary condition that a finite T* minimizes $A(T,L_s)$ under a fixed L_S can be obtained by differentiating $A(T,L_s)$ with respect to T and setting it equal to zero. If we let $\dfrac{dA(T,L_S)}{dT}$ equal to zero, we have:

$$(C_1 - C_0)\left[\begin{array}{l} \sum_{j=0}^{\infty} H^{(j+1)}(K)\int_0^T \exp\left(-\delta m_2(t)\right)P_{1j}(t)\,p\lambda(t)dt \\ \qquad - D(T)p\lambda(T)\sum_{j=0}^{\infty} H^{(j)}(K)\int_0^T \exp\left(-\delta m_2(t)\right)P_{1j}(t)dt \end{array}\right]$$
$$+\left[\,p(C_1 - C_0) + q\mu_y\right]\times\left(\sum_{j=0}^{\infty} H^{(j)}(K)\int_0^T \exp\left(-\delta m_2(t)\right)P_{1j}(t)\left[\lambda(T) - \lambda(t)\right]dt\right) = C_0 \tag{13}$$

where $D(T) = \sum_{j=0}^{\infty} H^{(j+1)}(K)P_{1j}(T)\Big/\sum_{j=0}^{\infty} H^{(j)}(K)P_{1j}(T)$. So, equation (13) will be rewritten as:

$$(C_1 - C_0)\left[\sum_{j=0}^{\infty} H^{(j+1)}(K)\int_0^T \exp\left(-\delta m_2(t)\right)P_{1j}(t)\,p\lambda(t)dt - D(T)p\lambda(T)\overline{Z}(T,L_s)\right]$$
$$+\left((C_1 - C_0) + \frac{q\mu_y}{p}\right)\times\left[\begin{array}{l} p\lambda(T)\overline{Z}(T,L_S) \\ \quad - \sum_{j=0}^{\infty} H^{(j)}(K)\int_0^T \exp\left(-\delta m_2(t)\right)P_{1j}(t)\,p\lambda(t)dt \end{array}\right] = C_0 \tag{14}$$

For simplification, let $U(T)$ denote the left-hand side of (14).

Theorem 1: If $\lambda(t)$ is an increasing function of t and $\lim_{T\to\infty} U(T) > C_0$, there are at least one finite optimal T*, $0 \le T^* < \infty$ such that $A(T^*,L_S) < A(T,L_S)$ for all T. And, if $\lambda(t)$ is strictly increasing and $U'(T) > 0$ for all T, then $U(T)$ is a strictly increasing function of T and the optimal T* is also unique. Furthermore, if $\lim_{T\to\infty} U(T) > C_0$ is not satisfied, then T* is infinite.

Proof: Here, $U(0) \equiv \lim_{T\to 0} U(T) = 0$ and

$$U(\infty) = \lim_{T\to\infty} U(T) = \lim_{T\to\infty}\left(\begin{array}{l} (C_1 - C_0)\left[\begin{array}{l}\left(\sum_{j=0}^{\infty} H^{(j+1)}(K)\int_0^T P_{1j}(t)\exp(-\delta m_2(t))\,p\lambda(t)dt\right) \\ \quad - D(T)p\lambda(T)\left(\sum_{j=0}^{\infty} H^{(j)}(K)\int_0^T P_{1j}(t)\exp(-\delta m_2(t))dt\right)\end{array}\right] \\ + \left(p(C_1 - C_0) + q\mu_y\right)\left(\sum_{j=0}^{\infty} H^{(j)}(K)\int_0^T P_{1j}(t)\exp(-\delta m_2(t))\left(\lambda(T) - \lambda(t)\right)dt\right) \end{array}\right)$$
$$= \left((C_1 - C_0) - \frac{\mu_y}{\delta}\right)\left(\frac{\delta q}{\delta q + p}\right)\sum_{j=0}^{\infty} H^{(j)}(K)\left(\frac{p}{\delta q + p}\right)^j + (C_1 - C_0)p\lambda(\infty)\left(1 - D(\infty) + \frac{q\mu_y}{p}\right) \tag{15}$$

where $\quad \lim\limits_{T \to \infty} \int_0^T \exp(-\delta m_2(t)) P_{1j}(t) p\lambda(t) dt = \left(\dfrac{p}{\delta q + p}\right)^{j+1}$. Since

