

Application of trapezoidal integration method in determining discrete time windows

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1. Modified Dirichlet window determined from the trapezoidal rule

In one of the essential works on time windows, the rectangle (Dirichlet) window is defined in the following way [2]:

- for the purpose of the finite Fourier transform (equation (21a)):

$$w(n) = 1 \quad \text{for} \quad n = -\frac{N}{2}, \dots, -1, 0, 1, \dots, \frac{N}{2} \quad (1)$$

- for the purpose of the discrete Fourier transform (DFT) (equation (21b)):

$$w(n) = 1 \quad \text{for} \quad n = 0, 1, \dots, N - 1 \quad (2)$$

Equation (1) includes $N+1$ points, whereas equation (2) only N points. It stems from the fact that while switching from the Fourier transform (expressed by an integral) to the DFT, none of the known numerical integration procedures is applied, but only the last element of the sum is deleted.

A classic method of numerical integration are Newton-Cotes quadratures with equally spaced points [1]. It consists in calculating an integral of $f(t)$ function defined in points

$$t_n = nh; \quad h = \text{const}; \quad n = 0, 1, \dots, N; \quad f_n = f(t_n)$$

Assuming a linear interpolation of the function the so called trapezoidal formula is obtained:

$$I_T = h \left(\frac{f_0}{2} + f_1 + f_2 + \dots + f_{N-1} + \frac{f_N}{2} \right) \quad (3)$$

From equation (3) the following form of a modified Dirichlet window is determined:

$$wm(n) = \begin{cases} 1 & \text{for } n = 1, 2, \dots, N - 1 \\ 0.5 & \text{for } n = 0 \text{ and } n = N \end{cases} \quad (4)$$

By analogy with equation (4), the Dirichlet window (equation(2)) is given in the form:

$$w(n) = \begin{cases} 1 & \text{for } n = 0, 1, 2, \dots, N - 1 \\ 0 & \text{for } n = N \end{cases} \quad (5)$$

By applying equation (4) the trapezoidal rule (3) can be given as:

$$I_T = h \sum_{n=0}^N wm(n) \cdot f_n \quad (6)$$

where: $wm(n)$ is defined by equation (4).

The form of the Dirichlet window $w(n)$ (equation(5)) as well as the modified Dirichlet window $wm(n)$ (equation(4)) for $N=16$ are shown in Fig.1.

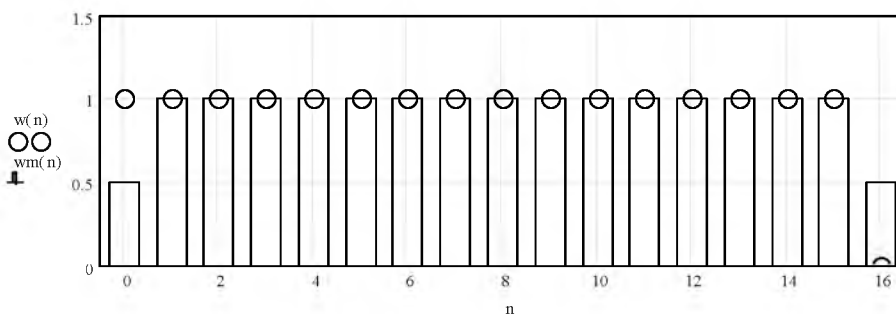


Fig. 1. Sample (N=16) Dirichlet window $w(n)$ (equation (5)) - marked with circles - and modified Dirichlet window $wm(n)$ (equation (4)) - marked with bars

Applying the DFT to the Dirichlet window:

$$W(\theta) = \sum_{n=0}^{N-1} w(n) \exp(-jn\theta) \quad (7)$$

a complex spectrum form is obtained ([2],equation (21c)):

$$W(\theta) = \exp(-j \frac{N-1}{2} \theta) \cdot \frac{\sin(\frac{N\theta}{2})}{\sin(\frac{\theta}{2})} \quad (8)$$

Whereas when using the trapezoidal rule the following equation is applied:

$$WM(\theta) = \sum_{n=0}^N wm(n) \exp(-jn\theta) \quad (9)$$

Having done the calculations the following equation is obtained

$$WM(\theta) = \exp(-j \frac{N\theta}{2}) \cdot \frac{\sin(\frac{N\theta}{2})}{\sin(\frac{\theta}{2})} \cos\left(\frac{\theta}{2}\right) \quad (10)$$

