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On a Certain Notion of Finite and a Finiteness Class in Set Theory without Choice

by

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Summary. We study the deductive strength of properties under basic set-theoretical operations of the subclass E-**Fin** of the Dedekind finite sets in set theory without the Axiom of Choice (**AC**), which consists of all E-finite sets, where a set X is called *E-finite* if for no proper subset Y of X is there a surjection $f : Y \to X$.

1. Introduction, terminology, and preliminary results

DEFINITION 1.1. **ZF** is the Zermelo-Fraenkel set theory minus the Axiom of Choice **AC**; and **ZFA** is **ZF** with the Axiom of Extensionality weakened in order to allow the existence of atoms.

The classical definition of a *finite set* is that a set X is finite if there exists a bijection $f : X \to n$ where n is a natural number ($n = \{m \in \omega : m < n\}$, where as usual ω denotes the set of all natural numbers). Otherwise, X is said to be *infinite*. In other words, X is finite if there exists an injection $f : X \to \omega$ and there is no injection $g : \omega \to X$. In this paper, we shall use the word "finite" in this classical sense and, as usual, *infinite* will mean "not finite".

We shall adopt the standard notation for comparability of sets as follows.

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The second- and third-named authors wish to dedicate this article to the memory of Horst Herrlich, who passed away on March 13, 2015.

Definition 1.2. Let X and Y be two sets.

$$\begin{array}{ll} X \leq Y & \text{if there exists an injection } f : X \to Y, \\ X \approx Y & \text{if there exists a bijection } f : X \to Y, \\ X < Y & \text{if } X \leq Y \text{ and } X \not\approx Y, \\ X \leq^* Y & \text{if } X = \emptyset \text{ or there exists a surjection } f : Y \to X, \\ X <^* Y & \text{if } X \leq^* Y \text{ and } X \not\approx Y. \end{array}$$

Definition 1.3. Let X be a set.

(1) X is called *A-finite* if X cannot be expressed as the disjoint union of two infinite sets. Otherwise, X is called *A-infinite.* If X is an infinite, A-finite set, then X is called *amorphous.*
(2) X is called *B-finite* if X has no infinite linearly orderable subsets. Otherwise, X is called *B-infinite.*
(3) X is called *C-finite* if there is no surjection $f : X \to \omega$. Equivalently (see [He, Lemma 4.11] or [Ta, pp. 94–95]), X is C-finite if there is no injection from $\mathcal{P}(X)$ (the power set of X) into a proper subset of $\mathcal{P}(X)$ (equivalently, X is C-finite if $\mathcal{P}(X)$ has no countably infinite subsets, i.e., $\mathcal{P}(X)$ is Dedekind-finite, see next item (4)). Otherwise, X is called *C-infinite.*
(4) X is called *D-finite* (or Dedekind-finite) if $\omega \not\leq X$. Equivalently, X is D-finite if there is no injection from X into a proper subset of X. Otherwise, X is called *D-infinite.* An infinite, D-finite set is called a *Dedekind set.*
(5) X is called *E-finite* if for no proper subset Y of X is there a surjection $f : Y \to X$. Otherwise, X is called *E-infinite.*

The notions of A-, B-, C-, and D-finite are due to Herrlich [Her]. We also note that A-finite was called Ia-*finite* by Lévy [L], and in the paper [Tr] by Truss the class of A-finite sets is denoted by Δ_1. In [Tr], the class of B-finite sets is denoted by Δ_3. C-finite is (equivalent to) III-*finite* in Lévy [L] ([1]), *weakly Dedekind finite* in Degen [De], and *almost finite* in Diel [Di]. In [Tr], the class of C-finite sets is denoted by Δ_4, and the class of D-finite sets by Δ.

The notion of E-finite has been suggested to us by Ulrich Felgner and it is the motivation for the research in this paper. In a way, the current paper continues the research in Herrlich–Howard–Tachtsis [HHT], in which properties of certain subclasses of the D-finite sets with respect to comparability of their elements and to boundedness of such classes were investigated. Here,

([1]) Lévy's formulation of III-finiteness is in particular the second clause of (3) in Definition 1.3, that is, X is III-finite if and only if $\mathcal{P}(X)$ is D-finite.

we specifically examine the class of all E-finite sets with respect to stability properties under basic set-theoretical operations.

DEFINITION 1.4. Let $n \geq 2$ be an integer. An *n-Russell set* is a set X which can be written as $X = \bigcup_{k\in\omega} X_k$ where

- for each $k \in \omega$, $X_k \approx n$,
- for i and j in ω, if $i \neq j$ then $X_i \cap X_j = \emptyset$, and
- no infinite subset of $\{X_k : k \in \omega\}$ has a choice function.

A 2-Russell set is also called a *Russell set.* It is clear that every n-Russell set, $n \in \omega \setminus 2$, is A-infinite, B-finite, C-infinite, and D-finite. We will prove in Theorem 2.2 that, in **ZF**, for every $n \geq 2$, every n-Russell set is E-finite.

Truss [Tr] introduced the following class of sets:

$$\Delta_5 = \{X : \text{there is no surjection } f : X \to X \cup \{a\}, \text{ where } a \notin X\},$$

and proved that $\Delta_5 \subset \Delta$, that is, every element of Δ_5 is a D-finite set.

It turns out that a set X is E-finite if and only if $X \in \Delta_5$ (hence, Δ_5 is the class of all E-finite sets, which in our paper will be denoted by E-**Fin**—see also Definition 1.9). Indeed, we have the following proposition.

PROPOSITION 1.5. *A set X is E-finite if and only if $X \in \Delta_5$.*

Proof. ($\to$) Let X be an E-finite set. Toward a contradiction assume that there is a surjection $f : X \to X \cup \{a\}$, $a \notin X$. Then $Y = X \setminus f^{-1}(\{a\})$ is a proper subset of X and $f \restriction Y$ is a function from Y onto X. Thus, X is E-infinite, a contradiction.

($\leftarrow$) This can be proved in a similar manner and is left to the reader. ∎

We also point out that E-finite is equivalent to *dually Dedekind finite* ([2]) of Degen [De] and to *strongly Dedekind finite* of Diel [Di].

In [Tr], the following result was proved, giving the relationship between the above notions of finite.

PROPOSITION 1.6. *The following hold:*

(1) *"Finite" implies "A-finite" implies "C-finite" implies "E-finite" implies "D-finite". None of these implications is reversible in* **ZF**.
(2) *"A-finite" implies "B-finite" implies "D-finite". None of these implications is reversible in* **ZF**.
(3) *"B-finite" does not imply "C-finite" and "C-finite" does not imply "B-finite" in* **ZF**. *Hence "E-finite" does not imply "B-finite" in* **ZF**.
(4) *"B-finite" does not imply "E-finite" in* **ZF**.

([2]) A set X is *dually Dedekind finite* if there is no non-injective surjection from X onto X.

The following definitions are from Herrlich [Her] (except for the class E-**Fin** in Definition 1.9(2)).

Definition 1.7. Let $\mathfrak{U}$ be a class of sets.

(1) $\mathfrak{U}$ is called *cardinality-determined* if $\mathfrak{U}$ contains, together with any set A, every set X with $X \approx A$.
(2) $\mathfrak{U}$ is called *hereditary* if $\mathfrak{U}$ contains, together with any set A, every set X with $X \leq A$.
(3) $\mathfrak{U}$ is called *cohereditary* if $\mathfrak{U}$ contains, together with any set A, every set X with $X \leq^* A$.
(4) $\mathfrak{U}$ is called *summable* if $\mathfrak{U}$ contains the union $\bigcup_{i\in I} A_i$ of any family $\{A_i : i \in I\}$ of members of $\mathfrak{U}$, indexed by a member I of $\mathfrak{U}$.
(5) $\mathfrak{U}$ is called *weakly summable* if $\bigcup \mathfrak{L} \in \mathfrak{U}$ for every member $\mathfrak{L}$ of $\mathfrak{U}$, all of whose members belong to $\mathfrak{U}$.
(6) $\mathfrak{U}$ is called *disjointly summable* if $\mathfrak{U}$ contains the union $\bigcup_{i\in I} A_i$ of any pairwise disjoint family $\{A_i : i \in I\}$ of members of $\mathfrak{U}$, indexed by a member I of $\mathfrak{U}$.
(7) $\mathfrak{U}$ is called *productive* if $\mathfrak{U}$ contains the product $\prod_{i\in I} A_i$ of any family $\{A_i : i \in I\}$ of members of $\mathfrak{U}$, indexed by a member I of $\mathfrak{U}$.
(8) $\mathfrak{U}$ is called *power-stable* if $\mathfrak{U}$ contains, together with any set A, also the power set $\mathcal{P}(A)$.

Definition 1.8. A class $\mathfrak{U}$ of sets is called a *finiteness class* if it satisfies the following conditions:

(1) $\mathfrak{U}$ contains every finite set.
(2) $\mathfrak{U}$ is hereditary.
(3) $\omega \notin \mathfrak{U}$.

We note that parts (2) and (3) of the definition imply that all finiteness classes consist of D-finite sets.

Definition 1.9. **Fin** is the class of all finite sets; and if $L \in \{A, B, C, D, E\}$, then L-**Fin** is the class of all L-finite sets.

