Vibrations of Periodic Sandwich Plates with Inert Core

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Abstract

In this note a free vibration analysis of periodic three-layered sandwich structures is performed. The equations of motion of such structures, which are derived basing on Kirchhoff's thin plate theory, contain periodic, non-continuous and highly oscillating coefficients, which makes them difficult to solve. In this work, the tolerance averaging technique is applied in order to transform the mentioned system of equations into a form with constant coefficients, which takes into account the effect of the microstructure size. The differences between two modelling procedures are shown and discussed. Eventually, formulas for free vibration frequencies of an exemplary 2D structure are derived and an analysis of influence of certain varying material properties is performed.

Keywords: *periodic sandwich plates, inert core, tolerance averaging, free vibrations, microstructure effect*

1. Introduction

Composites are more and more widely used in modern engineering. The possibility of combining several different materials into one heterogeneous structure, which material properties are outstanding when compared to 'classic' homogeneous materials, is very tempting for many researchers. All that is needed, is a proper model of such structures, which can be used in design and optimization process.

In this article three-layered sandwich structures are considered. A typical sandwich structure consists of external layers, which are made of materials characterised by high mechanical properties, hence, they are main bearing parts of the whole structure, and an inner layer, so called core, which is usually a light-weight, porous material, standing for thermal- and acoustic isolation. As a result, we obtain a highly durable structure, which, properly designed, can be used in many branches of engineering, such as aviation or even space ship construction.

On the other hand, sandwich structures also have disadvantages, such as vulnerability to local buckling, initial imperfections or concentrated loadings. Moreover, the mathematical models of such structures are complicated, with unclear and experimentally not proved assumptions connected with distribution of stresses and deflections. That is why, many different approaches towards the analysis of dynamic behaviour of such structures can be found in literature. Let us mention classic Euler-Bernoulli deflection hypothesis, Reissner-Mindlin's first order deformation theory, together with its extension to n^{th}-order deformation, or Zig-Zag hypothesis. For the exact description of above mentioned approaches, one should refer to Magnucki [1], Carrera [2] or Carrera and Brischetto [3], among others. In this work, let us concentrate on one of the most simple approach, which can be found in the works of Chonan [4], Oniszczuk [5] or Szcześniak [6], just to name few. In this approach, a three-layered sandwich structure is considered as a system of two Kirchhoff's type thin plates (outer layers), connected with each other by elastic Winkler's type material. Such assumption is well-fitted to our expectations, in which light-weight elastic core increase the stiffness of the structure by increasing its thickness, rather than being its bearing part.

In all above mentioned approaches, considered structures are characterised by constant geometry and are made of homogeneous or quasi-homogeneous materials. However, most recent sandwich structures contain certain varying geometry and/or material properties (especially the core can take very complicated shapes). As a result, governing equations of such structures have non-continuous and highly-oscillating coefficients, which make them difficult to solve. An answer to this problem can be the application of finite element method analysis. However, the optimization process with the use of such approach can be much time-consuming and ineffective. That is why, in this work one can find a mathematical model describing the vibrations of sandwich structure, which every layer can be characterised by periodic microstructure.

Solution to such problem was investigated by many researchers, for example by Brillouin [7], Mead [8] or Kohn and Vogelius [9], who created the basis of the asymptotic homogenisation method for plates. However, these models neglect the influence of microstructure on the behaviour of considered structures. The main aim of this paper is to derive a simple and useful model, which allows us to take into account this effect, with the use of the tolerance averaging technique, presented by Woźniak and Wierzbicki [10] or Woźniak et al. (eds.) [11], [12]. Eventually, as a result of two modelling procedures (*tolerance modelling* and *asymptotic-tolerance modelling*), free vibration frequencies of an exemplary rectangular sandwich plate are calculated.

2. Modelling foundations

Let $Ox_1x_2x_3$ be an orthogonal Cartesian coordinate system, where $\boldsymbol{x}\equiv(x_1,x_2)$, $x_3\equiv z$, and let us denote t as a time coordinate. The three-layered plate under consideration is assumed to have spans L_1 and L_2 in x_1 and x_2-axis directions, respectively, and total thickness $H(\boldsymbol{x})$. Hence, it can be stated, that undeformed structure occupies the region $\Omega\equiv[0,L_1]\times[0,L_2]\times[-\frac{1}{2}H(\boldsymbol{x}),\frac{1}{2}H(\boldsymbol{x})]$.