Figure 2 shows logarithmic characteristics (in decibels) of discrete time windows $w(n)$ and $wm(n)$:

$$LW(\theta) = 20 \log \left[\frac{W(\theta)}{W(0)} \right] ; LWM(\theta) = 20 \log \left[\frac{WZ(\theta)}{WZ(0)} \right] \quad (11)$$

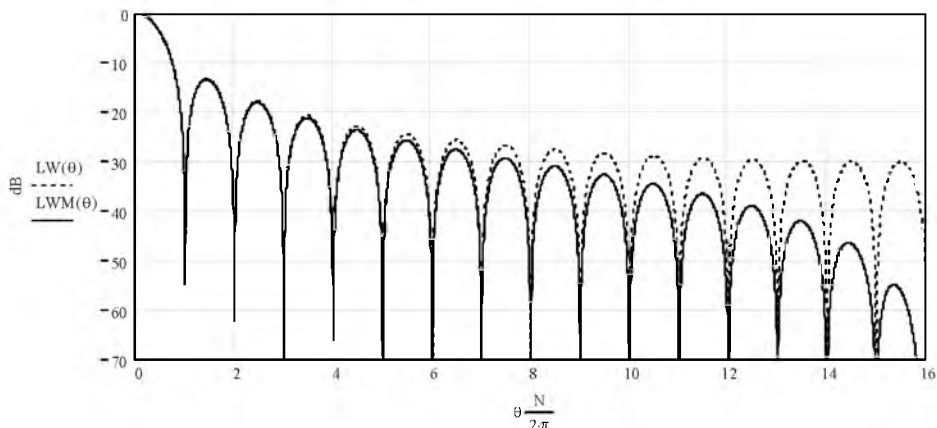


Fig. 2. Logarithmic characteristics (in decibels) of time windows for $N=32$: the Dirichlet window $LW(\theta)$ (dotted line) and the modified Dirichlet window $LWM(\theta)$ (solid line)

The amplitude characteristic of the modified window (solid line) demonstrates faster falling than the classic window characteristic. An analogous situation occurs for all time windows which at interval edges take the value different from zero.

2. Hamming window

By applying the DFT:

$$XH(\theta) = \sum_{n=0}^{N-1} Hm(n) \exp(-jn\theta) \quad (12)$$

to the Hamming window

$$Hm(n) = 0,54 - 0,46 \cdot \cos\left(\frac{2\pi n}{N}\right) \quad (13)$$

we obtain:

$$XH(\theta) = 0,54 \cdot W(\theta) - 0,23 \cdot \left(W\left(\theta - \frac{2\pi}{N}\right) + W\left(\theta + \frac{2\pi}{N}\right) \right) \quad (14)$$

where $W(\theta)$ - the rectangle window spectrum (equation(8)).

By implementing the trapezoidal rule:

$$XHM(\theta) = \sum_{n=0}^N wm(n) Hm(n) \exp(-jn\theta) \quad (15)$$

we obtain:

$$XHM(\theta) = 0.54 \cdot WM(\theta) - 0.23 \cdot \left(WM\left(\theta - \frac{2\pi}{N}\right) + WM\left(\theta + \frac{2\pi}{N}\right) \right) \quad (16)$$

where $WM(\theta)$ - spectrum of a modified rectangle window (equation (10)).

Figure 3 shows logarithmic characteristics (in decibels) of the Hamming window $LH(\theta)$ and the modified Hamming window $LHM(\theta)$.

