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BISIMULATION RELATION FOR SELECTED TYPES OF PROBABILISTIC AND QUANTUM AUTOMATA

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Abstract. The first step to make transitional systems more efficient is to minimize the number of their states. A bisimulation relation is a mathematical tool that helps in searching for equivalent systems, what is useful in the minimization of algorithms. For two transition systems bisimulation is a binary relation associating systems which behave in the same way in the sense that one system simulates the other and vice-versa. The definition for classical systems is clear and simple, but what happens with nondeterministic, probabilistic and quantum systems? This will be the main topic of this article.

1. Introduction

During the last fifty years many scientists have been searching for new computation models. They have developed probabilistic automata, models of finite automata over infinite words, timed automata, hybrid automata, etc. We can find their ontological review in the article [5]. In 1997 Kondacs and Watrous formulated the model of 1-way quantum finite automata (1QFA) [4]; in the same year, independently, Moore and Crutchfield defined the quantum finite automata [6]. Later, the model of quantum automata was evolved by Ambainis in many works (see e.g. [1]). This article present the definition of the bisimulation relation for different types of automata. The main focus will be on a finite reactive probabilistic automaton and a one-way quantum finite automaton.

2. Definitions of models

A transition system is a four-tuple $TS = (S, E, T, s_0)$, where S is a set of states with the initial state s_0, E is a set of events, $T \subseteq S \times E \times S$ is a transition relation (as usual, the transition (s, a, s_1) is written as $s \xrightarrow{a} s_1$) [5].

The more complex example of a transition system is a **nondeterministic finite automaton** which is a tuple $NFA = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states with the start state q_0, Σ is a finite set of input symbols, δ is a transition partial function $\delta : Q \times \Sigma \mapsto 2^Q$, $F \subseteq Q$ is a set of final states [3].

A Markov chain is the transition system, in which the probability of reaching the given state is considered. **A finite Markov chain** is a pair $MC = (Q, \delta)$, where Q is a set of states, δ is a transition function $(\delta : Q \mapsto \mathcal{D}(Q)$, where $\mathcal{D}(Q)$ is a discrete probability distribution) [10].

If $q \in Q$ and $\delta(q) = P$ with $P(q') = p > 0$, then the Markov chain is said to go from the state q to the state q' with probability p. We can find the different notations of the same phenomenon: $q \rightsquigarrow P$, $q \overset{p}{\rightsquigarrow} q'$, $\delta(q) = P$, $\delta(q)(q') = p$. Let us consider further extension of this model. **A finite reactive probabilistic automaton** is a tuple $PA = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is a finite set of input symbols, $\delta : Q \times \Sigma \mapsto \mathcal{D}(Q)$ is a transition partial function, $q_0 \in Q$ is an initial state, $F \subseteq Q$ is a set of final (accepting) states [10].

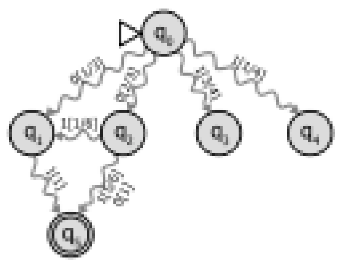

Figure 1: The PA example

After each step, a probabilistic automaton is in a superposition of states: $p_0q_0 + p_1q_1 + ... + p_nq_n$, where $p_0 + p_1 + ... + p_n = 1$.

To define a quantum automaton we need a brief introduction to the theory of quantum computing. In quantum mechanics the possible states of n-level quantum mechanical system are represented by unit vectors (called *"the state vectors"*) residing in a complex Hilbert space H_n (called *"the state space"*).

For the description of this system an ortonormal basis is used: $|x_1\rangle, |x_2\rangle, \ldots, |x_n\rangle$, where the basis vectors $|x_i\rangle$ are called the basis states. Any quantum state can be expressed by a superposition of basis states: $\alpha_1|x_1\rangle + \alpha_2|x_2\rangle + \cdots + \alpha_n|x_n\rangle$, where α_i is a complex number known as a probability amplitude. The probability of observing the state x_i is equal to $|\alpha_i|^2$, with the normalization $|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 = 1$. Time evolution of quantum system is represented by a unitary matrix (it has an inverse equal to its conjugate transpose). This is a stronger condition than that in the probabilistic systems, it causes a phenomenon of interference effects and guarantees that the time evolution of quantum state is reversible [2].