Proposition 1.10. *The following statements hold:*

(1) ([Her]) *Let $\mathfrak{U}$ be a finiteness class. Then:*
    (a) *$\mathfrak{U}$ is summable if and only if $\mathfrak{U}$ is weakly summable.*
    (b) *If $\mathfrak{U}$ is summable, then $\mathfrak{U}$ is disjointly summable.*
    (c) *If $\mathfrak{U}$ is summable, then $\mathfrak{U}$ is cohereditary.*
    (d) *If $\mathfrak{U}$ is productive, then $\mathfrak{U}$ is power-stable.*
(2) ([Her], [Tr]) **Fin** *and L-**Fin**, where $L \in \{A, B, C, D, E\}$, are finiteness classes.*

(3) ([Her]) **Fin** *is the smallest* (*with respect to inclusion*) *finiteness class. Further,* **Fin** *is the only productive and the only power-stable finiteness class.*
(4) ([Her]) D-**Fin** *is the largest* (*with respect to inclusion*) *finiteness class. Further,* D-**Fin** *is disjointly summable in* **ZF**, *but it is relatively consistent with* **ZF** *that* D-**Fin** *is not summable.*
(5) ([Her]) B-**Fin** *is disjointly summable in* **ZF**, *but it is relatively consistent with* **ZF** *that* B-**Fin** *is not summable.*
(6) ([Her]) C-**Fin** *is, in* **ZF**, *the largest cohereditary finiteness class and the largest summable finiteness class.*
(7) ([Tr]) *A finite union of* E*-finite sets is* E*-finite. It is relatively consistent with* **ZF** *that there are two* E*-finite sets whose product is not* E*-finite.*
(8) ([Tr]) *A finite union of* A*-finite sets is* A*-finite if and only if* A-**Fin** $=$ **Fin**.
(9) ([Tr]) *A finite union or product of* B*-finite* (*resp.* C*-finite*) *sets is* B*-finite* (*resp.* C*-finite*).

In order to prove that a finite union of E-finite sets is E-finite, Truss [Tr, Theorem 1(iv)] used the following result (labeled as Lemma 4 in [Tr]) by Lindenbaum and Tarski [LT]. Since Truss refers to [LT] for a proof and we had no access to that paper, we include our own proof here.

PROPOSITION 1.11. *Let* X, Y, Z *be three sets such that* $X \cap Y = X \cap Z = \emptyset$ *and* $(X \cup Y) \leq^* X \cup Z$. *Then* Y *can be partitioned into sets* A *and* B *such that* $B \leq^* Z$ *and* $X \cup A \leq^* X$.

Proof. Without loss of generality assume that $Y \cap Z = \emptyset$ (if not, then the proof goes through with obvious minor changes). If $X \cup Y = \emptyset$, then there is nothing to prove. Otherwise, there exists a surjection $f : X \cup Z \to X \cup Y$. Let B be the set of all $y \in Y$ for which there exists a finite sequence

$$(a(0), a(1), \ldots, a(n)) \tag{1.1}$$

such that

(a) $a(i+1) = f(a(i))$ for each $i < n$.
(b) $a(0) \in Z$.
(c) $a(i) \in X$ for each $0 < i < n$.
(d) $a(n) = y$.

If $B = \emptyset$, then $B \leq^* Z$ automatically. Otherwise, select $b^* \in B$ and define a surjection $g : Z \to B$ as follows: $g(a(0)) = a(n)$ for every finite sequence of the form (1.1), and $g(z) = b^*$ for all the remaining elements of Z.

Next we define a surjection $h : X \to X \cup A$, where $A = Y \setminus B$, as follows: For each $y \in A$, define a sequence $(X(y, n))$ of non-empty, pairwise disjoint subsets of X by recursion as follows:

$$X(y,0) = f^{-1}(\{y\}), \qquad X(y,n+1) = f^{-1}(X(y,n)).$$

Then $\{X(y) : y \in A\}$, where $X(y) = \bigcup\{X(y,n) : n \in \omega\}$, is a pairwise disjoint family of subsets of X. Define $X^* = \bigcup\{X(y) : y \in A\}$ and a map $h : X \to X \cup A$ by requiring $h(x) = f(x)$ if $x \in X^*$, and $h(x) = x$ otherwise. Then h is a surjection, finishing the proof of the proposition. ∎

In this paper, we also plan to investigate variations (with respect to finiteness classes and basically with respect to E-**Fin**) of the following principle **SP**, called the *shrinking principle* in [BS]:

SP: *For every family $\{A_i : i \in I\}$ of sets there exists a family $\{B_i : i \in I\}$ of pairwise disjoint subsets B_i of A_i such that $\bigcup_{i \in I} B_i = \bigcup_{i \in I} A_i$.*

It is known (see [BS] or [Her, p. 13, Exercise E 5] or the proof of Theorem 2.12) that **SP** is, in **ZF**, equivalent to **AC**. If $\mathfrak{U}$ is a class of sets, then we let

$\mathbf{SP}^{\mathfrak{U}}$: *For every family $\{A_i : i \in I\}$ of sets, where $I \in \mathfrak{U}$, there exists a family $\{B_i : i \in I\}$ of pairwise disjoint subsets B_i of A_i such that $\bigcup_{i \in I} B_i = \bigcup_{i \in I} A_i$.*

$\mathbf{SP}_{\mathfrak{U}}$: *For every family $\{A_i : i \in I\}$ of sets, where $A_i \in \mathfrak{U}$ for all $i \in I$, there exists a family $\{B_i : i \in I\}$ of pairwise disjoint subsets B_i of A_i such that $\bigcup_{i \in I} B_i = \bigcup_{i \in I} A_i$.*

$\mathbf{SP}^{\mathfrak{U}}_{\mathfrak{U}}$: *For every family $\{A_i : i \in I\}$ of sets, where $I \in \mathfrak{U}$ and $A_i \in \mathfrak{U}$ for all $i \in I$, there exists a family $\{B_i : i \in I\}$ of pairwise disjoint subsets B_i of A_i such that $\bigcup_{i \in I} B_i = \bigcup_{i \in I} A_i$.*

For a class $\mathfrak{U}$ of sets, we also consider the following restricted choice forms:

$\mathbf{AC}^{\mathfrak{U}}$: *Every family $\{A_i : i \in I\}$ of non-empty sets, where $I \in \mathfrak{U}$, has a choice function.*

$\mathbf{AC}_{\mathfrak{U}}$: *Every family $\{A_i : i \in I\}$ of sets, where $A_i \in \mathfrak{U} \setminus \{\emptyset\}$ for all $i \in I$, has a choice function.*

$\mathbf{AC}^{\mathfrak{U}}_{\mathfrak{U}}$: *Every family $\{A_i : i \in I\}$ of sets, where $I \in \mathfrak{U}$ and $A_i \in \mathfrak{U} \setminus \{\emptyset\}$ for all $i \in I$, has a choice function.*

The following result has been established in [Her, Theorem 12] (see also [He, E 15, p. 51]).

Proposition 1.12. $\mathbf{SP}^{D\text{-}\mathbf{Fin}}$ *holds if and only if* $\mathbf{SP}^{D\text{-}\mathbf{Fin}}_{D\text{-}\mathbf{Fin}}$ *holds if and only if "D-**Fin** = **Fin**" if and only if "D-**Fin** is summable".*

In the current paper, we show that the situation with the finiteness class E-**Fin** is *strikingly different*! Among other results, we shall prove that $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$ implies "E-**Fin** is summable" (Theorem 2.13), but the implication is not reversible in **ZFA** (Theorem 2.14). In fact, we show something stronger: $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ implies "E-**Fin** is summable" (Theorem 2.13), but the implication is not reversible in **ZFA** (Theorem 2.14).

2. Main results

2.1. Stability properties for the class E-Fin and related propositions. The subsequent lemma (Truss' Lemma 8 of [Tr, p. 192]) will play a crucial role in our investigation.

LEMMA 2.1. *If $X = \bigcup\{X_i : i \in \omega\}$ where each X_i is C-finite, and X is E-infinite, then there is a surjection $f : X \to X \cup \{a\}$, $a \notin X$, and a strictly increasing function $n : \omega \to \omega$ such that for each $i \in \omega$,*

$$f^{-(i+1)}(\{a\}) \subseteq X_{n(i)}.$$

THEOREM 2.2. *Let $m \geq 2$ be an integer. Then every m-Russell set is E-finite.*

Proof. Toward a contradiction, we assume that there exists an m-Russell set $X = \bigcup\{X_i : i \in \omega\}$ which is E-infinite. Then the hypotheses of Lemma 2.1 apply to X, so let $f : X \to X \cup \{a\}$, $a \notin X$, and $n : \omega \to \omega$ be the two functions obtained from the conclusion of that lemma. For each $i \in \omega$, let $Y_i = f^{-(i+1)}(\{a\})$. Then for all $i \in \omega$, f restricted to Y_{i+1} is a function from Y_{i+1} onto Y_i. Since for all $i \in \omega$, $Y_i \subseteq X_{n(i)}$ and $X_{n(i)}$ is finite, we conclude that for all $i \in \omega$, $Y_i \leq Y_{i+1}$. Since $X_i \approx m$ for all $i \in \omega$, it follows that $Y_i \leq m$ for all $i \in \omega$, so there is an $n_0 \in \omega$ such that $Y_i \approx Y_j$ for all $i, j \geq n_0$. Therefore for all $i \geq n_0$, f restricted to Y_{i+1} is an injective function from Y_{i+1} onto Y_i, hence f^{-1} restricted to Y_i is an injective function from Y_i onto Y_{i+1}.

Choose $t \in Y_{n_0}$. Then the set

$$Y = \bigcup\{\{t\}, f^{-1}(\{t\}), f^{-2}(\{t\}), f^{-3}(\{t\}), \ldots\}$$

is a countably infinite subset of X. This contradicts the fact that X is an m-Russell set and completes the proof of the theorem. ∎

We continue now with the investigation of stability properties under set theoretical operations for the class E-**Fin**. The subsequent simple Theorem 2.3, and in particular the equivalence between its parts (a) and (d), is one of the key results for this paper.

