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TSUNAMI-DRIVEN PLANE ACOUSTIC WAVES IN THE ATMOSPHERE

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Abstract: A problem of analytical-numerical modeling of the plane wave propagation from an ocean surface to the atmosphere is considered. We are interested in the thermosphere impact of tsunami waves. We suppose that the transport of energy and the momentum from lower atmosphere to thermosphere heights is due to acoustic waves.

A set of expressions for the atmosphere variables (pressure, velocity, entropy) as a function of the atmosphere parameters, time and height is derived and illustrated by plots. The surface water wave parameters, typical for tsunami also enter final expressions for the atmosphere and ionosphere variables.

Keywords: analytical-numeric modeling, atmosphere waves, tsunami, ionosphere effect

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1. Introduction

Detection and prediction of tsunami waves is an urgent task of modern geophysics [1, 2]. A set of studies aimed at exploring the ocean-atmosphere-ionosphere links is distinguished among the different approaches to the problem, which makes it possible to monitor this formidable phenomenon using satellite communication systems [3]. In this case, the main hardware control element is a system of receivers of satellite communication signals, which, in effect, gives a tomographic picture of the disturbance of the total electron concentration in a region dangerous from the point of view of activity of underwater earthquakes and, accordingly, tsunami.

The transmitting link of the disturbance from surface ocean waves is the atmosphere. A tsunami wave disturbs acoustic and internal gravitational waves in the atmosphere, the propagation of which at the heights of the thermosphere is accompanied by the transport of plasma along the magnetic lines of force, which affects the electron concentration. There are a number of works devoted to the generation and propagation of internal gravitational waves from a disturbed sea surface (see the literature review). Thus, it becomes necessary to study the formulation and solution of the problem of purely acoustic perturbations and their ionospheric effect. Such a problem is the propagation of a plane atmospheric wave, which obviously does not contain internal gravitational waves.

Thus, the solution of the boundary regime propagation problem proves to be important both in the diagnostics of atmospheric effects (see the literature [1] review) and for detection and prediction of tsunami waves. An interesting aspect of studying this problem are the mathematical features of the solution. In particular, the inverted dispersion relation is complex, and the range of variation of the variables covers only the half-space, which requires special attention when working with integral transformations.

In [4], an example of the general Cauchy problem for evolution equations in $1+1$ dimensions is studied extensively. The propagation of the boundary regime is formulated in terms of operators and it is illustrated by the simplest one-dimensional diffusion equation.

The main theory of acoustic and internal gravitational waves is considered in the paper [5], taking into account the entropy mode, relations relating perturbations characteristic of acoustic and entropy modes in an accelerated liquid or in a liquid subject to a constant mass force are derived. They allow us to decompose the complete perturbation vector and the total energy into acoustic and non-acoustic parts uniquely at any time. Three quantities are required for this purpose, for example, complete perturbations in entropy, pressure, and velocity. Estimates are made with respect to the content of acoustic and non-acoustic parts of the total energy, excluding its kinetic part. The work includes the consideration of the one-dimensional evolution of the exponential atmosphere perturbation.

The diagnostics and decomposition of atmospheric disturbances in a plane flow are considered in the work [6]. The study considers a situation in which the steady-state equilibrium gas temperature can depend on the vertical coordinate due to external forces. The relations connecting perturbations are analytically established. These perturbations determine the acoustic and entropy modes in an arbitrary stratified gas subject to a constant mass. These diagnostic relationships relate the acoustic and entropy regimes and are independent of time.

The method proposed in [7] separates gasdynamic perturbations propagating in a one-dimensional atmosphere into components propagating up and down and stationary components of the waves.

There are strong arguments in [8] that the phenomena that occur in the oceans are an important source of waves in the thermosphere. Theoretical analysis of the authors led to the assumption that waves of infragravitation (*i.e.* surface gravity waves in the ocean with periods of more than 30 s) can emit acoustic gravity waves and take into account a significant part of the wave activity observed in the thermosphere with periods between 5 and 3 hours. This article describes an experimental demonstration of thermospheric waves controlled by the ocean using the data of two deepwater assessments and reporting of Tsunamis stations located at radars on the east coast of the United States and Dynasonde located on the island of Wallops, Virginia.

The observations show large perturbations of the electron density in the F region ($\sim 100\%$), as well as perturbations of the total electron content (TEC) ($\sim 30\%$), which appear to correlate with the tsunami. The article [1] gives values of the characteristic velocity and horizontal wavelength of the perturbations – $\sim 200\,\mathrm{m/s}$ and $\sim 400\,\mathrm{km}$, respectively.

The basic theory of hydrodynamics of tsunami waves at all stages (from origin to ashore) is described in the book [9]. In the book attention is also paid to the conclusion of a simplified model of the soliton interaction and atmospheric disturbance. There is also an attractive trend to develop simple aysmptotic solutions for tsunami wave solvable in explicit form [10], convenient in the boundary regime modeling.

We consider propagation of plane acoustic waves from the ocean surface. Basically, we focused on the effects of tsunami waves in the thermosphere. We assume that the transfer of energy and angular momentum from the lower layers of the atmosphere to the heights of the thermosphere is carried out by means of acoustic waves.

Within the confines of this paper, the following tasks were set:

1. Develop methods for analytical solution of problems, as well as modeling the propagation of atmospheric waves.
2. Identify and solve the problem of vertical propagation of the boundary regime corresponding to acoustic waves in an exponential atmosphere.
3. Formulate the boundary regime corresponding to the excitation of atmospheric waves by the water surface motion.
4. Choose the parameters, shape and scale of the boundary condition corresponding to tsunami waves. A critical review of the literature data.
5. Develop methods for an approximate solution of a problem based on the Fourier transform and expand the dispersion relation with respect to a small parameter.
6. Construct an approximate solution, derive formulas for a wide range of heights, suitable for graphic illustration and calculations of the ionospheric effect.