A one-way quantum finite automaton (defined by Kondacs and Watrous) is a tuple $1QFA = (Q, \Sigma, \delta, q_0, Q_a, Q_r)$, where Q is a finite set of states, Σ is a finite set of input symbols, δ is a transition partial function, $q_0 \in Q$ is an initial state, $Q_a \subset Q$ and $Q_r \subset Q$ are sets of accepting and rejecting states.

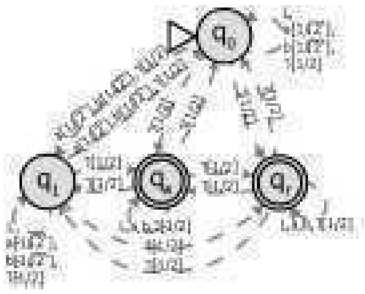

Figure 2: The 1-way QFA example

Q_a and Q_r are called non-halting states; $Q_n = Q \backslash (Q_a \cup Q_r)$. The symbols $\lfloor$ and $\rceil$ mark the beginning and the end of the word on the tape. The working alphabet of automaton is $\Gamma = \Sigma \cup \{\lfloor, \rceil\}$ [4].

The transition function $\delta : Q \times \Gamma \times Q \mapsto \mathbb{C}$ represents the amplitude with which an automaton being currently in a state $|q\rangle$, reading the symbol σ, will change a state to $|q'\rangle$. For $\sigma \in \Gamma$, V_σ is a linear transformation defined by: $V_\sigma(|q\rangle) = \sum_{q' \in Q} \delta(q, \sigma, q')|q'\rangle$ [1], [4].

3. Bisimulation

First, we must ask the question: when are two processes (states) behavioraly equivalent? Secondly, what does it mean for two systems to be equal with respect to their communication structures? The bisiumlation relation will allow us to find the answers.

Two transition systems $TS_1 = (T, \Sigma, \delta_T, t_0)$ and $TS_2 = (S, \Sigma, \delta_S, s_0)$ are bisimilar iff there is a relation $R \subseteq S \times T$ such that $(s_0, t_0) \in R$ and for all pairs $(s, t) \in R$ and for all $\sigma \in \Sigma$ the following holds: whenever $\delta_T(t, \sigma) = t'$, then there exists $s' \in S$ such that $\delta_S(s, \sigma) = s'$ and $(s', t') \in R$, and whenever $\delta_S(s, \sigma) = s'$, then there exists $t' \in T$ such that $\delta_T(t, \sigma) = t'$, and $(s', t') \in R$. The states s and t are called bisimilar which is denoted by $s \approx t$ [7], [10].

There is a simple way to determine whether two systems are bisimilar – by playing a game. This is a game between two persons: the Player and the Opponent. The Player tries to prove that systems are bisimilar, the Opponent intends otherwise. The Opponent opens the game by choosing a transition from the initial state of one of the systems. The Player has to find an equally labelled transition from the initial state of the second system, new states are the starting points for the next turn. If one of the players cannot move – the other wins this turn of the game. The Player loses abundantly if there are no corresponding transition for Opponent's move. The Player wins any infinite turn of the game or any repeated configuration.

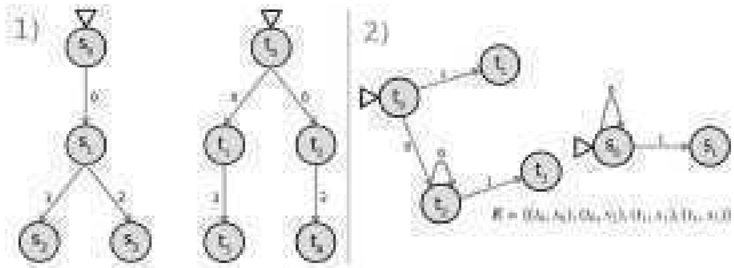

Figure 3: Example of nonbisimilar and bisimilar automata

In the first case, after reading the symbol 0 the Player must be in a state t_1 or t_2, then the road runs out to him, accordingly, or for the symbol 2 or 1. Systems are not bisimilar.

In the second example, we see that for each state and each symbol the Player will always find a corresponding way in the second automaton, so the automata are bisimilar.

4. Bisimulation for probabilistic and quantum systems

To define a bisimulation relation for probabilistic and quantum automata, one can wonder how to compare distributions of probabilities. For this purpose we use the following definitions.