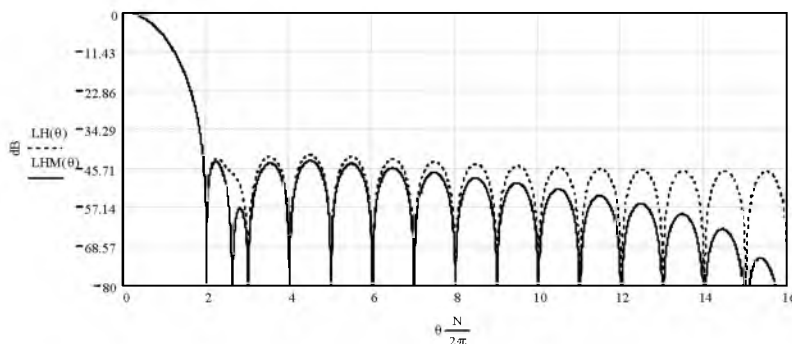


Fig. 3. Logarithmic characteristics (in decibels) of time windows for $N=32$: the Hamming window $LH(\theta)$ (dotted line) and the modified Hamming window $LHM(\theta)$ (solid line)

3. Blackman-Harris window

As another example the Blackman-Harris window will be examined [2]:

$$B(n) = a_0 - a_1 \cdot \cos\left(\frac{2\pi n}{N}\right) + a_2 \cos\left(\frac{4\pi n}{N}\right) \quad (17)$$

In paper [2] the following coefficient values were provided

$$a_0 = 0.44959; a_1 = 0.49364; a_2 = 0.05677 \quad (18)$$

for which window (17) is characterized by the fact that the maximum of the first sidelobe equals -61 dB.

In accordance with equations:

$$B(\theta) = \sum_{n=0}^{N-1} B(n) \exp(-jn\theta) \quad (18)$$

$$BM(\theta) = \sum_{n=0}^N wm(n) B(n) \exp(-jn\theta) \quad (19)$$

a complex spectra of the Blackman-Harris window (equation (18)) and the modified Blackman-Harris window (equation (19)) were derived. Figure 3 presents the logarithmic characteristics (equation (11)) of the Blackman-Harris window $LB(\theta)$ and the modified Blackman-Harris window $LBM(\theta)$.

Much like in Fig. 4, the $LBM(\theta)$ amplitude characteristic demonstrates substantially faster falling than the $LB(\theta)$ characteristic.

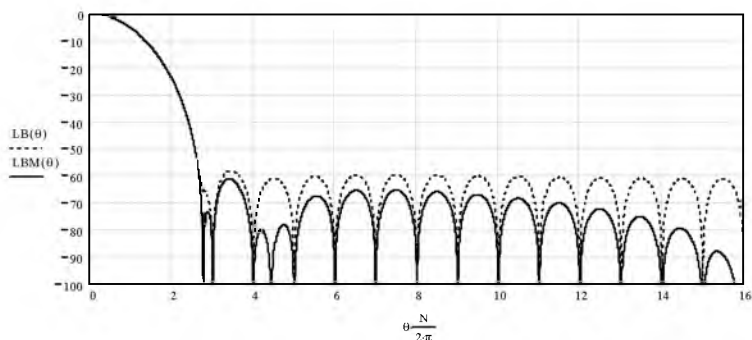


Fig. 4. Logarithmic characteristics (in decibels) of time windows for $N=32$: the Blackman-Harris window $LB(\theta)$ (dotted line) and the modified Blackman-Harris window $LBM(\theta)$ (solid line)

Paper [2] provides yet another set of coefficients:

$$a_0 = 0.42323 ; a_1 = 0.49755 ; a_2 = 0.07922 \quad (20)$$

yielding the window characterized by the maximum of the first sidelobe equal -67 dB. Calculations yield amplitude characteristics demonstrated in Fig. 5.

Paper [3] draws attention to an inaccuracy concerning sidelobe maximum values given in paper [2]. Namely, instead of -61 dB there should be $-62,05$ dB and instead of -67 dB there should be $-70,83$ dB. Therefore in this paper the value of the sidelobe maximum was determined in the function of window length N . The results of the calculations are presented in Table 1.