2. Basic equations

Consider the problem of propagation of acoustic waves in an exponentially stratified atmosphere. The pressure and density of the unperturbed atmosphere are described by the law:

$$\overline{p}(z) = p_0 \exp(-z/H) = \rho_0 g H \exp(-z/H); \quad \overline{\rho}(z) = \rho_0 \exp(-z/H) \tag{1}$$

Here: $\overline{p}$ – background pressure; $\overline{\rho}$ – background density; H – height of homogeneous atmosphere; z – current height value.

The conventional system of equations of 3D hydrothermodynamics is written as:

$$\frac{\partial \vec{V}}{\partial t} = -\frac{\overrightarrow{\nabla} p'}{\overline{\rho}} + \vec{a}\frac{\rho'}{\overline{\rho}} \tag{2}$$

$$\frac{\partial p'}{\partial t} = -\vec{V}\cdot\left(\overrightarrow{\nabla}\overline{p}\right) - \gamma\overline{p}\left(\overrightarrow{\nabla}\cdot\vec{V}\right) \tag{3}$$

$$\frac{\partial \rho'}{\partial t} = -\vec{V}\cdot\left(\overrightarrow{\nabla}\overline{\rho}\right) - \overline{\rho}\left(\overrightarrow{\nabla}\cdot\vec{V}\right) \tag{4}$$

where $\vec{V}$ – the velocity of the gas flow; $\gamma = C_p/C_v$; $\vec{a}$ – the force field vector the components of which in the case of gravitational forces are: $a_x = 0$, $a_y = 0$, $a_z = g$.

Further, it is convenient to enter a new value ϕ':

$$\phi' = p' - \gamma\frac{\overline{p}}{\overline{\rho}}\rho' \tag{5}$$

and go to the new variables:

$$P = p'\cdot\exp(z/2H),\ \Phi = \phi'\cdot\exp(z/2H),\ \overrightarrow{U} = \vec{V}\cdot\exp(-z/2H) \tag{6}$$

Next we consider the one-dimensional case. Finally the system of hydrothermodynamics takes the form:

$$\frac{\partial U_z}{\partial t} = \frac{1}{\rho_0}\left(\frac{\gamma-2}{2\gamma H} - \frac{\partial}{\partial z}\right)P + \frac{\Phi}{\gamma H\rho_0} \tag{7}$$

$$\frac{\partial P}{\partial t} = -\gamma g H\rho_0\left(\frac{\partial U_z}{\partial z}\right) - g\rho_0\frac{\gamma-2}{2}U_z \tag{8}$$

$$\frac{\partial \Phi}{\partial t} = -(\gamma-1)\rho_0 g U_z \tag{9}$$

Let us rewrite the system in terms of dimensionless functions and variables. For this purpose, we use the uniform atmosphere height H and the speed of sound $c = \sqrt{\gamma g H}$ as the dimension parameters which gives us the time scale $H/c = \sqrt{\frac{H}{\gamma g}}$ so that the new dimensionless variables are ξ, τ, defined by rescaling $z = H\xi$, $t = H/c\cdot\tau = \sqrt{\frac{H}{\gamma g}}\tau$. The functions are redefined as $U = cu = u\sqrt{\gamma g H}$, $P = p_0 p$ and, since Φ has the pressure as dimension as well (because $\phi' = p' - \gamma\frac{\overline{p}}{\overline{\rho}}\rho'$ and $\Phi = \phi'\cdot e^{z/2H}$) and $\Phi = p_0\phi$.

In the end we have a hydrothermodynamic system for the one-dimensional case in dimensionless quantities:

$$\frac{\partial u_z}{\partial \tau} = \frac{\gamma-2}{2\gamma^2}p - \frac{1}{\gamma}\frac{\partial p}{\partial \xi} + \frac{\phi}{\gamma^2} \tag{10}$$

$$\frac{\partial p}{\partial \tau} = -\gamma\frac{\partial u_z}{\partial \xi} - \frac{\gamma-2}{2}u_z \tag{11}$$

$$\frac{\partial \phi}{\partial \tau} = -(\gamma-1)u_z \tag{12}$$

We pose the mathematical problem of the boundary regime propagation. Consider the system of Equations (8) on the half-line $\xi \in [0;\infty)$. We transform System (10), expressing u_z from the last equation of the system:

$$u_z = -\frac{1}{(\gamma-1)}\frac{\partial\phi}{\partial\tau} \tag{13}$$

substituting it into other equations of the system and transferring the derivatives with respect to ξ to the left.

$$\frac{\partial p}{\partial\xi} = \frac{\gamma}{(\gamma-1)}\frac{\partial^2\phi}{\partial\tau^2} + \frac{\gamma-2}{2\gamma}p + \frac{\phi}{\gamma} \tag{14}$$

$$\frac{\partial^2\phi}{\partial\xi\partial\tau} = \frac{(\gamma-1)}{\gamma}\frac{\partial p}{\partial\tau} - \frac{\gamma-2}{2\gamma}\frac{\partial\phi}{\partial\tau} \tag{15}$$

Since the system of Equations (14) and (15) contains only two derivatives with respect to z, only two boundary conditions (16) will uniquely determine the solution of the system.

$$u_z(0,\tau) = F(\tau), \quad P(0,\tau) = G(\tau) \tag{16}$$

where F,G – arbitrary functions.