$U(0) = 0 < C_0$, and if $\lim\limits_{T \to \infty} U(T) > C_0$, then there exist at least one

finite optimal T^* such that $U(T^*) = c_0$, i.e., T^* satisfies (14). Such a T^* is a candidate which minimizes $A(T, L_S)$. Furthermore, if $\lambda(t)$ is strictly increasing and

$$U'(T) = (C_1 - C_0 - 1) \times \left(\sum_{j=0}^{\infty} H^{(j+1)}(K) P_{1j}(T) \exp(-\delta m_2(T)) p\lambda(T) \right)$$
$$- \left[D'(T) p\lambda(T) + D(T) p\lambda'(T) \right] \times \overline{Z}(T, L_s) > 0$$

then $U(T)$ is a strictly increasing function of t, and hence, the optimal T^* is also unique. And, the resulting optimal long-term expected cost per unit time will be:

$$A(T^*, L_S) = U(T^*)$$

Conversely, if $U'(T) > 0$ and $U(\infty) < C_0$ or $U'(T) \le 0$, then T^* is infinite.
In our model, we can have the following special cases:

Case 1. If $L_s \to \infty$, then $A(T, L_S)$ will be reduced simply to

$$A(T, \infty) = \frac{\begin{bmatrix} C_0 + (C_1 - C_0) \sum_{j=0}^{\infty} \left(H^{(j)}(K) - H^{(j+1)}(K) \right) \int_0^T \exp(-m_2(t)) P_{1j}(t) p\lambda(t) dt \\ + \mu_y \sum_{j=0}^{\infty} H^{(j)}(K) \int_0^T P_{1j}(t) \exp(-m_2(t)) q\lambda(t) dt \end{bmatrix}}{\sum_{j=0}^{\infty} H^{(j)}(K) \int_0^T P_{1j}(t) \exp(-m_2(t)) dt}$$

This is reduced to a periodical replacement policy with minimal repair at minor failure and with replacement at serious failure.

Case 2. If $L_S \to \infty$ and $p = 1$, then $\delta = 1$. $A(T, L_S)$ will be reduced to

$$A(T, \infty) = \left[C_0 + (C_1 - C_0) \sum_{j=0}^{\infty} \left(H^{(j)}(K) - H^{(j+1)}(K) \right) \int_0^T \lambda(t) P_{1j}(t) dt \right] \bigg/ \sum_{j=0}^{\infty} H^{(j)}(K) \int_0^T P_{1j}(t) dt$$

which is the same as in Zuckerman (1980) and model 1 in Ito and Nakagawa (2011).

4. Numerical Example

In this numerical example, we consider that external shocks occur randomly at a non-homogeneous Poisson process with an intensity function $\lambda(t) = \lambda t^{\beta-1}$, $\lambda > 0$, $\beta > 0$. We assume that the shape parameter is set at $\beta = 2$, and that $\lambda(t) = \lambda t$ is an increasing function of t. Suppose that the amount of damage X_i due to the *i-th* type I shock has an exponential distribution with finite mean $\mu_x = 12$, $i = 1, 2, 3, \ldots$. And, the failure level K of this system is set to be 100. So, the convolution $H^{(j)}(100)$ is computed as follows:

$$H^{(j)}(100) = \int_0^{100} \frac{1}{\Gamma(j)\mu_x} \left(\frac{x}{\mu_x} \right)^{j-1} e^{-\frac{x}{\mu_x}} dx = \int_0^{100/\mu_x} \frac{y^{j-1} e^{-y}}{(j-1)!} dy$$

Preventive and failure replacements occur at cost of $C_0 = 1000$ and $C_1 = 1500$, respectively, while the cost of consecutive minimal repairs are *i.i.d.* normal distribution with finite mean $\mu_y = 50$ and standard deviation $\sigma_y = 10$. If th predetermined limit L_S is set to be 62.82, so

$$\delta = P(Y_i > 62.82) = P\left(\frac{Y_i - \mu_y}{\sigma_y} > \frac{62.82 - 50}{10} \right) = 0.1 .$$