THEOREM 2.3. *The following statements are pairwise equivalent:*

(a) E-**Fin** *is summable.*

(b) E-**Fin** *is weakly summable.*
(c) E-**Fin** *is cohereditary.*
(d) E-**Fin** $= C$-**Fin**.

Proof. (a)↔(b) holds for every finiteness class (Proposition 1.10(a)), hence for E-**Fin** too.

(b)→(c). This follows from (a)↔(b) and from Proposition 1.10(c).

(c)→(d). By Proposition 1.6(1) we have C-**Fin** $\subseteq E$-**Fin**. Suppose now, toward a contradiction, that there is an E-finite, C-infinite set X. Then $\omega \leq^* X$, hence by (c), $\omega \in E$-**Fin**, a contradiction.

(d)→(a). This follows from our assumption and Proposition 1.10(6). ■

THEOREM 2.4. *The following three statements can be added to the list of the pairwise equivalent statements of Theorem* 2.3*:*

(1) E-**Fin** *is disjointly summable.*
(2) *There are no E-finite countably infinite unions of non-empty sets.*
(3) *There are no E-finite countably infinite unions of non-empty E-finite sets.*

Proof. "E-**Fin** is summable" → (1). This is straightforward.

(1)→(2). We first prove the following lemma.

LEMMA 2.5. *"E-**Fin** is disjointly summable" implies "E-**Fin** is closed under finite products".*

Proof. It suffices to show that our assumption implies that the product of two E-finite sets is E-finite. Then one proceeds via a straightforward induction. So let A and B be two E-finite sets and let $\mathcal{U} = \{A \times \{b\} : b \in B\}$. Clearly, $\mathcal{U}$ is an E-finite ($\mathcal{U} \approx B$, $B \in E$-**Fin**, and E-**Fin** is hereditary in **ZF** (Proposition 1.10(2)) pairwise disjoint family of E-finite sets (A is E-finite, hence $A \times \{b\}$ ($\approx A$) is E-finite for all $b \in B$). Thus, by our assumption, $\bigcup \mathcal{U} = A \times B$ is E-finite. ■

We return now to the proof of (1)→(2). Toward a contradiction, we assume that there is an E-finite set X which is a union of a countably infinite family of non-empty sets. Then $\mathcal{P}(X)$ has a countably infinite subset and is therefore D-infinite. As noted in Definition 1.3(3), X is C-infinite and therefore there is a countably infinite partition $\{X_i : i \in \omega\}$ of X into non-empty sets. Since X is E-finite and $X_i \subseteq X$ for all $i \in \omega$, each X_i is E-finite.

Let $Y(0) = X_0$, and for $i \in \omega \setminus 1$ and $x \in X_i$, let

$$Y(x) = X_0 \times X_1 \times \cdots \times X_{i-1} \times \{x\}.$$

Then $Y(x) \neq \emptyset$ for all $x \in X \setminus X_0$, and by Lemma 2.5, $Y(x) \in E$-**Fin** for all $x \in X \setminus X_0$. Furthermore, $\mathcal{Y} = \{Y(x) : x \in X \setminus X_0\} \cup \{Y_0\}$ is pairwise disjoint and belongs to E-**Fin** since $X \in E$-**Fin**, and E-**Fin** is hereditary and contains all finite sets. Thus, by our assumption, $Z = \bigcup \mathcal{Y}$ is E-finite. Define a function $f : Z \setminus Y(0) \to Z$ as follows: Let $x \in X_n$ for some $n \geq 1$ and let $(x_0, x_1, \ldots, x_{n-1}) \in X_0 \times X_1 \times \cdots \times X_{n-1}$. Then define $f((x_0, \ldots, x_{n-1}, x)) = (x_0, \ldots, x_{n-1})$. It is straightforward to verify that f is a surjection, thus Z is E-infinite, a contradiction. This completes the proof of (1)→(2).

(2)→(3). This is straightforward.

(3) → "E-**Fin** is summable". Assume (3) holds. By Theorem 2.3, it suffices to show that E-**Fin** = C-**Fin**. Assume otherwise; then since C-**Fin** ⊆ E-**Fin** (see Proposition 1.6(1)), there exists a C-infinite, E-finite set X. Thus, X can be expressed as a countably infinite disjoint union $\bigcup\{X_n : n \in \omega\}$, where each X_n is non-empty and E-finite (E-**Fin** is hereditary, hence each subset X_n of the E-finite set X is E-finite). This contradicts our assumption, hence E-**Fin** is summable as required. ∎

Remark 2.6. We note that the statement "E-**Fin** is closed under finite products" *is independent of the axioms of* **ZF** + (¬**AC**) *set theory.* Indeed, in [Tr, Theorem 5(iii)], it is shown that "E-**Fin** = C-**Fin**" is relatively consistent with **ZF** + (¬**AC**), hence (by Proposition 1.10(9)), "E-**Fin** is closed under finite products" is also relatively consistent with **ZF** + (¬**AC**).

On the other hand, Truss [Tr, p. 193] constructs a permutation model of **ZFA** in which there exist two E-finite sets whose product is E-infinite. Then the latter result is transferred to **ZF** + (¬**AC**) via the Jech–Sochor Theorem (see also [J, Problem 1 in Chapter 6]). Thus, "E-**Fin** is *not* closed under finite products" is also relatively consistent with **ZF** + (¬**AC**).

Theorem 2.7. *Each of the following statements implies the one beneath it:*

1. E-**Fin** *is summable.*
2. *The Partial Kinna–Wagner Selection Principle for countable families of C-finite sets, each with at least two elements* (*i.e., for every family $\mathcal{A} = \{A_i : i \in \omega\}$ of C-finite sets, each with at least two elements, there is an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ and a function f with domain $\mathcal{B}$ such that for every $B \in \mathcal{B}$, $f(B)$ is a non-empty proper subset of B*).
3. *There are no n-Russell sets, for all integers $n \geq 2$.*

The implication "(2)→(3)*" is not reversible in* **ZFA**.

Proof. (1)→(2). Let $\mathcal{X} = \{X_i : i \in \omega\}$ be a countable family of C-finite sets, each with at least two elements. Without loss of generality, we assume that $\mathcal{X}$ is pairwise disjoint (if not, replace each X_i by $X_i \times \{i\}$; then since C-**Fin** is hereditary, it follows that $X_i \times \{i\}$ is C-finite).

Toward a contradiction, we assume that $\mathcal{X}$ does not admit a partial Kinna–Wagner selection function. By (1), it follows that $X = \bigcup \mathcal{X}$ is E-infinite, hence by Lemma 2.1, let $f : X \to X \cup \{a\}$, $a \notin X$, and let $n : \omega \to \omega$ be the two functions obtained by the conclusion of that lemma applied to the family $\mathcal{X}$. For each $i \in \omega$, set $Y_i = f^{-(i+1)}(\{a\})$. Since $Y_i \subseteq X_{n(i)}$ for all $i \in \omega$, and $\mathcal{X}$ has no partial Kinna–Wagner selection function, there is an $i_0 \in \omega$ such that

$$(\forall i \in \omega)\ [i \geq i_0 \ \Rightarrow\ Y_i = X_{n(i)}]. \tag{2.1}$$

Choose any $x \in Y_{i_0}$ $(= X_{n(i_0)})$. Since $X_{n(i_0+1)} = Y_{i_0+1} = f^{-1}(Y_{i_0})$ and f is a surjection, it follows that $f^{-1}(\{x\})$ is a non-empty proper subset of $X_{n(i_0+1)}$. In a similar fashion it follows, by induction, that for all $k \in \omega \setminus 1$, $f^{-k}(\{x\})$ is a non-empty proper subset of $X_{n(i_0+k)}$, so that

$$K = \bigcup \{\{x\}, f^{-1}(\{x\}), f^{-2}(\{x\}), f^{-3}(\{x\}), \ldots\}$$

is a Kinna–Wagner selection set for the infinite subfamily $\{X_{n(i)} : i \in \omega,\ i \geq i_0\}$ of $\mathcal{X}$, a contradiction.

(2)→(3). Use (2) and induction. We leave the details to the interested reader.

For the last assertion of the theorem we employ the Fraenkel–Mostowski (FM) model $\mathcal{N}6$ in [HR] (Lévy's Model I): The set of atoms $A = \bigcup\{P_n : n \in \omega\}$ is a disjoint union, where $P_0 = \{a_0\}$, $P_1 = \{a_1, a_2\}$, $P_2 = \{a_3, a_4, a_5\}$, $P_3 = \{a_6, a_7, a_8, a_9\}, \ldots$; in general for $n \in \omega$ with $n > 0$, $|P_n| = p_n$, where p_n is the nth prime. G is the group of all permutations of A generated by $\{\pi_n : n \in \omega\}$, where, if $P_n = \{a_{m+1}, a_{m+2}, \ldots, a_{m+p_n}\}$, then

$$\pi_n : a_{m+1} \mapsto a_{m+2} \mapsto \cdots \mapsto a_{m+p_n} \mapsto a_{m+1}, \quad \pi_n(x) = x \text{ for all } x \notin P_n.$$

$\mathcal{I}$ is the normal ideal of all finite subsets of A. It is known that the axiom of choice for families of n-element sets, $n \in \omega$, $n > 0$, holds in the model (see [HR] or [J, Theorem 7.11]). Hence, there are no n-Russell sets, $n \in \omega$, $n \geq 2$, in $\mathcal{N}6$. On the other hand, using standard techniques of FM models, it can be easily verified that the family $\{P_n : n \in \omega\}$ is countable in the model, consists of finite, hence C-finite, sets, and admits no partial Kinna–Wagner selection function in the model. Hence, our assertion is valid. ∎

THEOREM 2.8.