3. Solution of the mathematical problem of boundary regime propagation

We use the Fourier transform for the basic quantities of the system of Equations (14) and (15):

$$p(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega\tau}\tilde{p}(\xi,\omega)d\omega \tag{17}$$

$$\phi(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega\tau}\tilde{\phi}(\xi,\omega)d\omega \tag{18}$$

To do this we extend the functions onto the whole axis τ antisymmetrically. We substitute (17)–(18) into Equations (14) and (15):

$$\frac{\partial\tilde{p}(\xi,\omega)}{\partial\xi} = -\frac{\gamma}{(\gamma-1)}\omega^2\tilde{\phi}(\xi,\omega) + \frac{\gamma-2}{2\gamma}\tilde{p}(\xi,\omega) + \frac{1}{\gamma}\tilde{\phi}(\xi,\omega) \tag{19}$$

$$\frac{\partial\tilde{\phi}(\xi,\omega)}{\partial\xi} = \frac{(\gamma-1)}{\gamma}\tilde{p}(\xi,\omega) - \frac{\gamma-2}{2\gamma}\tilde{\phi}(\xi,\omega), \quad \text{if} \quad \omega \neq 0 \tag{20}$$

We also find the equation of the relationship between u_z and ϕ in ω-space:

$$u_z(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega\tau}\tilde{u}_z(\xi,\omega)d\omega \tag{21}$$

$$\tilde{u}_z(\xi,\omega) = -\frac{i\omega}{(\gamma-1)}\tilde{\phi}(\xi,\omega) \tag{22}$$

The system of Equations (19) and (20) is a linear homogeneous system of ordinary differential equations with constant coefficients that depend on the parameter.

The general solution of a linear homogeneous system of ordinary differential equations with constant coefficients depending on the parameter exists in the form:

$$p(\xi,\omega)=C_1(\omega)e^{\lambda_1(\omega)\xi}+C_2(\omega)e^{\lambda_2(\omega)\xi} \tag{23}$$

$$\phi(\xi,\omega)=C_3(\omega)e^{\lambda_1(\omega)\xi}+C_4(\omega)e^{\lambda_2(\omega)\xi} \tag{24}$$

where $\lambda_1\neq\lambda_2$.

We differentiate with respect to ξ Equation (19):

$$\frac{d^2\tilde{p}(\xi,\omega)}{d\xi^2}-\frac{\gamma-2}{2\gamma}\frac{d\tilde{p}(\xi,\omega)}{d\xi}-\frac{\gamma-1-\omega^2\gamma}{\gamma(\gamma-1)}\frac{d\tilde{\phi}(z,\omega)}{d\xi}=0 \tag{25}$$

We substitute the derivative $\frac{d\tilde{\phi}(\tilde{z},\omega)}{d\xi}$ from the Equation (20):

$$\frac{d^2\tilde{p}(\xi,\omega)}{d\xi^2}-\frac{\gamma-2}{2\gamma}\frac{d\tilde{p}(\xi,\omega)}{d\xi}-\frac{\gamma-1-\omega^2\gamma}{\gamma(\gamma-1)}\left[\frac{\gamma-1}{\gamma}\tilde{p}(\xi,\omega)-\frac{\gamma-2}{2\gamma}\tilde{\phi}(\xi,\omega)\right]=0 \tag{26}$$

We take the value $\tilde{\phi}(\xi,\omega)$ from the Equation (19):

$$\tilde{\phi}(\xi,\omega)=\frac{\gamma(\gamma-1)}{(\gamma-1-\omega^2\gamma)}\frac{d\tilde{p}(\xi,\omega)}{d\xi}-\frac{(\gamma-1)(\gamma-2)}{2(\gamma-1-\omega^2\gamma)}\tilde{p}(\xi,\omega) \tag{27}$$

$$\begin{aligned}&\frac{d^2\tilde{p}(\xi,\omega)}{d\xi^2}-\frac{\gamma-2}{2\gamma}\frac{d\tilde{p}(\xi,\omega)}{d\xi}-\frac{\gamma-1-\omega^2\gamma}{\gamma(\gamma-1)}\Bigg[\frac{\gamma-1}{\gamma}\tilde{p}(\xi,\omega)-\\&\frac{\gamma-2}{2\gamma}\left(\frac{\gamma(\gamma-1)}{(\gamma-1-\omega^2\gamma)}\frac{d\tilde{p}(\xi,\omega)}{d\xi}-\frac{(\gamma-1)(\gamma-2)}{2(\gamma-1-\omega^2\gamma)}\tilde{p}(\xi,\omega)\right)\Bigg]=0\end{aligned} \tag{28}$$

$$\frac{d^2\tilde{p}(\xi,\omega)}{d\xi^2}-\frac{\gamma^2-4\omega^2\gamma}{4\gamma^2}\tilde{p}(\xi,\omega)=0 \tag{29}$$

It is known that the solution (29) exists and is unique:

$$p(\xi,\omega)=C_1(\omega)e^{\lambda_1(\omega)\xi}+C_2(\omega)e^{\lambda_2(\omega)\xi} \tag{30}$$

where λ_1 and λ_2 – roots of the characteristic equation for Equation (29).

$$\lambda_1=\sqrt{1/4-\omega^2/\gamma};\quad \lambda_2=-\sqrt{1/4-\omega^2/\gamma} \tag{31}$$

Then, the solution takes the form:

$$p(\xi,\omega)=C_1(\omega)e^{\sqrt{1/4-\omega^2/\gamma}\xi}+C_2(\omega)e^{-\sqrt{1/4-\omega^2/\gamma}\xi} \tag{32}$$