Let us assume, that both outer layers are Kirchhoff's type thin plates. Moreover they are made of the same set of materials and they have the same geometry, hence, all material and mechanical properties of these layers are the same, cf. Figure 1. Let us introduce their bending stiffness $B_{\alpha\beta\gamma\delta}(\boldsymbol{x},t)$ and mass density per unit area $\mu(\boldsymbol{x},t)$ as:

$$B_{\alpha\beta\gamma\delta}(\boldsymbol{x},z)=\int_{-h(\boldsymbol{x})/2}^{h(\boldsymbol{x})/2}C_{\alpha\beta\gamma\delta}(\boldsymbol{x},z)z^2dz,\qquad \mu(\boldsymbol{x},z)=\int_{-h(\boldsymbol{x})/2}^{h(\boldsymbol{x})/2}\rho(\boldsymbol{x},z)dz, \tag{1}$$

where $h(\boldsymbol{x})$ is the thickness of the outer layers, $C_{\alpha\beta\gamma\delta}(\boldsymbol{x},z)$ is their elastic modulus tensor and $\rho(\boldsymbol{x},z)$ is their mass density. Both outer layers are connected by an elastic Winkler's type material, so called *core*, characterized by elasticity modulus $k(\boldsymbol{x})$, mass density $\rho_c(\boldsymbol{x},z)$ and thickness $h_c(\boldsymbol{x})$.

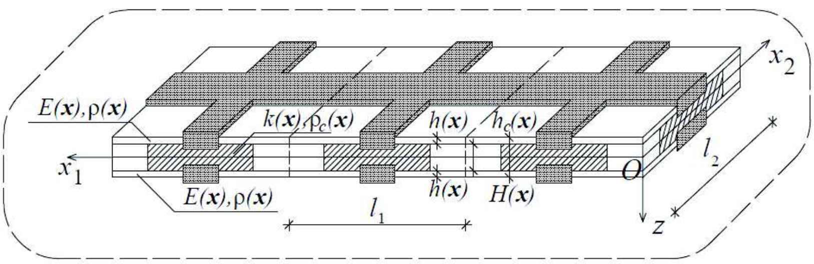

Figure 1. A part of periodic sandwich plate

The whole structure is build of small, repeatable elements, called *periodicity cells*. Every cell has dimensions l_1 and l_2 in x_1- and x_2-axis direction, respectively, while its diameter is referred to as to the *microstructure parameter l*. It is assumed, that dimensions of the plate and the microstructure parameter must satisfy following normalizing conditions: $h(\boldsymbol{x}) << l << \min(L_1, L_2)$, hence, the outer layers of the structure can be treated as thin plates not only in a macro-scale, but also when a single periodicity cell is considered.

Let us follow the simplified approach presented by Szcześniak [6]. According to the Kirchhoff's type thin plate theory, governing equations of this structure takes the form:

$$\begin{aligned} \partial_{\alpha\beta}(B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_1) + \mu\ddot{u}_1 + k(u_1 - u_2) = f_1, \\ \partial_{\alpha\beta}(B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_2) + \mu\ddot{u}_2 + k(u_2 - u_1) = f_2, \end{aligned} \tag{2}$$

where $u_1(\boldsymbol{x},t)$, $u_2(\boldsymbol{x},t)$ are deflections of upper and lower outer layers along z-axis direction, respectively, and $f_1(\boldsymbol{x},t)$, $f_2(\boldsymbol{x},t)$ are their loadings, defined as:

$$f_1 \equiv p_1 - \tfrac{1}{2}\mu_c\ddot{u}_1, \qquad f_2 \equiv p_2 - \tfrac{1}{2}\mu_c\ddot{u}_2, \qquad \mu_c = \int_{-h_c(\boldsymbol{x})/2}^{h_c(\boldsymbol{x})/2} \rho_c(\boldsymbol{x},z)dz, \tag{3}$$

where $p_1(\boldsymbol{x},t)$, $p_2(\boldsymbol{x},t)$ are external loadings applied to outer layers of the structure. It should be emphasized, that coefficients in system of equations (2) are periodic, non-continuous and highly oscillating. In order to derive a system of governing equations with constant coefficients, the tolerance averaging technique will be used.