Let $R \subseteq S \times T$ be a relation between sets S and T. Let $P_1 \in \mathcal{D}(S)$ and $P_2 \in \mathcal{D}(T)$ be probability distributions. Define $P_1 \equiv_R P_2$ iff there exists a distribution $Pr \in \mathcal{D}(S \times T)$ such that $Pr(s,T) = P_1(s)$ for any $s \in S$, $Pr(S,t) = P_2(t)$ for any $t \in T$, $Pr(s,t) \neq 0$ iff $(s,t) \in R$ [10].

Let R be an equivalence relation on the set S and let $P_1, P_2 \in \mathcal{D}(S)$ be probability distributions. Then $P_1 \equiv_R P_2 \iff \forall C \in S/R : P_1(C) = P_2(C)$, where C is an abstract class [10].

Let R be an equivalence relation on the set S, A be an arbitrary set, and let $P_1, P_2 \in \mathcal{D}(S)$ be probability distributions. Then $P_1 \equiv_{R,A} P_2 \iff \forall C \in S/R, \forall a \in A : P_1(a, C) = P_2(a, C)$ [10].

An equivalence relation on a set of states Q of a Markov chain (Q, δ) will be a bisimulation relation iff $\forall (q, t) \in R$ the following holds: if $\delta(q) = P_1$, then there exists $\delta(t) = P_2$ such that $P_1 \equiv_R P_2$.

Let $PA_1 = (S, \Sigma, \delta_S)$ and $PA_2 = (T, \Sigma, \delta_T)$ be two probabilistic automata, then there exists a bisimulation relation $R \subseteq S \times T$ if for all pairs $(s, t) \in R$ and for all $\sigma \in \Sigma$ we have: if $\delta_S(s, \sigma) = P_1$, then there exists a probability distribution P_2 such that for some $t \in T$ there exists $\delta_T(t, \sigma) = P_2$ and $P_1 \equiv_{R,\Sigma} P_2$ [10].

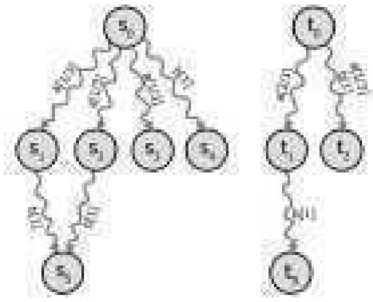

Figure 4: Bisimilar PA

Finally let us go to the bisimulation of the quantum automata, in this case we have to compare the linear operators.

For the given operator V_σ we define $v_\sigma(S) = \sum_{q' \in S} |\delta(q, \sigma, q')|^2$ (the sum of squares of the values of ruthless amplitudes), where $S \subseteq Q$.

Let R be an equivalence relation on the set S, A be an arbitrary set, and V_1, V_2 be unitary operators corresponding to transitions of the quantum system. Then $V_1 \equiv_{R,A} V_2 \iff \forall C \in S/R,\ \forall a \in A : v_{1a}(C) = v_{2a}(C)$.

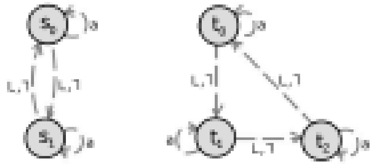

Figure 5: Bisimilar 1-way QFA

Let $1QFA_1 = (S, \Sigma, \delta_S)$ and $1QFA_2 = (T, \Sigma, \delta_T)$ be two one-way quantum finite automata. Then there exists a bisimulation relation $R \subseteq S \times T$ if for all pairs $(s, t) \in R$ and for all $\sigma \in \Sigma$ we have: if $V_{1\sigma}(|s\rangle) = \sum_{s' \in S} \delta_S(s, \sigma, s')|s'\rangle$, then there exists $V_{2\sigma}(|t\rangle) = \sum_{t' \in T} \delta_T(t, \sigma, t')|t'\rangle$ such that $V_1 \equiv_{R,\Sigma} V_2$.

5. Summary

A bisimulation relation can be a great tool to search for systems that simulate each other, and therefore their behavior is analogous to the same symbols, actions, impulses.

The simple way for checking whether or not two classical systems are bisimilar is a game, but for probabilistic and quantum systems we have to consider the sum of probabilities and amplitudes.

Bisimulation can also be a foundation for relations useful, for example, in minimization of systems [8], [9].

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