In turn, solutions for $\tilde{\phi}(\xi,\omega)$ will be expressed in terms of the solution $p(\xi,\omega)$:

$$\begin{aligned}&\tilde{\phi}(\xi,\omega)=\frac{\gamma(\gamma-1)}{(\gamma-1-\omega^2\gamma)}\frac{d\tilde{p}(\xi,\omega)}{d\xi}-\frac{(\gamma-1)(\gamma-2)}{2(\gamma-1-\omega^2\gamma)}\tilde{p}(\xi,\omega)=\\&\frac{\gamma(\gamma-1)}{(\gamma-1-\omega^2\gamma)}\frac{d}{d\xi}\left[C_1(\omega)e^{\sqrt{1/4-\omega^2/\gamma}\xi}+C_2(\omega)e^{-\sqrt{1/4-\omega^2/\gamma}\xi}\right]-\\&\frac{(\gamma-1)(\gamma-2)}{2(\gamma-1-\omega^2\gamma)}\left[C_1(\omega)e^{\sqrt{1/4-\omega^2/\gamma}\xi}+C_2(\omega)e^{-\sqrt{1/4-\omega^2/\gamma}\xi}\right]\end{aligned} \tag{33}$$

$$\tilde{\phi}(\xi,\omega)=\frac{2\gamma(\gamma-1)\sqrt{1/4-\omega^2/\gamma}-(\gamma-1)(\gamma-2)}{2\left(\gamma-1-\omega^2\gamma\right)}C_1(\omega)e^{\sqrt{1/4-\omega^2/\gamma}\xi}+ \\ \frac{-2\gamma(\gamma-1)\sqrt{1/4-\omega^2/\gamma}-(\gamma-1)(\gamma-2)}{2\left(\gamma-1-\omega^2\gamma\right)}C_2(\omega)e^{-\sqrt{1/4-\omega^2/\gamma}\xi} \tag{34}$$

or

$$\tilde{\phi}(\xi,\omega)=C_3(\omega)e^{\sqrt{1/4-\omega^2/\gamma}\xi}+C_4(\omega)e^{-\sqrt{1/4-\omega^2/\gamma}\xi} \tag{35}$$

where

$$C_3(\omega)=\frac{2\gamma(\gamma-1)\sqrt{1/4-\omega^2/\gamma}-(\gamma-1)(\gamma-2)}{2\left(\gamma-1-\omega^2\gamma\right)}C_1(\omega) \tag{36}$$

$$C_4(\omega)=\frac{-2\gamma(\gamma-1)\sqrt{1/4-\omega^2/\gamma}-(\gamma-1)(\gamma-2)}{2\left(\gamma-1-\omega^2\gamma\right)}C_2(\omega) \tag{37}$$

Suppose that $F(t)$ and $G(t)$ are given on the boundary for velocity and pressure:

$$u_z(0,\tau)=F(\tau), \quad P(0,\tau)=G(\tau) \tag{38}$$

Let us write the boundary regimes in the ω-space:

$$\tilde{u}_z(0,\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega\tau}F(\tau)d\tau \tag{39}$$

$$\tilde{p}(0,\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega\tau}G(\tau)d\omega \tag{40}$$

Then, the boundary regime for $\tilde{\phi}$ in ω-space:

$$\tilde{\phi}(0,\omega)=\frac{i(\gamma-1)}{\omega}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega\tau}F(\tau)d\omega \tag{41}$$

We set $z=0$ in solutions (24) and (29) and equate them to the boundary regimes in the ω-space:

$$\tilde{p}(0,\omega)=C_1(\omega)+C_2(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega\tau}G(\tau)d\omega \tag{42}$$

and

$$\tilde{\phi}(0,\omega)=C_3(\omega)+C_4(\omega)= \\ \frac{2\gamma(\gamma-1)\sqrt{1/4-\omega^2/\gamma}-(\gamma-1)(\gamma-2)}{2(\gamma-1-\omega^2\gamma)}C_1(\omega)+ \\ \frac{-2\gamma(\gamma-1)\sqrt{1/4-\omega^2/\gamma}-(\gamma-1)(\gamma-2)}{2\left(\gamma-1-\omega^2\gamma\right)}C_2(\omega)= \\ \frac{i(\gamma-1)}{\omega}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega\tau}F(\tau)d\omega \tag{43}$$

4. Physical specification of the statement of the boundary regime propagation problem

The phenomenon of propagation of acoustic disturbance caused by surface waves is due to the transfer of energy and momentum from moving water masses to atmospheric gas. The interface between two media, which defines the boundary regime, is mobile due to the nature of its motion, which is at least two-dimensional.

However, considering waves of very long length, which include tsunami waves, allows some simplification and transition to a one-dimensional model. Similarly, assuming that the vertical displacement of the water surface in the tsunami wave is small in comparison with the height of the homogeneous atmosphere, we confine ourselves to solving the problem with a fixed boundary at the point $\xi=0$ and the boundary regime of gas velocity and pressure simulating the effect of the movement of water masses on the atmospheric gas.

In this case, we do not set a rigid condition that the vertical component of the gas velocity at the point $\xi = 0$ turns to zero. Instead, we ignore the wave propagating downward, and, in the linear model, the entropy mode. The amplitude of the wave propagating upwards can be determined in terms of the amplitude of the velocity of motion of the surface of the water, considering the perturbation of pressure, relying on thermodynamic considerations. Taking into account that the model is approximate, it can be said, the engineering character, adjustments of the model parameters are possible, relying on the practice of specific cases of the tsunami phenomenon and the corresponding measurements of the ionospheric effect.