(1) *"E-**Fin** = **Fin**" if and only if "E-**Fin** is power-stable" if and only if "E-**Fin** is productive".*
(2) *Each of the following statements implies the one beneath it:*
   (a) D-**Fin** = **Fin**.
   (b) E-**Fin** = **Fin**.
   (c) *For every infinite set X, $2 \times \mathcal{P}(X) \approx \mathcal{P}(X)$.*
   (d) *For every infinite set X, $\mathcal{P}(X)$ is D-infinite (if and only if "C-**Fin** = **Fin**").*
   (e) A-**Fin** = **Fin**.

*Hence, "E-**Fin** = **Fin**" is not provable in* **ZF**. *In addition,* (c)*, hence* (d) *and* (e)*, does not imply* (b) *in* **ZFA**.

(3) *"E-**Fin** = **Fin**" if and only if "(C-**Fin** = **Fin**) + (E-**Fin** is summable)". Hence, "E-**Fin** = **Fin**" implies each one of the statements listed in Theorem* 2.7.

(4) *"E-**Fin** is summable" does not imply "E-**Fin** = **Fin**" in* **ZFA**. *Hence, "E-**Fin** is summable" implies neither "D-**Fin** = **Fin**" nor "For all infinite cardinals m, $2m = m$" in* **ZFA**.

(5) *The axiom of multiple choice* (**MC**) *implies, in* **ZFA**, *"C-**Fin** = **Fin**".*

(6) *The Kinna–Wagner selection principle* (**KW**) *implies, in* **ZF**, *"C-**Fin** = **Fin**".*

(7) *"C-**Fin** = **Fin**" does not imply "E-**Fin** = **Fin**", in both* **ZFA** *and* **ZF**.

Proof. (1). This follows from Proposition 1.10(3).

(2) (a)→(b). This follows from Proposition 1.6(1).

(b)→(c). Let X be an infinite set. By our assumption, X is E-infinite, hence there is a surjection $f : X \to X \cup \{a\}$, $a \notin X$. It is clear that $\mathcal{P}(X \cup \{a\}) \approx 2 \times \mathcal{P}(X)$ for $a \notin X$. Define a function $g : \mathcal{P}(X \cup \{a\}) \to \mathcal{P}(X)$ by setting $g(A) = f^{-1}(A)$ for all $A \subseteq X \cup \{a\}$. Then g is injective, thus (by the Cantor–Bernstein theorem, which is provable in **ZF**) $2 \times \mathcal{P}(X) \approx \mathcal{P}(X)$.

(c)→(d). This can be proved similarly to the proof of Theorem 2.3 in Halpern and Howard [HH, p. 488]. The details are left to the reader. For the second assertion, see Definition 1.3(3).

(d)→(e). This follows from our assumption and Proposition 1.6(1). The reader is also refered to [Her, Proposition 21] for several characterizations of "A-**Fin** = **Fin**".

For the second assertion of (2), note that the existence of an amorphous set—hence the existence of an infinite E-finite set—is relatively consistent with **ZF** (see [J]).

For the third and also the last assertion of (2), we note first that in Howard and Spišiak [HS, Theorem 2.1], it is shown that, under $\mathbf{AC}_2$ (the axiom of choice for families of pairs), "C-**Fin** = **Fin**" implies that for every infinite set X, $2 \times \mathcal{P}(X) \approx \mathcal{P}(X)$ (in particular, the authors in [HS] show that, under $\mathbf{AC}_2$ + (C-**Fin** = **Fin**), for every infinite set X, there is a set $\mathbf{D} \neq \emptyset$ such that $2^\omega \times \mathbf{D} \approx \mathcal{P}(X)$; $\mathbf{D}$ is a suitable partition of $\mathcal{P}(X)$).

Consider now the FM model $\mathcal{N}6$ in [HR]; see the proof of Theorem 2.7 for the description of the model. Since **MC** (the axiom of multiple choice) holds in $\mathcal{N}6$ (see [HR]), by (5) every infinite set in the model is C-infinite, i.e., C-**Fin** = **Fin** in $\mathcal{N}6$. Further, as $\mathbf{AC}_2$ holds in $\mathcal{N}6$ (see [HR] or [J]), by the above result of Howard and Spišiak, $2 \times \mathcal{P}(X) \approx \mathcal{P}(X)$ for every infinite set

$X \in \mathcal{N}6$. However, the family $\{P_n : n \in \omega\}$, where $A = \bigcup\{P_n : n \in \omega\}$ is the set of atoms, does not admit a partial Kinna–Wagner selection function (see [HR]), hence since each P_n is a finite set, by the proof of (2)→(3) of Theorem 2.7 we deduce that the infinite set A is E-finite, thus E-**Fin** $\neq$ **Fin** in $\mathcal{N}6$.

(3) This follows from (b)→(d) of (2) and Theorem 2.3.

(4) In Mostowski's linearly ordered permutation model (model $\mathcal{N}3$ in [HR]), E-**Fin** $=$ C-**Fin** and E-**Fin** $\neq$ **Fin** (see [Tr, pp. 203–204]); the set A of atoms is an infinite E-finite, hence D-finite, set in $\mathcal{N}3$. Thus, D-**Fin** $\neq$ **Fin** in $\mathcal{N}3$ and, by Theorem 2.3, E-**Fin** is summable in $\mathcal{N}3$. On the other hand, it is well-known that the infinite set A of atoms is a D-finite set in $\mathcal{N}3$, hence "D-**Fin** $=$ **Fin**" fails in $\mathcal{N}3$.

The last assertion of (4) follows from the fact that "For all infinite cardinals m, $2m = m$" implies every infinite set is D-infinite (see [HH]).

(5) **MC** is the principle: For every family $\{X_i : i \in I\}$ of non-empty sets there exists a function F with domain I such that for each $i \in I$, $F(i)$ is a non-empty finite subset of X_i.

It is known (see [HR]) that **MC** is equivalent to the proposition: Every infinite set has a well-orderable partition into non-empty finite sets (this is Lévy's characterization of **MC**) and that **MC** is equivalent to **AC** in **ZF**. Using Lévy's characterization of **MC**, it can be readily verified now that, under **MC**, every infinite set is C-infinite, hence C-**Fin** $=$ **Fin**.

(6) The principle **KW** is the proposition: For every family $\mathcal{A}$ of sets there is a function f (called a *Kinna–Wagner selection function*) such that for all $A \in \mathcal{A}$, if $2 \leq A$, then $\emptyset \neq f(A) \subsetneq A$.

Now, **KW** implies "For every infinite set X, $\mathcal{P}(X)$ is D-infinite": to see this it suffices to suitably adopt the proof of the Lemma (**KW** implies every infinite set can be linearly ordered) of Felgner [F, p. 123]. Since the statement "For every infinite set X, $\mathcal{P}(X)$ is D-infinite" is equivalent to "C-**Fin** $=$ **Fin**", the result follows.

(7) In the second Fraenkel permutation model (model $\mathcal{N}2$ in [HR]) **MC** is true (see [HR]). Thus, by (5), also C-**Fin** $=$ **Fin** in $\mathcal{N}2$. On the other hand, it is known that the infinite set A of atoms is a Russell set, hence, by Theorem 2.2, A is E-finite. Thus, E-**Fin** $\neq$ **Fin** in $\mathcal{N}2$.

Now, in the basic Cohen model $\mathcal{M}1$, **KW** holds (see [HR]). Thus, by (5), also C-**Fin** $=$ **Fin** in $\mathcal{M}1$. On the other hand, the infinite set A of the countably many added generic reals is E-finite (see [Tr]), hence, in $\mathcal{M}1$, E-**Fin** $\neq$ **Fin**. ∎

Remark 2.9. (1) The fact that C-**Fin** $=$ **Fin** in $\mathcal{N}2$ has been shown differently in [Tr] via techniques of FM models and properties of $\mathcal{N}2$. However, the result of Theorem 2.8(5) is more general (recall that $\mathcal{N}2 \models$ **MC**), thus providing a strengthening of the aforementioned result by Truss.

(2) Let $\mathbf{KW}(\mathbb{R})$ be the Kinna–Wagner selection principle restricted to families of sets of reals, each with at least two elements. It is clear that $\mathbf{KW}(\mathbb{R})$ is a theorem of **ZF**, thus it can be proved without using any form of choice. As in Theorem 2.8(6), it can be verified that $\mathbf{KW}(\mathbb{R})$ implies that for every infinite set X of reals, $\mathcal{P}(X)$ is D-infinite. It follows that, *in ZF, every infinite set of reals is C-infinite*. A different proof of this fact has been given in Degen [De, Corollary 15]. A third proof will be given in our Theorem 2.11 below.

(3) We point out that "E-**Fin** $= D$-**Fin**" *if and only if* "D-**Fin** $=$ **Fin**". Indeed, assuming that E-**Fin** $= D$-**Fin**, let X be a D-finite set, and toward a contradiction, assume that it is infinite (i.e., X is a Dedekind set in the terminology of Definition 1.3(4)). Let $Y = \bigcup\{Y_n : n \in \omega\}$ where $Y_n = \{f \in X^n : f \text{ is an injection}\}$. It is easy to see that Y is D-finite, hence by our assumption, it is also E-finite. However, it is not hard to verify that Y is E-infinite. Define $f : Y \setminus Y_0 \to Y$ by setting $f(s) = \emptyset$ for all $s \in Y_1$, and for $(x_1, \ldots, x_n) \in Y$, $n > 1$, let $f((x_1, \ldots, x_n)) = (x_1, \ldots, x_{n-1})$. Clearly, f is a surjection (see also [Tr, Lemma 6, p. 190]), a contradiction.