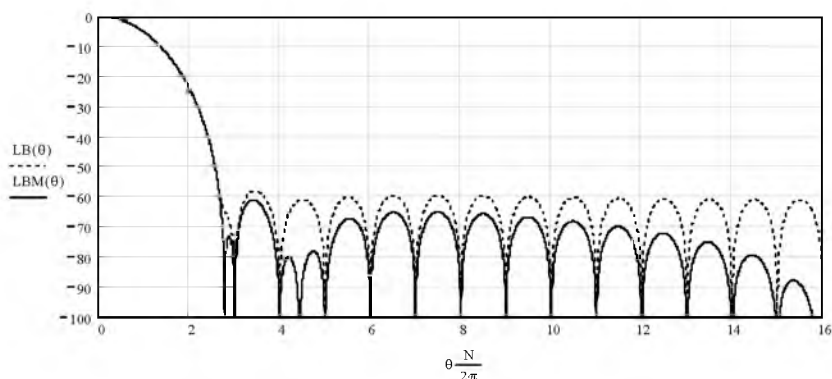


Fig. 5. Logarithmic characteristics (in decibels) of time windows for $N=32$: the Blackman-Harris window $LB(\theta)$ (dotted line, equations (17),(20)) and the modified Blackman-Harris window $LBM(\theta)$ (solid line)

Table 1. Maximum values of the first sidelobe in relation to window length N

Window length N	- 61 dB	- 67 dB
	Maximum of the first sidelobe	
16	-53,7	-62
32	-58,2	-66,7
64	-60,7	-70,6
128	-61,7	-70,8
256	-62	-70,8

Table 1 recognizes the fact that the window under consideration is a discrete one – there is a considerable dependence of the maximum value on the window length N. The values proposed in paper [3] are obtained for N=256, i.e. they refer to continuous time windows.

4. Polynomial windows

Paper [4] describes the rules of a polynomial window construction. A relation describing a polynomial window of the properties similar to those of the Hamming window is given below - assuming an identical maximum value of the first sidelobe (-42.87 dB for N=32):

$$P(n) = 1 + 0.93494 \left[2 \cdot \left| 1 - \frac{2n}{N} \right|^3 - 3 \left(1 - \frac{2n}{N} \right)^2 \right] \quad (21)$$

Figure 6 demonstrates the logarithmic characteristics (in decibels) of time windows for $N = 32$: the Hamming window $LH(\theta)$ (dotted line) and the polynomial window $LP(\theta)$ (solid line).

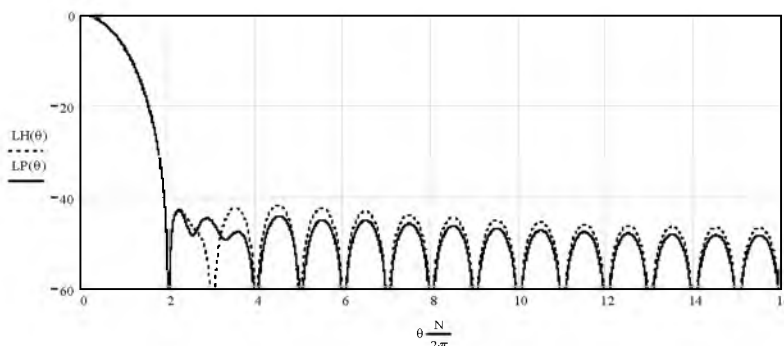


Fig. 6. Logarithmic characteristics (in decibels) of time windows for $N=32$: the Hamming window $LH(\theta)$ (dotted line) and the polynomial window $LP(\theta)$ (solid line)

In accordance with the accepted assumption, the equality of maximum values of the first sidelobe is observed, however, for the following lobes the maximum for the polynomial window ranges between 1.5 and 2 dB below the maximum for the Hamming window.

Figure 7 shows the logarithmic characteristics (in decibels) of the polynomial window $LP(\theta)$ ($N=32$) and the modified polynomial window $LPM(\theta)$. Faster falling of the characteristic of the window determined on the basis of the trapezoidal formula is observed.

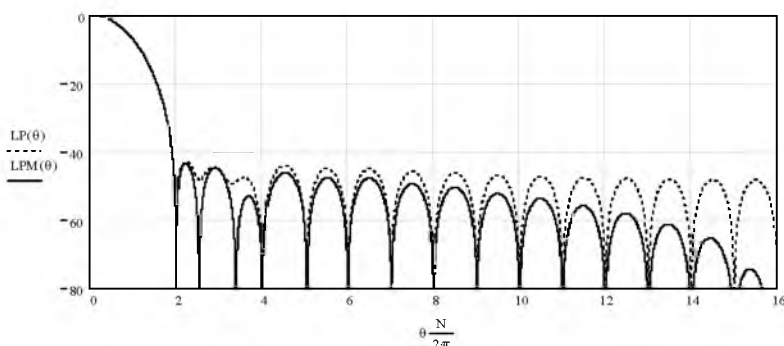


Fig. 7. Logarithmic characteristics (in decibels) of time windows for $N=32$: the polynomial window $LP(\theta)$ (dotted line) and the modified polynomial window $LPM(\theta)$ (solid line)