5. A particular solution of the problem of the boundary regime propagation

Let the boundary regime be given as a pulse in the form of a modulated Gaussian:

$$u_z(0,\tau)=Ae^{\frac{-\tau^2}{\alpha}}\sin(\omega_0\tau) \tag{44}$$

$$p(0,\tau)=Be^{\frac{-\tau^2}{\alpha}}\sin(\omega_0\tau) \tag{45}$$

Let us refine the parameters α and ω_0. In [9] it is indicated that the period of tsunami waves lies within the limits of 5 to 60 minutes (in dimensional quantities). We find the values of the lower bound of the period in dimensionless quantities:

$$t=\frac{H}{c}\tau \tag{46}$$

$$\frac{H}{c}=\frac{H}{\sqrt{\gamma g H}}=\sqrt{\frac{H}{\gamma g}} \tag{47}$$

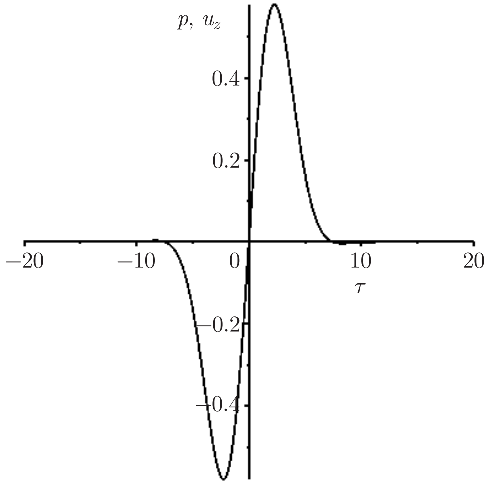

Figure 1. $e^{\frac{-\tau^2}{15}}\sin(0.42\tau)$

According to [11], the height of a homogeneous atmosphere at an interval of $0-100$ kilometers above sea level is approximately $H=6\,\text{km}$.

$$\frac{H}{c}=\sqrt{\frac{6000}{3/2\cdot 9.8}}\approx 20 \tag{48}$$

This means that 20 seconds in a dimension system equals 1 dimensionless unit of time. Then, the lower bound for the period in the dimensionless unit is $T=15$.

The model of the boundary regime is represented in the form of a single oscillation, *i.e.*, the time of the pulse – hence its period. Thus, the parameter α is chosen so that the pulse duration is equal to $T=15$. Then in turn ω_0 is selected according to the formula:

$$\omega_0=\frac{2\pi}{T}=\frac{2\pi}{15}\approx 0.42 \tag{49}$$

Let us find the Fourier transforms of the boundary conditions (44) and (45):

$$\tilde{u}_z(0,\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega\tau}u_z(0,\tau)d\tau=\frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega\tau}e^{\frac{-\tau^2}{\alpha}}\sin(\omega_0\tau)d\tau \tag{50}$$

$$\tilde{p}(0,\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega\tau}p(0,\tau)d\tau=\frac{B}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega\tau}e^{\frac{-\tau^2}{\alpha}}\sin(\omega_0\tau)d\tau \tag{51}$$

$$\tilde{u}_z(0,\omega)=A\frac{\sqrt{\alpha}}{i\sqrt{8}}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) \tag{52}$$

$$\tilde{p}(0,\omega)=B\frac{\sqrt{\alpha}}{i\sqrt{8}}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) \tag{53}$$

Using (42), we find:

$$\tilde{\phi}(0,\omega)=A\frac{(\gamma-1)}{\omega}\frac{\sqrt{\alpha}}{\sqrt{8}}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) \tag{54}$$

Then:

$$\tilde{p}(0,\omega)=C_1(\omega)+C_2(\omega)=B\frac{\sqrt{\alpha}}{i\sqrt{8}}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) \tag{55}$$

$$\begin{aligned}\tilde{\phi}(0,\omega)=&\frac{2\gamma(\gamma-1)\sqrt{1/4-\omega^2}-(\gamma-1)(\gamma-2)}{2(\gamma-1-\omega^2\gamma^2)}C_1(\omega)+\\&\frac{-2\gamma(\gamma-1)\sqrt{1/4-\omega^2}-(\gamma-1)(\gamma-2)}{2(\gamma-1-\omega^2\gamma^2)}C_2(\omega)=\\&\frac{(\gamma-1)}{\omega}A\frac{\sqrt{\alpha}}{\sqrt{8}}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)\end{aligned} \tag{56}$$

We find:

$$\begin{aligned}C_1(\omega)=&\left[\left(\frac{1}{2}+\frac{(\gamma-2)}{4\gamma\sqrt{1/4-\omega^2}}\right)\frac{B}{i}+\right.\\&\left.\frac{(\gamma-1-\omega^2\gamma^2)}{2\omega\gamma\sqrt{1/4-\omega^2}}A\right]\sqrt{\frac{\alpha}{8}}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)\end{aligned} \tag{57}$$

$$\begin{aligned}C_2(\omega)=&\left[\left(\frac{1}{2}-\frac{(\gamma-2)}{4\gamma\sqrt{1/4-\omega^2}}\right)\frac{B}{i}-\right.\\&\left.\frac{(\gamma-1-\omega^2\gamma^2)}{2\omega\gamma\sqrt{1/4-\omega^2}}A\right]\sqrt{\frac{\alpha}{8}}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)\end{aligned} \tag{58}$$