(4) In [HR, Note 94] the following definition is given: Let F be a definition of finite (say for example, in our paper, standard finite, or A-, B-, C-, D, or E-finite). Then a set A is *F''-finite* if $\mathcal{P}(A)$ is F-finite. Also, a set A is called *V-finite* if $A = \emptyset$ or $A < 2 \times A$. In [HR, Note 94, p. 280], the question on the relationship between E-finite and V''-finite is raised. The proof of (b)→(c) of Theorem 2.8(2) gives a partial answer to this question. In particular, V''-finite implies E-finite. The latter implication has also been established in [Cr, Theorem 5].

2.2. C-finiteness, E-finiteness, and sets of reals

THEOREM 2.10. *Consider the basic Cohen model* (*model $\mathcal{M}1$ in* [HR]) *and let $A = \{a_i : i \in \omega\} \in \mathcal{M}1$ be the D-finite set of the countably many added generic reals. Then:*

(1) *For every natural number $n \geq 1$, A^n is E-finite. Hence, the existence of an infinite, E-finite subset of $\mathbb{R}$ (or of $\mathbb{R}^n$, $n \in \omega$, or of $\mathbb{R}^\omega$) is relatively consistent with* **ZF**.
(2) *"E-**Fin** is summable" fails in $\mathcal{M}1$. Hence, "E-**Fin** is disjointly summable" also fails in $\mathcal{M}1$ (see Theorem 2.4).*
(3) $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *fails in $\mathcal{M}1$. Hence, in view of Theorem 2.13(4) below, the shrinking principle $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ also fails in $\mathcal{M}1$.*

Proof. (1) The fact that A is E-finite in $\mathcal{M}1$ has been proved in [Tr, p. 199]. We only show that A^2 is also E-finite in $\mathcal{M}1$; the proof that any other finite power n of A is E-finite is similar, and we leave the details to the interested reader. (We remind the reader that "E-**Fin** is not closed under finite products" is consistent relative to **ZF**—see Proposition 1.10. Furthermore, a careful reading of the proof of Proposition 1.11 should reveal why one should not be tempted to use this result for the proof.) For our proof, we follow the terminology in [J]. Toward a contradiction, we assume that, in $\mathcal{M}1$, there exists a surjection $f : A \times A \to (A \times A) \cup \{a\}$ where $a \notin A \times A$. Let $\dot{f}$ be a hereditarily symmetric name of f with support E (a finite subset of ω), that is, every permutation π of ω which fixes E pointwise fixes $\dot{f}$. Let p be a forcing condition in the generic filter G such that

$$p \Vdash \text{“}\dot{f} \text{ is a surjection from } (A \dot{\times} A) \text{ to } ((A \times A)^{\cdot} \cup \{a\})\text{”}.$$

Since every finite set is E-finite, it follows that there are infinitely many points $(x, y) \in A \times A$ such that $f((x, y)) \neq (x, y)$. Furthermore, since f is a surjection, it cannot be the case that for all $x \in A \setminus \bar{E}$, where $\bar{E} = \{a_e : e \in E\}$, $A \times \{x\}$ and $\{x\} \times A$ are mapped onto $A \times \{u\}$ or $\{u\} \times A$, where $u \in \bar{E}$. In addition, using standard forcing arguments and the fact that A is E-finite, it follows that it is not the case that for all $x \in A \setminus \bar{E}$, $A \times \{x\}$ is mapped (via f) onto $\{x\} \times A$ and vice versa. We leave the fairly easy checking of the details to the interested reader.

In view of the above and the fact that E and $\mathrm{Dom}(p)$ are finite, we may conclude that there exist $i, j, r, s \in \omega$ such that $f((a_i, a_j)) = (a_r, a_s)$ and either $r \neq i, j$, $r > n$ for all $n \in E$, and $(r, m) \notin \mathrm{Dom}(p)$ for all $m \in \omega$, or $s \neq i, j$, $s > n$ for all $n \in E$, and $(s, m) \notin \mathrm{Dom}(p)$ for all $m \in \omega$. Without loss of generality assume that $r \neq i, j$, $r > n$ for all $n \in E$, and $(r, m) \notin \mathrm{Dom}(p)$ for all $m \in \omega$.

Let q be a forcing condition in G such that $q \leq p$ and

$$q \Vdash \dot{f}((\dot{a}_i, \dot{a}_j)) = (\dot{a}_r, \dot{a}_s), \tag{2.2}$$

and let $k \in \omega \setminus \{r\}$ be such that $k > n$ for all $n \in E$ and $(k, m) \notin \mathrm{Dom}(q)$ for all $m \in \omega$. Consider the permutation π of ω which swaps k and r but fixes each of the remaining natural numbers. Since $k, r \notin E$, π fixes E pointwise and so $\pi(\dot{f}) = \dot{f}$. Further, q and $\pi(q)$ are compatible forcing conditions, hence $q \cup \pi(q)$ is a well-defined extension of q. From (2.2), we obtain $\pi(q) \Vdash \dot{f}((\dot{a}_i, \dot{a}_j)) = (\dot{a}_k, \dot{a}_s)$, and as $q \cup \pi(q) \leq q$ and $q \cup \pi(q) \leq \pi(q)$, we also have

$$q \cup \pi(q) \Vdash \dot{f}((\dot{a}_i, \dot{a}_j)) = (\dot{a}_r, \dot{a}_s) \quad \text{and}$$
$$q \cup \pi(q) \Vdash \dot{f}((\dot{a}_i, \dot{a}_j)) = (\dot{a}_k, \dot{a}_s).$$

This is a contradiction.

(2) It is known (see [Co] or [HR]) that A is dense in $\mathbb{R}$. Since A is E-finite, it follows that $A^+ = \{x \in A : x > 0\}$ is E-finite (recall that E-**Fin** is hereditary). Furthermore, $A^+ = \bigcup\{A \cap (n, n+1) : n \in \omega\}$ and $A \cap (n, n+1) \neq \emptyset$ for all $n \in \omega$. Hence, A^+ is C-infinite, thus E-**Fin** $\neq$ C-**Fin**, and consequently, by Theorem 2.3, E-**Fin** is not summable.

(3) Set $A^+ = \{x \in A : x > 0\}$ and $\mathcal{U} = \{A^+(x) : x \in A^+\}$, where for $x \in A^+$, $A^+(x) = (x, \infty) \cap A = \{y \in A : x < y\}$. Then $\mathcal{U}$ is indexed by an E-finite set and each element of $\mathcal{U}$ is a non-empty E-finite set (recall that A is dense in $\mathbb{R}$). We show that $\mathcal{U}$ has no choice function in $\mathcal{M}1$. Assume otherwise and let f be a choice function for $\mathcal{U}$ which belongs to $\mathcal{M}1$. Let $\dot{f}$ be a hereditarily symmetric name of f with support E and let p be a forcing condition in the generic filter G such that

$$p \Vdash \text{“}\dot{f} \text{ is a choice function for } \dot{\mathcal{U}}\text{”}.$$

Then there exist $i, j \in \omega$ such that $f(A^+(a_i)) = a_j$, $i \neq j$, $i, j > n$ for all $n \in E$, and $(j, m) \notin \mathrm{Dom}(p)$ for all $m \in \omega$. Let $q \in G$ (the generic filter) be such that $q \leq p$ and

$$q \Vdash \dot{f}(\dot{A^+(a_i)}) = \dot{a_j}, \tag{2.3}$$

and let $k \in \omega \setminus \{j\}$ be such that $k > n$ for all $n \in E$ and $(k, m) \notin \mathrm{Dom}(q)$ for all $m \in \omega$. Considering the transposition $\pi = (j, k)$, we may show similarly to part (1) that $q \cup \pi(q)$ is a well-defined extension of q which forces a contradiction (i.e., $f(A^+(a_i)) = a_j$ and $f(A^+(a_i)) = a_k$). ■

THEOREM 2.11. *In* **ZF**, *every infinite set of reals is C-infinite, hence by Proposition* 1.6(1), *A-infinite.*

Proof. Let A be an infinite set of reals. Without loss of generality assume that $\omega \not\leq A$. (If $\omega \leq A$, then A is clearly C-infinite.) We consider the following cases:

CASE 1: *A is unbounded to the right.* Then for infinitely many $n \in \mathbb{N}$ (= the set of positive integers) the sets $A(n) = A \cap [n, n+1)$ are not empty. Thus, there exists a strictly increasing sequence $(n(k))$, $k \in \mathbb{N}$, such that each $A(n(k))$ is non-empty. Then the map $f : A \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} k & \text{if } a \in A(n(k)), \\ 0 & \text{otherwise}, \end{cases}$$

is surjective.

CASE 2: *A is unbounded to the left.* Replace A by $g[A]$, where $g(x) = -x$ for all $x \in A$. Then proceed as in Case 1.

CASE 3: *A is bounded.* Then A has an accumulation point, say p.

CASE 3a: *p is an accumulation point of A from the right.* Then replace the set $A^* = \{a \in A : p < a\}$ by $\{1/(a - p) : a \in A^*\}$ and proceed as in Case 1.