Figure 8 demonstrates the comparison between the logarithmic characteristics (in decibels) of time windows for $N=32$: the modified Hamming window $LHM(\theta)$ (dotted line) and the modified polynomial window $LPM(\theta)$ (solid line). The maximum of the following sidelobes for the modified polynomial window falls about 2 dB below the maximum of the lobes for the modified Hamming window.

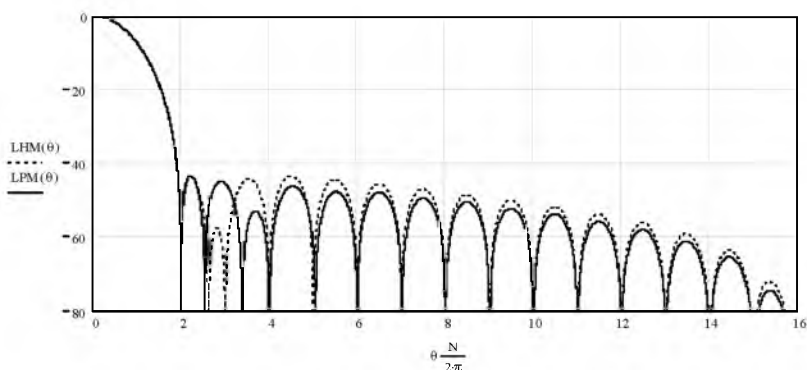


Fig. 8. Logarithmic characteristics (in decibels) of the modified Hamming window $LHM(\theta)$ (dotted line) and the modified polynomial window $LPM(\theta)$ (solid line) for $N = 32$

5. Gaussian window

As yet another example the Gaussian window was considered [2]:

$$G(n) = \exp\left(-\left(2\left(1 - \frac{2n}{N}\right)\right)^2\right) \quad (22)$$

Figure 9 shows characteristics determined by the following methods: the DFT-dotted line $LG(\theta)$, and the trapezoidal formula – solid line $LGM(\theta)$. Similarly to all the previously examined cases, the logarithmic characteristic of the modified Gaussian window ($LGM(\theta)$) falls faster than the logarithmic characteristic of the Gaussian window ($LG(\theta)$).

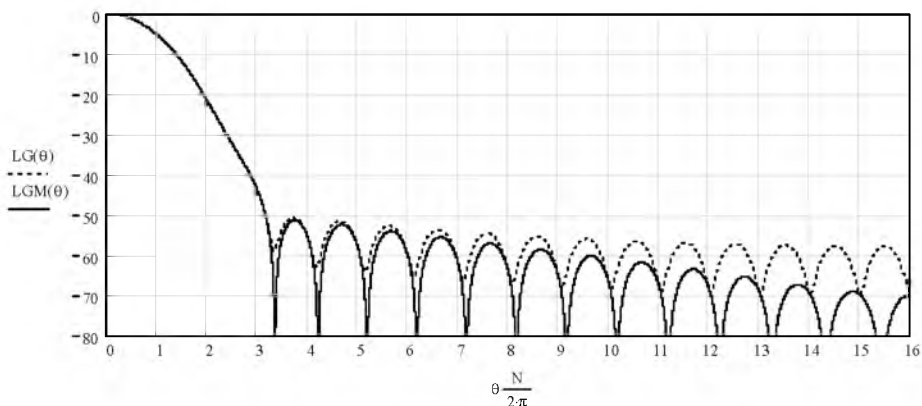


Fig. 9. Logarithmic characteristics (in decibels) of the Gaussian window $LG(\theta)$ (dotted line) and the modified Gaussian window $LGM(\theta)$ (solid line) for $N=32$

6. Harmonic signal

The final example is a harmonic signal of the form:

$$x(n) = \cos(n \cdot \theta + \phi) \quad (23)$$

By applying the following expressions:

$$X(\theta) = \sum_{n=0}^{N-1} x(n) \exp(-jn\theta) \quad (24)$$

$$XM(\theta) = \sum_{n=0}^N wm(n)x(n) \exp(-jn\theta) \quad (25)$$

a complex spectra $X(\theta)$ and $XM(\theta)$ are determined.