We substitute the resulting coefficients (57) and (58) in the solution (32). We also use the inverse Fourier transform to return to the τ-representation:

$$p(\xi,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega\tau}\tilde{p}(\xi,\omega)d\omega \tag{59}$$

so we have:

$$\begin{aligned}p(\xi,\tau)=&\frac{1}{4}\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty}e^{i\left(\omega\tau+\sqrt{\omega^2-1/4}\xi\right)}\left[\left(\frac{1}{2}+\frac{(\gamma-2)}{4\gamma\sqrt{\omega^2-1/4}}\right)\frac{B}{i}+\right.\\&\left.\frac{(\gamma-1-\omega^2\gamma^2)}{2i\omega\gamma\sqrt{\omega^2-1/4}}A\right]\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)d\omega+\\&\frac{1}{4}\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty}e^{i\left(\omega\tau-\sqrt{\omega^2-1/4}\xi\right)}\left[\left(\frac{1}{2}-\frac{(\gamma-2)}{4\gamma\sqrt{\omega^2-1/4}}\right)\frac{B}{i}-\right.\\&\left.\frac{(\gamma-1-\omega^2\gamma^2)}{2i\omega\gamma\sqrt{\omega^2-1/4}}A\right]\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)d\omega\end{aligned} \tag{60}$$

Formula (60) is the main result of the paper. (60) is the exact solution of the problem of propagation of a boundary regime of the form (44) and (45). The term containing $e^{i\left(\omega\tau+\sqrt{\omega^2-1/4}\xi\right)}$ describes a wave moving downward, and the term containing $e^{i\left(\omega\tau-\sqrt{\omega^2-1/4}\xi\right)}$ describes a wave moving upward.

6. Processing the exact solution

The calculation of the integrals in (60) is complicated by the presence of $e^{i\sqrt{\omega^2-1/4}\xi}$ and $e^{-i\sqrt{\omega^2-1/4}\xi}$, so we will carry out their approximation. By virtue of the fact that the contribution to the $\omega\in[-0.5;0.5]$ corresponds to the damping of the wave, so the approximation accuracy on this section can be considered not essential, we choose an approximation of the form of $|\omega|$.

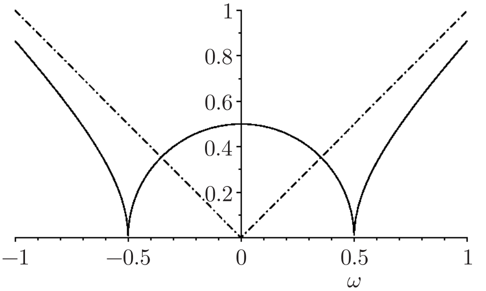

Figure 2. Solid line – $\sqrt{|\omega^2-1/4|}$; dot line – $|\omega|$

Next we divide formula (60) into three parts:

$$p(\xi,\tau)=p_1(\xi,\tau)+p_2(\xi,\tau)+p_3(\xi,\tau) \tag{61}$$

where

$$p_1(\xi,\tau)=\frac{B\sqrt{\alpha}}{8i\sqrt{\pi}}\int\limits_{-\infty}^{\infty}e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)e^{i|\omega|\xi}d\omega+$$
$$\frac{B\sqrt{\alpha}}{8i\sqrt{\pi}}\int\limits_{-\infty}^{\infty}e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)e^{-i|\omega|\xi}d\omega \tag{62}$$

$$p_2(\xi,\tau)=\frac{1}{4}\sqrt{\frac{\alpha}{\pi}}\int\limits_{-\infty}^{\infty}e^{i\omega\tau}\left(\frac{(\gamma-2)}{4\gamma i\sqrt{\omega^2-1/4}}\right)\frac{B}{i}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)e^{i|\omega|\xi}d\omega+$$
$$\frac{1}{4}\sqrt{\frac{\alpha}{\pi}}\int\limits_{-\infty}^{\infty}e^{i\omega\tau}\left(-\frac{(\gamma-2)}{4\gamma i\sqrt{\omega^2-1/4}}\right)\frac{B}{i}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)e^{-i|\omega|\xi}d\omega \tag{63}$$

and

$$p_3(\xi,\tau)=\frac{1}{4}\sqrt{\frac{\alpha}{\pi}}\int\limits_{-\infty}^{\infty}e^{i\omega\tau}\left[\frac{(\gamma-1-\omega^2\gamma^2)}{2i\omega\gamma\sqrt{\omega^2-1/4}}A\right]\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)e^{i|\omega|\xi}d\omega+$$
$$\frac{1}{4}\sqrt{\frac{\alpha}{\pi}}\int\limits_{-\infty}^{\infty}e^{i\omega\tau}\left[-\frac{(\gamma-1-\omega^2\gamma^2)}{2i\omega\gamma\sqrt{\omega^2-1/4}}A\right]\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)e^{-i|\omega|\xi}d\omega \tag{64}$$

It is easy to show that:

$$\int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{i|\omega|\xi} d\omega + \int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{-i|\omega|\xi} d\omega =$$
$$\int_{0}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{i\omega\xi} d\omega + \int_{-\infty}^{0} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{-i\omega\xi} d\omega +$$
$$\int_{0}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{-i\omega\xi} d\omega + \int_{-\infty}^{0} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{i\omega\xi} d\omega =$$
$$\int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{i\omega\xi} d\omega + \int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{-i\omega\xi} d\omega \tag{65}$$

Thus:

$$p_1(\xi,\tau) = \frac{B\sqrt{\alpha}}{8i\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{i\omega\xi} d\omega +$$
$$\frac{B\sqrt{\alpha}}{8i\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{-i\omega\xi} d\omega \tag{66}$$