CASE 3b: *p is an accumulation point of A from the left.* Replace A by $g[A]$, where $g(x) = -x$, $x \in A$, and proceed as in Case 3a. ■

2.3. Certain choice and shrinking principles and stability properties of the class E-Fin. We start this section by proving that the shrinking principle $\mathbf{SP}_{E\text{-}\mathbf{Fin}}$ is equivalent, in **ZF**, to the full **AC**, thus equivalent to the full shrinking principle **SP** (see Section 1). In fact, we prove something more, namely $\mathbf{SP}_{L\text{-}\mathbf{Fin}}$ is equivalent to **AC** for every $L \in \{0, A, B, C, D, E\}$, where for $L = 0$, 0-**Fin** stands for the class **Fin**.

Theorem 2.12. *In* **ZF**, **AC** *is equivalent to* $\mathbf{SP}_{L\text{-}\mathbf{Fin}}$ *for every* $L \in \{0, A, B, C, D, E\}$. *Thus, in* **ZF**, **SP** *is equivalent to* $\mathbf{SP}_{L\text{-}\mathbf{Fin}}$ *for every* $L \in \{0, A, B, C, D, E\}$.

Proof. Fix any $L \in \{0, A, B, C, D, E\}$, assume $\mathbf{SP}_{L\text{-}\mathbf{Fin}}$ holds, and let $\mathcal{A} = \{A_i : i \in I\}$ be a pairwise disjoint family of non-empty sets. Set $X = \bigcup \mathcal{A}$ and without loss of generality assume that $X \cap I = \emptyset$. For each $x \in X$, let $i(x)$ be the unique $i \in I$ such that $x \in A_i$ and let $A(x) = \{x, i(x)\}$. Set $\mathcal{U} = \{A(x) : x \in X\}$. Clearly, for every $x \in X$, $A(x)$, being a finite set, is L-finite for every $L \in \{0, A, B, C, D, E\}$. Let, by our assumption, $\mathcal{V} = \{B(x) : x \in X\}$ be a shrinking of the cover $\mathcal{U}$ of $X \cup I$. It is clear that for every $i \in I$, there is a unique $x_i \in X$ such that $B(x_i) = \{x_i, i\}$. Then $f = \{(i, x_i) : i \in I\}$ is a choice function of $\mathcal{A}$, finishing the proof. ∎

Theorem 2.13. *The following hold in* **ZF**:

(1) *"E-**Fin** = **Fin**" implies both* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$ *and* $\mathbf{AC}^{E\text{-}\mathbf{Fin}}$. *Hence, under "$E$-**Fin** = **Fin**",* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$ *is equivalent to* $\mathbf{AC}^{E\text{-}\mathbf{Fin}}$.
(2) $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$ *implies* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *implies "There are no E-finite countable disjoint unions of non-empty sets".*
(3) $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *implies "E-**Fin** is summable". Hence,* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$ *implies "E-**Fin** is summable".*
(4) $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *is equivalent to "*$\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *+ (E-**Fin** is summable)".*
(5) $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *implies "A-**Fin** = **Fin**". Hence, by (4),* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *also implies "A-**Fin** = **Fin**".*
(6) $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *fails in the basic Fraenkel model. Hence, by (4),* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ *also fails in that model.*
(7) *"(E-**Fin** = **Fin**) +* $\mathbf{AC}_{\mathbf{fin}}$*" is equivalent to* $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$, *where* $\mathbf{AC}_{\mathbf{fin}}$ *is the axiom of choice for families of non-empty finite sets. Hence,* $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$ *implies "E-**Fin** is summable" and, by (4), it also implies* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$.
(8) $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$ *implies both* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$ *and* $\mathbf{AC}^{E\text{-}\mathbf{Fin}}$.

Proof. (1) This is straightforward.

(2) The first implication is obvious. For the second implication, assume $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ holds and, toward a contradiction, assume that $X = \bigcup\{A_i : i \in \omega\}$ is an E-finite countable disjoint union of non-empty sets. Use now the proof of Theorem 2.12 suitably, in order to define a choice function for the family

$\{A_i : i \in \omega\}$ (simply replace, in that proof, I with ω and notice that $\mathcal{U} = \{A(x) : x \in X\}$ is an (E-finite)-indexed family of finite, hence E-finite, sets; the rest of the proof remains the same). It follows that X is D-infinite, a contradiction (recall that E-**Fin** $\subseteq D$-**Fin**).

(3) Assume the hypotheses hold. In view of Theorem 2.3, it suffices to show that E-**Fin** is a subclass of C-**Fin**. Assume on the contrary that there exists an infinite E-finite set X which is C-infinite. Then X has a countable partition $\{A_n : n \in \omega\}$ where $A_n \neq \emptyset$ for all $n \in \omega$. Since X and A_n, $n \in \omega$, are E-finite, this contradicts (2).

(4) ($\rightarrow$) "$\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}} \rightarrow \mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$" can be proved as in Theorem 2.12, using the result of (3).

($\leftarrow$) Let $\mathcal{A} = \{A_i : i \in I\}$ be a family such that I is E-finite and A_i is E-finite for all $i \in I$. We will find a shrinking $\mathcal{B} = \{B_i : i \in I\}$ of $\mathcal{A}$. By our assumption, $X = \bigcup \mathcal{A}$ is E-finite. For each $x \in X$, let $J(x) = \{i \in I : x \in A_i\}$ and set

$$\mathcal{J} = \{J(x) : x \in X\}.$$

Then $J(x) \neq \emptyset$ for all $x \in X$, and since X is E-finite and $J(x)$ is E-finite for all $x \in X$ (for $J(x) \subseteq I$, I is E-finite, and E-**Fin** is hereditary), we may apply $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ to the family $\mathcal{J}$ in order to obtain a choice function $f_{\mathcal{J}}$ for $\mathcal{J}$. Then the family $\mathcal{B} = \{B_i : i \in I\}$, where

$$B_i = \{x \in A_i : f_{\mathcal{J}}(x) = i\},$$

is clearly a shrinking of $\mathcal{A}$.

(5) Assume $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ and, towards a contradiction, assume that there exists an amorphous set X.

We assert that for every natural number n, $[X]^n$ (the set of all n-element subsets of X) is E-finite. Indeed, let $n \in \omega$ and define $\mathrm{Inj}(X^n)$ to be the set of all injections from n into X. Since X is A-finite, it is also C-finite, and since C-**Fin** is closed under finite products and hereditary (see Proposition 1.10(2) and (9)), it follows that X^n is C-finite and consequently $\mathrm{Inj}(X^n)$ is C-finite. Now, it is clear that $[X]^n \leq^* \mathrm{Inj}(X^n)$, and since C-**Fin** is (the largest) cohereditary (finiteness) class (see Proposition 1.10(6)), it follows that $[X]^n$ is C-finite, thus E-finite.

To end the proof, pick any integer $n > 1$. By $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$, the family $\{Y : Y \in [X]^n\}$ has a choice function. But then, by [Tr, Theorem 3, p. 196], the set X is not amorphous, a contradiction.

(6) In the basic Fraenkel model $\mathcal{N}1$ in [HR] (see also the proof of Theorem 2.15 for the model's description), the set A of atoms is amorphous, hence "A-**Fin** = **Fin**" fails in that model. By (5), it follows that both $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ and $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ fail in $\mathcal{N}1$.

(7) ($\rightarrow$) This is clear.

(←) "$\mathbf{AC}_{E\text{-}\mathbf{Fin}} \to \mathbf{AC}_{\mathbf{fin}}$" is clear. To prove that $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$ implies "E-$\mathbf{Fin} = \mathbf{Fin}$", let, toward a contradiction, X be an infinite E-finite set. Since E-$\mathbf{Fin}$ is hereditary, $\mathcal{P}(X) \subset E$-$\mathbf{Fin}$. Then, by $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$, $\mathcal{P}(X) \setminus \{\emptyset\}$ has a choice function, hence X is well-orderable, a contradiction (for, every E-finite set is D-finite). Thus, X is finite and E-$\mathbf{Fin} = \mathbf{Fin}$ as required.

(8) This follows from (7) and (1). ∎

Theorem 2.14.

(1) *"E-$\mathbf{Fin}$ is summable" implies none of the principles* $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$, $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$, $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$, $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$, *in* $\mathbf{ZFA}$ *set theory.*
(2) $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$ *does not imply* $\mathbf{AC}$ *in* $\mathbf{ZFA}$. *Hence, it does not imply* $\mathbf{SP}_{E\text{-}\mathbf{Fin}}$ *in* $\mathbf{ZFA}$ *either.*
(3) $\mathbf{AC}_{\mathbf{fin}}$ *does not imply any of* $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$, $\mathbf{AC}^{E\text{-}\mathbf{Fin}}$, $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$, *in* $\mathbf{ZF}$.

Proof. (1) We employ the Mostowski linearly ordered model (model $\mathcal{N}3$ in [HR]). Recall that $\mathcal{N}3$ is constructed by starting with a ground model $\mathcal{M}$ of $\mathbf{ZFA} + \mathbf{AC}$ with a countable set A of atoms together with a dense linear ordering $<$ of A without endpoints. (Hence, A is order isomorphic to the set $\mathbb{Q}$ of all rational numbers with the order it inherits from $\mathbb{R}$). The group G of permutations of A used to define the model is the group of all order automorphisms of $(A, <)$. For each finite set $E \subseteq A$, let $\text{fix}_G(E) = \{\phi \in G : (\forall e \in E)\ (\phi(e) = e)\}$ be the pointwise stabilizer of E. Let Γ be the normal filter of subgroups of G generated by the filter base $\{\text{fix}_G(E) : E \in \mathcal{P}_{\text{fin}}(A)\}$. $\mathcal{N}3$ is the FM model determined by $\mathcal{M}$, G and Γ. If $x \in \mathcal{N}3$ and $E \in \mathcal{P}_{\text{fin}}(A)$ is such that $\text{fix}_G(E) \subseteq \text{Sym}_G(x)$ $(= \{\phi \in G : \phi(x) = x\})$, then we say that E is a *support* of x.