Figure 10 presents the amplitude of the complex spectrum $|X(\theta)|$ (dotted line) and $|XM(\theta)|$ (solid line). For the complex spectrum $X(\theta)$ and $XM(\theta)$ logarithmic amplitude characteristics are introduced:

$$LX(\theta) = 20 \log \left[\left| \frac{X(\theta)}{X(\theta_1)} \right| \right] ; \quad LXM(\theta) = 20 \log \left[\left| \frac{XM(\theta)}{XM(\theta_1)} \right| \right] \quad (26)$$

Figure 11 demonstrates the logarithmic amplitude characteristics of a harmonic signal $x(n) = \cos(0,25\pi n)$ determined on the basis of equation (26): the dotted line $|LX(\theta)|$; the solid line $|LXM(\theta)|$.

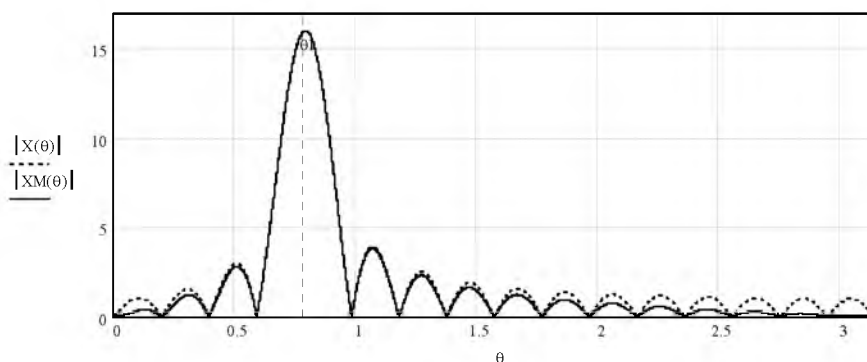


Fig. 10. Complex spectrum amplitude of the harmonic signal $x(n) = \cos(0,25\pi n)$: dotted line $|X(\theta)|$ determined as a DFT; solid line $|XM(\theta)|$ - on the basis of equation (25)- the modified complex spectrum of the signal

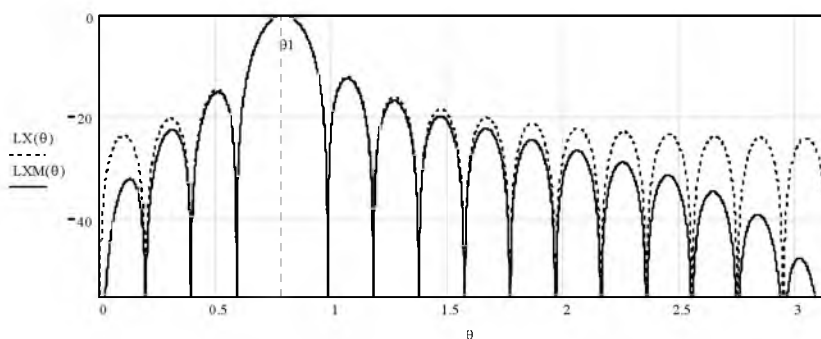


Fig. 11. Logarithmic amplitude characteristics of the harmonic signal $x(n) = \cos(0,25\pi n)$ determined on the basis of equation (26): dotted line $|LX(\theta)|$; solid line $|LXM(\theta)|$

Both figures 10 and 11 demonstrate faster falling of the modified spectrum of the harmonic signal determined using the trapezoidal formula (equation (25)).

Figure 12 compares the logarithmic amplitude characteristics of the harmonic signal $x(n) = \cos(0,5\pi n)$ determined using equation (26): dotted line $|LX(\theta)|$ (equation (24)) and solid line $|LXM(\theta)|$ (equation (25)).