3. Basic modelling assumptions of the tolerance averaging technique

The whole modelling procedure with the use of the tolerance averaging technique uses several introductory concepts, such as: *an averaging operator,* a *slowly varying function,* a *tolerance-periodic function* or a *highly oscillating function.* The idea standing behind those concepts, as well as a detailed description of the tolerance averaging technique, can be found in a various literature, for example by Woźniak and Wierzbicki [10] or by Woźniak et al. (eds.) [11], [12].

Let us introduce the definition of the *averaging operator*, which for an arbitrarily chosen basic periodicity cell $\Delta(\boldsymbol{x})$ can be formulated as follows:

$$< \partial^k f > (\boldsymbol{x}) = \frac{1}{|\Delta|} \int_{\Delta(\boldsymbol{x})} \tilde{f}^{(k)}(\boldsymbol{x}, y)dy, \tag{4}$$

where $\tilde{f}^{(k)}(\boldsymbol{x}, y)$ is periodic approximation of k^{th} gradient of certain function $f(\boldsymbol{x})$.

There are two main assumptions of the tolerance averaging technique. The first of them is *the micro-macro decomposition*, which stands, that the deflections of outer plates u_1, u_2 can be formulated as sums of macrodeflections $w_1(\boldsymbol{x},t)$, $w_2(\boldsymbol{x},t)$ and products of mode shape functions $g_1{}^A(\boldsymbol{x})$, $g_2{}^B(\boldsymbol{x})$ and fluctuation amplitudes $v_1{}^A(\boldsymbol{x},t)$, $v_2{}^B(\boldsymbol{x},t)$:

$$\begin{aligned} &u_1(\boldsymbol{x},t) = w_1(\boldsymbol{x},t) + g_1{}^A(\boldsymbol{x}) v_1{}^A(\boldsymbol{x},t), \\ &u_2(\boldsymbol{x},t) = w_2(\boldsymbol{x},t) + g_2{}^B(\boldsymbol{x}) v_2{}^B(\boldsymbol{x},t), \qquad A,B = 1,\ldots,N. \end{aligned} \tag{5}$$

Both macrodeflections $w_1(\boldsymbol{x},t)$, $w_2(\boldsymbol{x},t)$ and fluctuation amplitudes $v_1{}^A(\boldsymbol{x},t)$, $v_2{}^B(\boldsymbol{x},t)$ are basic unknowns, additionally assumed to be slowly varying functions for every t.

The second assumption contain *the tolerance averaging approximations*. By introducing certain given ‘a priori’ *tolerance parameter* δ and keeping in mind properties of functions mentioned as introductory concepts, it is possible to prove the following equations:

$$\begin{aligned} &<\Phi>(x) = <\overline{\Phi}>(x) + O(\delta), \qquad <\Phi F>(x) = <\Phi>(x) F(x) + O(\delta), \\ &<\Phi \partial_\alpha (gF)>(x) = <\Phi \partial_\alpha g>(x) F(x) + O(\delta), \end{aligned} \tag{6}$$

where Φ is tolerance-periodic function, $\overline{\Phi}$ is periodic approximation of Φ, F is slowly varying function, g is highly oscillating function and $O(\delta)$ is negligibly small term, $0<\delta<<1$.

4. Tolerance modelling procedure and model equations

The starting point of the tolerance modelling procedure is the system of equations (2) together with denotations (3). By applying *the averaging operator* to (2) and transforming it with the use of both *the micro-macro decompositions* and *the tolerance averaging approximations*, the averaged form of system of equations (2) can be obtained in the form:

$$\begin{aligned} &\partial_{\alpha\beta}(<B_{\alpha\beta\gamma\delta}>\partial_{\gamma\delta} w_1 + <B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta} g_1^A> v_1^A) + <\mu + \tfrac{1}{2}\mu_c> \ddot{w}_1 + \\ &+ <k>(w_1 - w_2) + \underline{<k g_1^A>} v_1^A - \underline{<k g_2^B>} v_2^B = <p_1>, \\ &<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} g_1^A>\partial_{\gamma\delta} w_1 + <B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} g_1^A \partial_{\gamma\delta} g_1^B> v_1^A + \underline{<(\mu + \tfrac{1}{2}\mu_c) g_1^A g_1^B>} \ddot{v}_1^A + \\ &+ \underline{<k g_1^B>}(w_1 - w_2) + \underline{<k g_1^A g_1^B>} v_1^A - \underline{<k g_2^A g_1^B>} v_2^A = \underline{<p_1 g_1^B>}, \\ &\partial_{\alpha\beta}(<B_{\alpha\beta\gamma\delta}>\partial_{\gamma\delta} w_2 + <B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta} g_2^A> v_2^A) + <\mu + \tfrac{1}{2}\mu_c> \ddot{w}_2 + \\ &+ <k>(w_2 - w_1) + \underline{<k g_2^A>} v_2^A - \underline{<k g_1^B>} v_1^B = <p_2>, \\ &<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} g_2^B>\partial_{\gamma\delta} w_2 + <B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} g_2^A \partial_{\gamma\delta} g_2^B> v_2^A + \underline{<(\mu + \tfrac{1}{2}\mu_c) g_2^A g_2^B>} \ddot{v}_2^A + \\ &+ \underline{<k g_2^B>}(w_2 - w_1) + \underline{<k g_2^A g_2^B>} v_2^A - \underline{<k g_1^A g_2^B>} v_1^A = \underline{<p_2 g_2^B>}. \end{aligned} \tag{7}$$

The above system of equations constitute *the tolerance model of the periodic sandwich structures under consideration*. It is the system of $2N$+2 partial differential equations with constant coefficients, where the exact number of equations depends on the amount of assumed mode shape functions $g_1{}^A$, $g_2{}^B$, A,B = 1,...,N. System of equation (7) should be followed by four boundary conditions for every macrodeflection and a two initial conditions for every unknown function. It can be also observed, that only the underlined terms in (7) are dependent on *the microstructure parameter l*.

5. Asymptotic-tolerance modelling procedure and model equations

The asymptotic-tolerance model can be obtained in two steps, which are described for example by Woźniak et al. [12] or for plates by Kaźmierczak and Jędrysiak [13]. In the first step, the asymptotic solution to the problem is derived. In our considerations it can be obtained by omitting the underlined terms in equations (7). As a result, we arrive at:

$$\begin{aligned}
&\partial_{\alpha\beta}(<B_{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}w_1+<B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}g_1^A>v_1^A)+<\mu+\tfrac{1}{2}\mu_c>\ddot{w}_1+<k>(w_1-w_2)=<p_1>,\\
&<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}g_1^A>\partial_{\gamma\delta}w_1+<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}g_1^A\partial_{\gamma\delta}g_1^B>v_1^A=0,\\
&\partial_{\alpha\beta}(<B_{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}w_2+<B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}g_2^A>v_2^A)+<\mu+\tfrac{1}{2}\mu_c>\ddot{w}_2+<k>(w_2-w_1)=<p_2>,\\
&<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}g_2^B>\partial_{\gamma\delta}w_2+<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}g_2^A\partial_{\gamma\delta}g_2^B>v_2^A=0.
\end{aligned}\tag{8}$$

By analyzing the above system of equations, one can observe, that it can be easily transformed into a system of two equations with unknowns macrodeflections. Hence, macro-scale vibrations can be estimated regardless of the micro-scale fluctuations.

In the second step, an additional micro-macro decomposition, with the use of already known macrodeflections w_1^0, w_2^0, is applied to system of equations (2):

$$u_1(\boldsymbol{x},t)=w_1^0(\boldsymbol{x},t)+\hat{g}_1{}^A(\boldsymbol{x})V_1{}^A(\boldsymbol{x},t), u_2(\boldsymbol{x},t)=w_2^0(\boldsymbol{x},t)+\hat{g}_2{}^B(\boldsymbol{x})V_2{}^B(\boldsymbol{x},t).\tag{9}$$