Since $\lambda(t) = \lambda t$ is increasing in *t*, so $\lambda(\infty) \to \infty$, furthermore,

$$\sum_{j=0}^{\infty} H^{(j)}(100) \left(\frac{p}{\delta q + p} \right)^j = 1 + \sum_{j=1}^{\infty} \left(\int_0^{100/12} \frac{1}{(j-1)!} (y)^{j-1} e^{-y} dy \right) \times \left(\frac{p}{\delta q + p} \right)^j$$
$$= 1 + \left(\frac{p}{\delta q} \right) \times \left(1 - e^{-\left(\frac{25\delta q}{3(\delta q + p)} \right)} \right)$$

tends to a fixed number and $D(\infty) \to 1$, then $U(\infty) \to \infty$, as $t \to \infty$. That means that $\lim\limits_{T \to \infty} U(T) > C_0$. According to the conditions of ***Theorem 1***, we can see that the optimal T^* will be finite and unique.

We use Mathematical software "MAPLE" to compute $A(T, L_S)$ under different combinations of parameters λ, p and δ. In order to understand the effects on T^* and $A(T^*, L_S)$ from different parameters, we consider two cases:
Case 1: p=0.9, 0.8, 0.7, 0.6, 0.5 and λ=1.0, 1.5, 2.0, 2.5, 3.0, δ=0.1.
Case 2: δ=0.1, 0.15, 0.2, 0.25, 0.3, p=0.7, λ=2.

The results of cases 1 and 2 are showed in Tables 1 and 2, respectively. From tables, 1 and 2, we have the following conclusions:
(1) When λ is greater, it is shown that the optimal preventive period T^* decreases but the minimum long-term expected cost per unit time $A(T^*, L_S)$ increases. The greater λ implies that the failures of the system occurred easily, so the optimal T^* must be shorter to prevent the occurrence of random failures.
(2) When p is smaller (and $(1-p)$ is greater), we can see that the optimal preventive period T^* increases but the minimum $A(T^*, L_S)$ is firstly decreasing and then increasing. The smaller p implies that the probability of replacing the system at serious failure is smaller, so the optimal T^* can be longer.
(3) When δ is greater, it will be shown that the optimal preventive period T^* and the minimum long-term expected cost per unit time $A(T^*, L_S)$ increase simultaneously. The greater δ implies that the possibility of replacing of the system from serious failure is reduced slightly, so the optimal T^* can be extended slightly.

Table 1. Optimal T^* and $A(T^*, L_S)$ at different values of λ and p.

λ	$p=0.9$		$p=0.8$		$p=0.7$		$p=0.6$		$p=0.5$	
	T^*	$A(T^*, L_S)$	T^*	$A(T^*, L_S)$	T^*	$A(T^*, L_S)$	T^*	$A(T^*, L_S)$	T^*	$A(T^*, L_S)$
1.0	4.14	324.4449148	4.36	322.9600534	4.62	322.1957584	4.93	322.4162087	5.31	324.0059950
1.5	3.38	397.3623157	3.56	395.5436502	3.78	394.6077033	4.03	394.8774576	4.34	396.8248869
2.0	2.95	458.8561575	3.09	456.7344917	3.27	455.6532650	3.49	455.9652071	3.75	458.2141217
2.5	2.62	512.9920485	2.76	510.6440824	2.92	509.4369527	3.12	509.7845672	3.36	512.2985146
3.0	2.39	561.9551858	2.52	559.3825379	2.67	558.0590004	2.85	558.4410978	3.07	561.1954111

Table 2. Optimal T^* and $A(T^*, L_S)$ at different values of δ.

λ	$\delta=0.1$		$\delta=0.15$		$\delta=0.20$		$\delta=0.25$		$\delta=0.30$	
	T^*	$A(T^*, L_S)$	T^*	$A(T^*, L_S)$	T^*	$A(T^*, L_S)$	T^*	$A(T^*, L_S)$	T^*	$A(T^*, L_S)$
2.0	3.27	455.6532650	3.3	467.2935446	3.32	479.1248321	3.35	491.1283410	3.38	503.2854405

5. Conclusions

In this paper, a periodical replacement policy with repair-cost limit under cumulative damage model is introduced, in which we derived the long-term expected cost per unit time $A(T, L_S)$ by incorporating costs due to replacement and minimal repair. This research verifies that under some specific conditions, the optimal period T^* to minimize $A(T, L_S)$ will be finite and unique. This work can be extended to consider multi-unit system or include the concept of imperfect repair.

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