By the proof of Theorem 2.8(4), the class E-$\mathbf{Fin}$ is summable in the model $\mathcal{N}3$. Further, in view of Theorem 2.13, it suffices to show that $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ fails in $\mathcal{N}3$. To this end, we first recall that in [L] it is shown that A is C-finite in $\mathcal{N}3$, hence also E-finite in $\mathcal{N}3$. Now consider the family

$$\mathcal{U} = \{(x, \infty) : x \in A\}$$

where $(x, \infty) = \{y \in A : x < y\}$. Then for each $x \in A$, (x, ∞) is an E-finite set since $(x, \infty) \subset A$ and E-$\mathbf{Fin}$ is hereditary. We assert that $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ fails for the family $\mathcal{U}$. Assume to the contrary that f is a choice function for $\mathcal{U}$ with support $E = \{e_1 < \cdots < e_n\} \subset A$. Let $x \in A$ be such that $e_n < x$ (recall that A has no endpoints) and assume that $f((x, \infty)) = z$. Let $r \in A$ be such that $z < r$. Let ϕ be an order automorphism of A such that ϕ fixes $E \cup \{x\}$ pointwise and $\phi(z) = r$. Then

$$((x, \infty), z) \in f \ \to\ (\phi((x, \infty)), \phi(z)) \in \phi(f) \ \to\ ((x, \infty), r) \in f. \tag{2.4}$$

Since $r \neq z$, (2.4) implies that f is not a function, a contradiction.

(2) Consider the Howard/Rubin Model I (model $\mathcal{N}38$ in [HR]). In this model, "D-**Fin** = **Fin**", hence "E-**Fin** = **Fin**", holds, and $\mathbf{AC_{fin}}$ also holds (see [HR]). Thus, by Theorem 2.13(6), $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$ also holds in $\mathcal{N}38$. However, **AC** fails in that model (see [HR]), hence the independence result.

(3) In Cohen's basic model $\mathcal{M}1$, the Boolean Prime Ideal Theorem **BPI** (Every Boolean algebra has a prime ideal) holds (see [HR]), hence $\mathbf{AC_{fin}}$ also holds in $\mathcal{M}1$. Follow now the proof of Theorem 2.10(3) in order to verify that $\mathbf{AC}_{E\text{-}\mathbf{Fin}}$ and $\mathbf{AC}^{E\text{-}\mathbf{Fin}}$ fail in $\mathcal{M}1$. The fact that $\mathbf{SP}^{E\text{-}\mathbf{Fin}}$ fails in $\mathcal{M}1$ follows from Theorems 2.13(3) and 2.10(2). ∎

2.4. E-finiteness and two more definitions of finite. We start this section by first pointing out that the proof of Theorem 2.13(5) immediately yields Theorem 2.15(1) below.

THEOREM 2.15.

(1) *In* **ZF**, *for every amorphous set X and for every natural number n, the set $[X]^n$ is C-finite, thus E-finite.*
(2) *The statement "For every A-finite set X, the set $\mathcal{P}_{\text{fin}}(X)$ of all finite subsets of X is E-finite" is consistent relative to* $\mathbf{ZFA} + \neg\mathbf{AC}$. *In particular, it holds in the basic Fraenkel model.*
(3) *In* **ZF**, *for every amorphous set X, $\mathcal{P}_{\text{fin}}(X)$ is E-finite if and only if $\mathcal{P}(X)$ is E-finite.*

Proof. (2) The basic Fraenkel permutation model $\mathcal{N}1$ in [HR] is constructed by starting with a ground model $\mathcal{M}$ of $\mathbf{ZFA} + \mathbf{AC}$ with a countable set A of atoms. The group G of permutations of A used to define the model is the group of all permutations of A. For each finite set $E \subseteq A$, let $\text{fix}_G(E)$ be the pointwise stabilizer of E, i.e., $\text{fix}_G(E) = \{\phi \in G : (\forall e \in E)\ (\phi(e) = e)\}$. Let Γ be the normal filter of subgroups of G generated by the filter base $\{\text{fix}_G(E) : E \in \mathcal{P}_{\text{fin}}(A)\}$. $\mathcal{N}1$ is the FM model determined by $\mathcal{M}$, G and Γ. If $x \in \mathcal{N}1$ and $E \in \mathcal{P}_{\text{fin}}(A)$ is such that $\text{fix}_G(E) \subseteq \text{Sym}_G(x) = \{\phi \in G : \phi(x) = x\}$, then as usual E is called a support of x. In $\mathcal{N}1$, A is an amorphous set, as can be easily checked.

We argue now that the statement "For every A-finite set X, the set $\mathcal{P}_{\text{fin}}(X)$ is E-finite" is valid in $\mathcal{N}1$. Let $X \in \mathcal{N}1$ be an A-finite set. If X is finite, then there is nothing to show. So assume that X is an amorphous set. Since X is not well orderable, it follows by the Lemma on page 389 of Blass [B] that, in $\mathcal{N}1$, there is an infinite subset B of A (hence, B is a cofinite subset of A) and an injection $f : B \to X$. Without loss of generality assume that $B \subseteq X$ and that $X \cap A = B$. Since X is amorphous, it follows that $D = X \setminus B$ is a finite set. Assume that $D \approx m$ for some $m \in \omega$.

Toward a contradiction, suppose that $\mathcal{P}_{\text{fin}}(X) = \bigcup\{[X]^n : n \in \omega\}$ is E-infinite. By (1), for all $n \in \omega$, $[X]^n$ is C-finite. From this, as well as from our assumption that $\mathcal{P}_{\text{fin}}(X)$ is an E-infinite set, we derive by Lemma 2.1 that there exist a surjection $f : \mathcal{P}_{\text{fin}}(X) \to \mathcal{P}_{\text{fin}}(X) \cup \{a\}$, $a \notin \mathcal{P}_{\text{fin}}(X)$, and a strictly increasing function $n : \omega \to \omega$ such that $f^{-(i+1)}(\{a\}) \subseteq [X]^{n(i)}$ for each $i \in \omega$. Let $K \subset A$ be a finite support for f and for each element of D. Since the function $n : \omega \to \omega$ is strictly increasing, there exists an $i \in \omega$ such that $n(i) > |K| + m$ (where m is the cardinality of $D = X \setminus B$). Therefore (for such an $i \in \omega$), there exists a set $Y \in \mathcal{P}_{\text{fin}}(X)$ such that $n(i) \approx Y$, hence $Y \nsubseteq K \cup D$, and $Y \in \bigcup\{f^{-(i+1)}(\{a\}) : i \in \omega\}$. Thus, $\emptyset \neq Y \setminus (K \cup D) \subset A$, and since K is finite, out of all such Y's we can pick a $Y \in [X]^{n(i)}$ such that $|Y \cap K|$ is maximal. Let $Z \in [X]^{n(i+1)}$ be such that $f(Z) = Y$. Since n is strictly increasing we have $Y < Z$. We may continue now similarly to the proof of [Tr] that $\mathcal{P}_{\text{fin}}(A)$ is E-finite in $\mathcal{N}1$ in order to show that f cannot be a function. We leave the checking to the interested reader.

(3) Let X be an amorphous set such that $\mathcal{P}_{\text{fin}}(X)$ is E-finite. Since every subset of X is either finite or cofinite, it follows that $\mathcal{P}(X) \approx 2 \times \mathcal{P}_{\text{fin}}(X)$ (the mapping $f(A) = \langle 0, A\rangle$ if $A \subset X$ is finite and $f(A) = \langle 1, X \setminus A\rangle$ if $A \subseteq X$ is infinite, is clearly a bijection).

Since for $i \in 2 = \{0,1\}$, $\{i\} \times \mathcal{P}_{\text{fin}}(X) \approx \mathcal{P}_{\text{fin}}(X)$, and E-**Fin** is hereditary, the set $\{i\} \times \mathcal{P}_{\text{fin}}(X)$, $i \in 2$, is E-finite. Furthermore, as a finite union of E-finite sets is E-finite (Proposition 1.10(7)), it follows that $2 \times \mathcal{P}_{\text{fin}}(X) = (\{0\} \times \mathcal{P}_{\text{fin}}(X)) \cup (\{1\} \times \mathcal{P}_{\text{fin}}(X))$ is E-finite. Hence, $\mathcal{P}(X)$ is E-finite. ∎

Remark 2.16. We note that Theorem 2.15(2) enhances Truss' result in [Tr] that the amorphous set A of the atoms in the basic Fraenkel model $\mathcal{N}1$ is such that $\mathcal{P}_{\text{fin}}(A)$ is E-finite in the model.

Theorem 2.15 provides a good reason for introducing and studying the following two notions of finite (Definition 2.17) along with the corresponding finiteness classes. Our goal here is to investigate their relationship with the notions of finite already mentioned in this paper.

Definition 2.17. Let X be a set.

(a) X is called *F_D-finite* if $\mathcal{P}_{\text{fin}}(X)$ is D-finite.
(b) X is called *F_E-finite* if $\mathcal{P}_{\text{fin}}(X)$ is E-finite.

F_D-**Fin** and F_E-**Fin** denote the classes of F_D-finite and F_E-finite sets, respectively.