Following the tolerance modelling procedure, after several manipulations, we arrive at the system of differential equations for fluctuation amplitudes $V_1^A(\boldsymbol{x},t)$, $V_2^B(\boldsymbol{x},t)$:

$$\begin{aligned}
&<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}\hat{g}_1^A\partial_{\gamma\delta}\hat{g}_1^B>V_1{}^A+\underline{<(\mu+\tfrac{1}{2}\mu_c)\hat{g}_1^A\hat{g}_1^B>}\ddot{V}_1{}^A+\underline{<k\hat{g}_1^A\hat{g}_1^B>}V_1{}^A+\\
&-\underline{<k\hat{g}_2^A\hat{g}_1^B>}V_2{}^A=\underline{<p_1\hat{g}_1^B>}-<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}\hat{g}_1^A>\partial_{\gamma\delta}w_1^0-\underline{<k\hat{g}_1^B>}(w_1^0-w_2^0),\\
&<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}\hat{g}_2^A\partial_{\gamma\delta}\hat{g}_2^B>V_2{}^A+\underline{<(\mu+\tfrac{1}{2}\mu_c)\hat{g}_2^A\hat{g}_2^B>}\ddot{V}_2{}^A+\underline{<k\hat{g}_2^A\hat{g}_2^B>}V_2{}^A+\\
&-\underline{<k\hat{g}_1^A\hat{g}_2^B>}V_1{}^A=\underline{<p_2\hat{g}_2^B>}-<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}\hat{g}_2^B>\partial_{\gamma\delta}w_2^0-\underline{<k\hat{g}_2^B>}(w_2^0-w_1^0).
\end{aligned}\tag{10}$$

Systems of equations (8) and (10) together constitute *the asymptotic-tolerance model of the periodic sandwich structure under consideration.* As a result, using this modelling procedure allows to perform a simplified analysis of vibrations in only macro- or micro-scale without the necessity of evaluating both. The amount of boundary and initial conditions is the same as in the tolerance model.

6. Calculation example - the analysis of free vibrations

Let us consider a rectangular three-layered plate, which is simply supported on all four edges. It is assumed, that the relations between characteristic dimensions of the structure can be formulated as follows: $L_2/L_1=2$, $l_2/l_1=2$. The outer layers of the plate are assumed to be made of periodically varying isotropic materials, having different Young's modulus E_1, E_2 and densities ρ_1, ρ_2, but constant Poisson's ratio $\nu=0.2$ and thickness $h=0.1l_1$, cf. Figure 2.

Let us introduce only one mode-shape function, the same for both upper and lower outer layer. Moreover, in order to obtain comparable results, let it be the same function for both tolerance and asymptotic-tolerance models:

$$\begin{aligned}
&G\equiv g_1{}^1=g_2{}^1=\hat{g}_1{}^1=\hat{g}_2{}^1=l_1^2\cos(2\pi x_1/l_1)\cos(2\pi x_2/l_2)+c,\\
&c=l_1^2<\hat{\mu}\cos(2\pi x_1/l_1)\cos(2\pi x_2/l_2)>/<\hat{\mu}>, \qquad \hat{\mu}=\mu+\tfrac{1}{2}\mu_c.
\end{aligned}\tag{11}$$

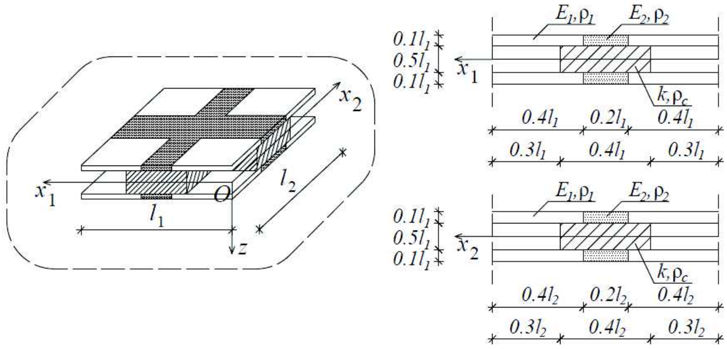

Figure 2. A periodicity cell of plate in the calculation example

By defining $A_{w_i}, A_{v_i}, A_{V_i}$ as amplitudes of unknowns, i = 1,2, λ_1, λ_2 as wave numbers and ω as a frequency, solutions to all governing equations can be assumed in the following forms, which satisfy boundary conditions:

$$w_i(\boldsymbol{x},t) = A_{w_i}\sin(\lambda_1 x_1)\sin(\lambda_2 x_2)e^{i\omega t},$$
$$v_i(\boldsymbol{x},t) = A_{v_i}\sin(\lambda_1 x_1)\sin(\lambda_2 x_2)e^{i\omega t}, \qquad V_i(\boldsymbol{x},t) = A_{V_i}\sin(\lambda_1 x_1)\sin(\lambda_2 x_2)e^{i\omega t}. \tag{12}$$