Figure 13 shows the logarithmic amplitude characteristics of the harmonic signal $x(n) = \cos(0,75\pi n)$ determined using equation (26): the dotted line $|LX(\theta)|$ - in accordance with the DFT, and the solid line $|LXM(\theta)|$ - in accordance with the trapezoidal formula.

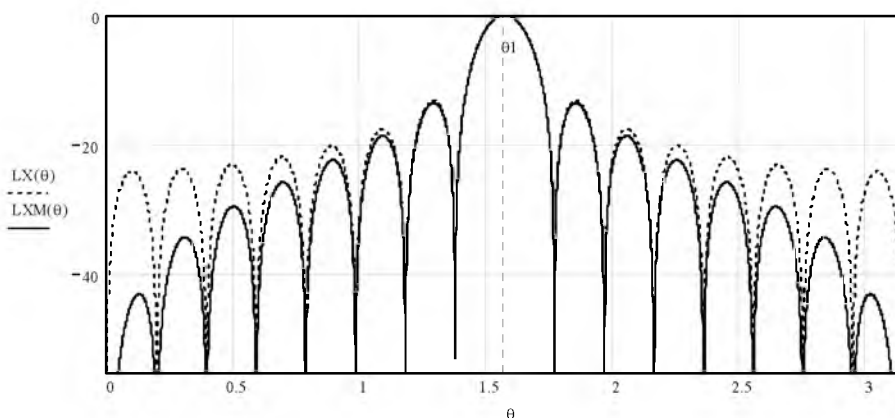


Fig. 12. Logarithmic amplitude characteristics of the harmonic signal $x(n) = \cos(0,5\pi n)$ determined using equation (26): dotted line $|LX(\theta)|$; solid line $|LXM(\theta)|$

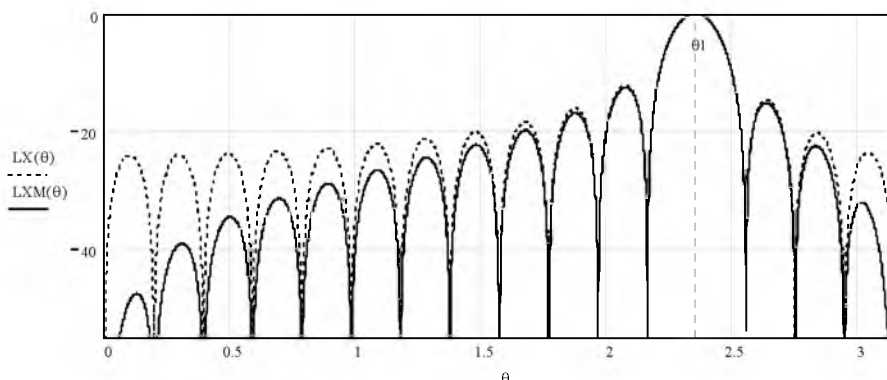


Fig. 13. Logarithmic amplitude characteristics of the harmonic signal $x(n) = \cos(0,75\pi n)$ determined using equation (26): dotted line $|LX(\theta)|$; solid line $|LXM(\theta)|$

7. Summary

For all the examined time windows: Dirichlet, Hamming, Blackman-Harris (- 61 dB, - 67dB), polynomial and Gaussian, faster falling of frequency characteristics of the modified window was observed (determined on the basis of the trapezoidal formula) compared to the frequency characteristics of the classic window (determined using the DFT method).

In case of the harmonic signal, also faster falling of the modified spectrum of the harmonic signal, determined by applying the trapezoidal formula (equation (25)), was observed compared to the spectrum of the signal determined by applying the DFT (equation (24)). However, it should be noted that Figures 10, 11, 12 were created for the harmonic signal (equation (23)) with phase $\phi = 0$.

When changing the value of the phase ϕ , the obtained results do not demonstrate such a clear advantage of the modified method over the classic one.

The author of this article has not considered all the aspects resulting from the application of the trapezoidal formula, however, his aim was to draw attention to the possibilities of using the trapezoidal formula in signal frequency analysis.

References

- [1] Fortuna Z., Macukow B., Wąsowski J.: *Metody numeryczne*. Warszawa, WNT 1993, s.166.
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