We leave the verification of the following proposition to the reader.

Proposition 2.18. *F_D-**Fin** and F_E-**Fin** are both finiteness classes and F_E-**Fin** $\subseteq F_D$-**Fin**.*

Theorem 2.19. *The following hold:*

(1) *In* **ZF**, *C-**Fin** $\subseteq F_D$-**Fin**.*

(2) *It is relatively consistent with* **ZF** *that* F_D-**Fin** $\nsubseteq$ E-**Fin** (*hence,* F_D-**Fin** $\nsubseteq$ C-**Fin** *and* F_D-**Fin** $\nsubseteq$ F_E-**Fin**) *and* E-**Fin** $\nsubseteq$ F_D-**Fin** (*hence,* E-**Fin** $\nsubseteq$ F_E-**Fin**). *However, under "*E-**Fin** *is summable",* E-**Fin** $\subseteq$ F_D-**Fin**.

(3) C-**Fin** $\nsubseteq$ F_E-**Fin** *is consistent relative to* **ZFA**.

(4) *In* **ZF**, F_E-**Fin** $\subseteq$ C-**Fin**. *Hence, in* **ZF**, F_E-**Fin** $\subseteq$ E-**Fin**.

(5) *"*E-**Fin** *is summable" implies that every infinite set is* F_E*-infinite. Thus, "*E-**Fin** *is summable" implies "*F_E-**Fin** = **Fin**".

(6) *In the Mostowski linearly ordered model* $\mathcal{N}3$*, every infinite set is* F_E*-infinite. In particular,* F_E-**Fin** = **Fin** *in* $\mathcal{N}3$.

(7) *"*E-**Fin** *is summable" fails in the basic Fraenkel model* $\mathcal{N}1$ *in [HR].*

(8) *"*E-**Fin** *is summable" does not imply "*F_E-**Fin** = E-**Fin**" *in* **ZFA**.

(9) *Assume "*E-**Fin** *is summable". Then "*F_E-**Fin** = E-**Fin**" *if and only if "*E-**Fin** = **Fin**".

(10) *Assume "*E-**Fin** *is summable". Then "*C-**Fin** $\subseteq$ F_E-**Fin**" *if and only if "*C-**Fin** = **Fin**".

(11) *"*D-**Fin** = **Fin**" *implies "*F_E-**Fin** = E-**Fin** = **Fin**".

(12) *"*F_D-**Fin** = F_E-**Fin**" *implies "*A-**Fin** = **Fin**". *The implication is not reversible in* **ZFA**.

Proof. (1) For the proof of "C-**Fin** $\subseteq$ F_D-**Fin**" in **ZF**, see Definition 1.3(3).

(2) Consider any model $\mathcal{M}$ of **ZF** which has an amorphous set A. (A is certainly F_D-finite and also C-finite). For each $n \in \omega$, let $B_n = \{f \in A^n : f \text{ is an injection}\}$ and let $B = \bigcup\{B_n : n \in \omega\}$. It can be easily verified that B is F_D-finite and E-infinite; for the latter, see Remark 2.9(3). Thus, in $\mathcal{M}$, F_D-**Fin** $\nsubseteq$ E-**Fin**, hence F_D-**Fin** $\nsubseteq$ F_E-**Fin**—note also that $\mathcal{P}_{\text{fin}}(B)$ is E-infinite (otherwise, since $B \leq \mathcal{P}_{\text{fin}}(B)$ and E-**Fin** is hereditary, B is E-finite, a contradiction)—and F_D-**Fin** $\nsubseteq$ C-**Fin**.

For "E-**Fin** $\nsubseteq$ F_D-**Fin**", consider the second Cohen model $\mathcal{M}7$ in [HR]. In $\mathcal{M}7$, there exists a Russell set $X = \bigcup\{X_n : n \in \omega\}$ (see [HR]), hence X is E-finite (Theorem 2.2) and X is clearly F_D-infinite.

Now, if E-**Fin** is summable, then by Theorem 2.3, E-**Fin** = C-**Fin**, hence by (1), it follows that E-**Fin** $\subseteq$ F_D-**Fin**.

(3) Consider the Mostowski linearly ordered model $\mathcal{N}3$ and let A be the set of atoms in that model. We know that A is C-finite (see [L]) and that E-**Fin** = C-**Fin** in $\mathcal{N}3$. Hence, by Theorem 2.3, the class E-**Fin** is cohereditary in $\mathcal{N}3$. It follows that A is F_E-infinite, otherwise since $\omega \leq^* \mathcal{P}_{\text{fin}}(A)$ we would infer that ω is E-finite, a contradiction.

(4) Let A be an F_E-finite set. Toward a contradiction, assume that A is C-infinite. Hence, A has a partition $\{A_i : i \in \omega\}$ into non-empty sets. We argue that A is F_E-infinite. Indeed, let $p \notin \mathcal{P}_{\text{fin}}(A)$ and consider the

function $g : \mathcal{P}_{\text{fin}}(A) \to \mathcal{P}_{\text{fin}}(A) \cup \{p\}$ which is defined as follows: $g(X) = p$ for every $X \in \mathcal{P}_{\text{fin}}(A_0)$, and $g(X) = X$ for every $X \in \mathcal{P}_{\text{fin}}(A_n)$ for all positive integers n. Now, for $X \in \mathcal{P}_{\text{fin}}(A) \setminus \bigcup_{n\in\omega} \mathcal{P}_{\text{fin}}(A_n)$, let $\text{tr}(X) = \{i_1 < \cdots < i_{n(X)}\}$ be the *trace* of X (i.e., $\text{tr}(X) = \{i \in \omega : X \cap A_i \neq \emptyset\}$) and define $g(X) = X \cap \bigcup_{j=1}^{n(X)-1} A_{i_j}$. It is easy to verify that g is surjective. This contradicts A being F_E-finite.

(5) Assume the hypothesis holds and let X be an infinite set. Toward a contradiction, assume that X is F_E-finite. Since E-**Fin** is hereditary, it follows that $[X]^n$ is E-finite for all $n \in \omega$. Now $\mathcal{P}_{\text{fin}}(X) = \bigcup\{[X]^n : n \in \omega\}$ is an E-finite countably infinite (disjoint) union of (infinite) E-finite sets. But this contradicts the result of Theorem 2.7 stating that if E-**Fin** is summable, then there are no E-finite countably infinite unions of non-empty E-finite sets.

(6) This follows from the fact that E-**Fin** is summable in the model $\mathcal{N}3$ (see the proof of Theorem 2.14(1)), hence by (5), every infinite set in $\mathcal{N}3$ is F_E-infinite.

(7) In $\mathcal{N}1$, F_E-**Fin** $\neq$ **Fin**; in [Tr, pp. 202–203] it is shown that $\mathcal{P}_{\text{fin}}(A)$, where A is the (infinite) set of atoms in $\mathcal{N}1$, is E-finite, thus A is F_E-finite in $\mathcal{N}1$. Thus, by (5), E-**Fin** is not summable in $\mathcal{N}1$.

(8) In $\mathcal{N}3$, E-**Fin** is summable and E-**Fin** $\neq$ **Fin** $=$ F_E-**Fin**.

(9), (10), and (11) follow from (1) and (5) and the fact that “D-**Fin** $=$ **Fin**” $\to$ “E-**Fin** $=$ **Fin**”.

(12) Let X be an A-finite set and, toward a contradiction, assume that X is amorphous. For each $n \in \omega$, let $Y_n = \{f \in X^n : f \text{ is an injection}\}$ and let $Y = \bigcup\{Y_n : n \in \omega\}$. Then Y is F_D-finite, hence by our assumption, Y is F_E-finite, hence, by (1), Y is E-finite, a contradiction.

For the second assertion of (12), consider Mostowski's model $\mathcal{N}3$. In $\mathcal{N}3$, A-**Fin** $=$ **Fin** (see [HR]). Now let A be the set of atoms of $\mathcal{N}3$. Since A is C-finite, it follows by (2) that A is F_D-finite. However, by (5) every infinite set in $\mathcal{N}3$, hence A, is F_E-infinite. ∎

3. Questions

1. Is the statement “If X is an amorphous set, then X is F_E-finite” provable in **ZF**?
2. Is the implication “(E-**Fin** is summable) $\to$ (A-**Fin** $=$ **Fin**)” provable in **ZF**? (Note that if Question 1 receives an affirmative answer, then in view of Theorem 2.19(5), the answer to Question 2 is also in the affirmative).
3. Is “(E-**Fin** $=$ **Fin**) $\wedge$ $\neg$(D-**Fin** $=$ **Fin**)” relatively consistent with **ZFA**? Is “E-**Fin** $=$ **Fin**” equivalent to “D-**Fin** $=$ **Fin**”?

4. Does $\mathbf{AC}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ imply $\mathbf{SP}^{E\text{-}\mathbf{Fin}}_{E\text{-}\mathbf{Fin}}$ or equivalently (see Theorem 2.13(4)) "E-**Fin** is summable"?
5. What is the deductive strength of "E-**Fin** is closed under finite products"? In particular, does "E-**Fin** is closed under finite products" imply "E-**Fin** is disjointly summable"? (Recall that we have proved (see Lemma 2.5) that the converse is true.)
6. The proof of Lemma 2.5 actually shows that, *in* **ZF**, *any disjointly summable finiteness class is closed under finite products*. Is the converse true? That is, is the statement "If $\mathfrak{U}$ is a finiteness class which is closed under finite products, then $\mathfrak{U}$ is disjointly summable" true in **ZF**?
7. Is there a largest finiteness class which is disjointly summable?

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