Let us neglect all external loadings. Free vibration frequencies are calculated by solving characteristic equations of homogeneous systems of equations (7) and (8), (10) and presented in dimensionless form, derived with the transformation below:

$$\overline{\omega} \equiv \omega[\rho_1 l_1^2 (E_1)^{-1}]^{0.5}. \tag{13}$$

Moreover, all calculations are performed for constant wave numbers: $\lambda_1=0.1/l_1$, $\lambda_2=0.1/l_2$. On charts in Figure 3 and 4 lower order frequencies are denoted as "a" and "b", while higher order frequencies as "c" and "d". Moreover, the tolerance model results are distinguished by subscript "1" and the asymptotic-tolerance model - by subscript "2".

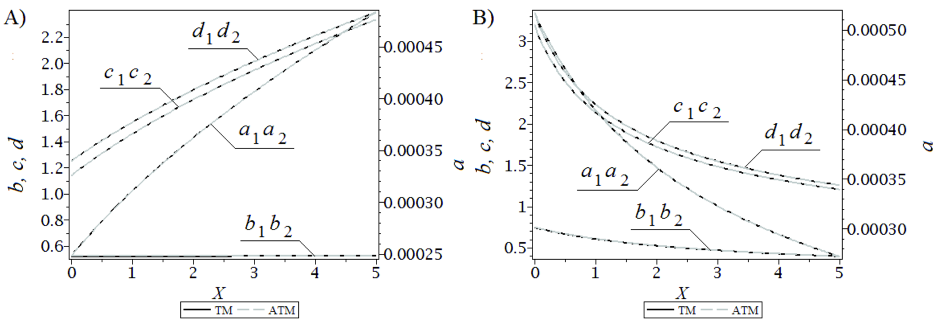

Figure 3. Dimensionless free vibrations frequencies' parameters $\overline{\omega}$ versus parameter X:
A) $E_2=XE_1$, $\rho_2=2\rho_1$, $k=0.03E_1/l_1$, $\rho_c=0.03\rho_1$,
B) $E_2=2E_1$, $\rho_2=X\rho_1$, $k=0.03E_1/l_1$, $\rho_c=0.03\rho_1$

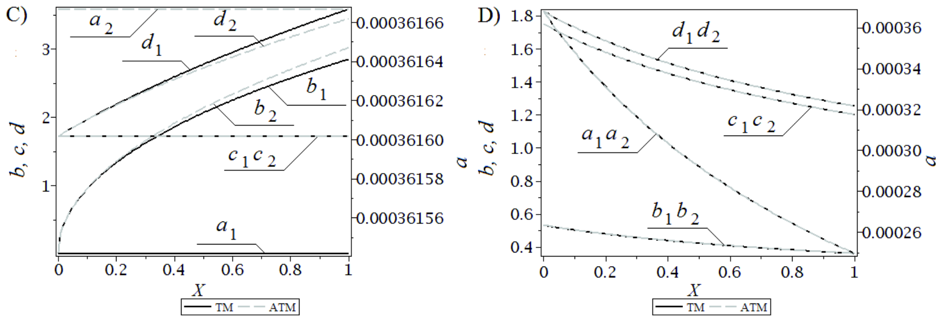

Figure 4. Dimensionless free vibrations frequencies' parameters $\overline{\omega}$ versus parameter *X*:
C) $E_2=2E_1$, $\rho_2=2\rho_1$, $k=XE_1/l_1$, $\rho_c=0.03\rho_1$,
D) $E_2=2E_1$, $\rho_2=2\rho_1$, $k=0.03E_1/l_1$, $\rho_c=X\rho_1$

7. Remarks

In this article, two averaged models describing vibrations of periodic three-layered plates are presented. The simple model of sandwich plate, described by Szcześniak [6], is extended and modified with the use of two modelling procedures of the tolerance averaging technique, so as structures with periodic microstructure can also be analyzed. As a result of these modifications, systems of governing equations with constant coefficients are obtained and solved.

Basing on the considered calculation examples, it can be observed that results of both models are comparable even for structures with much varying material properties. Hence, presented solutions can be used in the process of optimization of mechanical properties of considered sandwich structures, as a simple and convenient way of estimating the frequency of vibrations.

In the future investigations, the consistency of the proposed averaged models with finite element method will be presented. Moreover, a physical correctness of derived models will be described and justified.

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