We additionally transform (63) and (64) in such a way that:

$$p_{2\xi}(\xi,\tau) = \frac{B\sqrt{\alpha}}{4i\sqrt{\pi}}\left(\frac{(\gamma-2)}{4\gamma}\right) \int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{i\omega\xi} d\omega +$$
$$\frac{B\sqrt{\alpha}}{4i\sqrt{\pi}}\left(\frac{(\gamma-2)}{4\gamma}\right) \int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{-i\omega\xi} d\omega \tag{67}$$

and

$$p_{3\xi\tau}(\xi,\tau) = \frac{A\sqrt{\alpha}}{4\sqrt{\pi}} \frac{i(\gamma-1)}{2\gamma} \int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{i\omega\xi} d\omega -$$
$$\frac{A\sqrt{\alpha}}{4\sqrt{\pi}} \frac{i(\gamma-1)}{2\gamma} \int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{-i\omega\xi} d\omega -$$
$$\frac{iA\gamma\sqrt{\alpha}}{8\sqrt{\pi}} \frac{d^2}{d\tau^2}\left[\int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{i\omega\xi} d\omega\right] +$$
$$\frac{iA\gamma\sqrt{\alpha}}{8\sqrt{\pi}} \frac{d^2}{d\tau^2}\left[\int_{-\infty}^{\infty} e^{i\omega\tau}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) e^{-i\omega\xi} d\omega\right] \tag{68}$$

Using the Fourier-image table, it is easy to find

$$I(\tau+\xi) = \int_{-\infty}^{\infty} e^{i\omega(\tau+\xi)}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}} - e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right) d\omega = 4i\sqrt{\frac{\pi}{\alpha}} e^{\frac{-(\tau+\xi)^2}{\alpha}} \sin(\omega_0(\tau+\xi)) \tag{69}$$

and

$$I(\tau-\xi)=\int_{-\infty}^{\infty} e^{i\omega(\tau-\xi)}\left(e^{\frac{-\alpha(\omega-\omega_0)^2}{4}}-e^{\frac{-\alpha(\omega+\omega_0)^2}{4}}\right)d\omega=4i\sqrt{\frac{\pi}{\alpha}}e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right) \quad (70)$$

We rewrite (66)–(68) using (69) and (70):

$$p_1(\xi,\tau)=\frac{B}{2}e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)+\frac{B}{2}e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right) \quad (71)$$

$$p_{2\xi}(\xi,\tau)=B\frac{(\gamma-2)}{4\gamma}e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)+B\frac{(\gamma-2)}{4\gamma}e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right) \quad (72)$$

and

$$\begin{aligned}p_{3\xi\tau}(\xi,\tau)&=-A\frac{\gamma-1}{2\gamma}e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)-A\frac{\gamma-1}{2\gamma}e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right)-\\&\frac{A\gamma}{2}\frac{d^2}{d\tau^2}\left[e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)\right]+\frac{A\gamma}{2}\frac{d^2}{d\tau^2}\left[e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right)\right]\end{aligned} \quad (73)$$

Now we can back integrate (72) and (73):

$$\begin{aligned}p_2(\xi,\tau)=&B\frac{(\gamma-2)}{4\gamma}\int_0^{\xi} e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)d\xi+\\&B\frac{(\gamma-2)}{4\gamma}\int_0^{\xi} e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right)d\xi+K_1(\tau)\end{aligned} \quad (74)$$

and

$$\begin{aligned}p_3(\xi,\tau)=&-A\frac{\gamma-1}{2\gamma}\int_0^{\xi}\int_0^{\tau} e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)d\tau d\xi\\&-A\frac{\gamma-1}{2\gamma}\int_0^{\xi}\int_0^{\tau} e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right)d\tau d\xi-\\&\frac{A\gamma}{2}\frac{d^2}{d\tau^2}\int_0^{\xi}\int_0^{\tau}\left[e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)\right]d\tau d\xi+\\&\frac{A\gamma}{2}\frac{d^2}{d\tau^2}\int_0^{\xi}\int_0^{\tau}\left[e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right)\right]d\tau d\xi+\int_0^{\xi}N(\xi)d\xi+K_2(\tau)\end{aligned} \quad (75)$$

Now we have to determine the functions $K_1(\tau)$ and $K_2(\tau)$, obtained as a result of integration. We put $\xi=0$ in $p_1(\xi,\tau)$,$p_2(\xi,\tau)$ and $p_3(\xi,\tau)$. The substitution of $\xi=0$ into the limits leads to the fact that integration on the interval from 0 to 0 turns the integral to zero.

$$p(0,\tau)=p_1(0,\tau)+K_1(\tau)+K_2(\tau) \tag{76}$$

$$Be^{\frac{-\tau^2}{\alpha}}\sin(\omega_0\tau)=\frac{B}{2}e^{\frac{-\tau^2}{\alpha}}\sin(\omega_0(\tau))+\frac{B}{2}e^{\frac{-\tau^2}{\alpha}}\sin(\omega_0(\tau))+K_1(\tau)+K_2(\tau) \tag{77}$$

$$K_1(\tau)+K_2(\tau)=0 \tag{78}$$

We choose $K_1(\tau)$ and $K_2(\tau)$ so that:

$$K_1(\tau)=0, \quad K_2(\tau)=0 \tag{79}$$

Let us consider:

$$\begin{gathered}
p_{3\xi}(\xi,\tau)=-A\frac{\gamma-1}{2\gamma}\frac{\sqrt{\alpha\pi}}{4i}e^{\frac{-\alpha\omega_0^2}{4}}\left[\operatorname{erf}\left(\frac{\tau-\xi}{\sqrt{\alpha}}-\frac{i\omega_0\alpha}{2\sqrt{\alpha}}\right)-\operatorname{erf}\left(\frac{\tau-\xi}{\sqrt{\alpha}}+\frac{i\omega_0\alpha}{2\sqrt{\alpha}}\right)\right]- \\
A\frac{\gamma-1}{2\gamma}\frac{\sqrt{\alpha\pi}}{4i}e^{\frac{-\alpha\omega_0^2}{4}}\left[\operatorname{erf}\left(\frac{\tau+\xi}{\sqrt{\alpha}}-\frac{i\omega_0\alpha}{2\sqrt{\alpha}}\right)-\operatorname{erf}\left(\frac{\tau+\xi}{\sqrt{\alpha}}+\frac{i\omega_0\alpha}{2\sqrt{\alpha}}\right)\right]+ \\
\frac{A\gamma}{4i}\Big((i\omega_0-2(\tau-\xi)/\alpha)e^{\frac{-(\tau-\xi)^2}{\alpha}}e^{i\omega_0(\tau-\xi)}+(i\omega_0+2(\tau-\xi)/\alpha)e^{\frac{-(\tau-\xi)^2}{\alpha}}e^{-i\omega_0(\tau-\xi)}\Big)+ \\
\frac{A\gamma}{4i}\Big((i\omega_0-2(\tau+\xi)/\alpha)e^{\frac{-(\tau+\xi)^2}{\alpha}}e^{i\omega_0(\tau+\xi)}+(i\omega_0+2(\tau+\xi)/\alpha)e^{\frac{-(\tau+\xi)^2}{\alpha}}e^{-i\omega_0(\tau+\xi)}\Big)+N(\xi)
\end{gathered} \tag{80}$$

When $\tau\to\infty$ the wave additive to pressure tends to zero, and hence its derivative, *i.e.*:

$$p_{3\xi}(\xi,\infty)\to 0 \tag{81}$$

Substituting $\tau\to\infty$ in the Equation (80) we obtain

$$p_{3\xi}(\xi,\infty)\to N(\xi) \tag{82}$$

then

$$N(\xi)=0 \tag{83}$$

Thus, finally we have:

$$p_1(\xi,\tau)=\frac{B}{2}e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin(\omega_0(\tau+\xi))+\frac{B}{2}e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin(\omega_0(\tau-\xi)) \tag{84}$$

$$\begin{gathered}
p_2(\xi,\tau)=B\frac{(\gamma-2)}{4\gamma}\int_0^{\xi}e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin(\omega_0(\tau+\xi))\,d\xi+ \\
B\frac{(\gamma-2)}{4\gamma}\int_0^{\xi}e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin(\omega_0(\tau-\xi))\,d\xi
\end{gathered} \tag{85}$$

and

$$p_3(\xi,\tau) = -A\frac{\gamma-1}{2\gamma}\int\limits_0^{\xi}\int\limits_0^{\tau} e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)d\tau d\xi - A\frac{\gamma-1}{2\gamma}\int\limits_0^{\xi}\int\limits_0^{\tau} e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right)d\tau d\xi - \frac{A\gamma}{2}\frac{d^2}{d\tau^2}\int\limits_0^{\xi}\int\limits_0^{\tau}\left[e^{\frac{-(\tau+\xi)^2}{\alpha}}\sin\left(\omega_0(\tau+\xi)\right)\right]d\tau d\xi + \frac{A\gamma}{2}\frac{d^2}{d\tau^2}\int\limits_0^{\xi}\int\limits_0^{\tau}\left[e^{\frac{-(\tau-\xi)^2}{\alpha}}\sin\left(\omega_0(\tau-\xi)\right)\right]d\tau d\xi \tag{86}$$

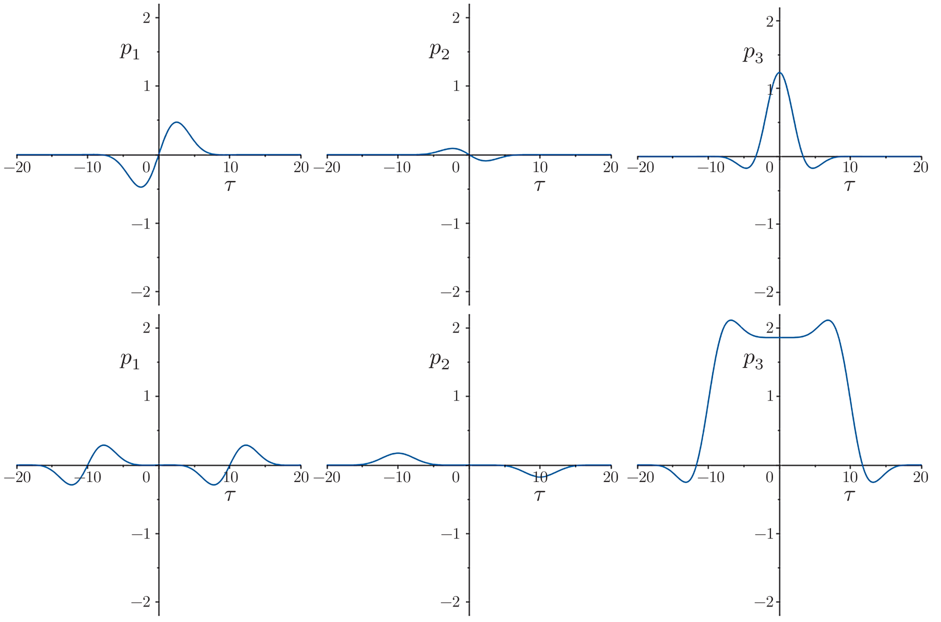

Figure 3. p_1, p_2, p_3 at $\xi = 1$ (top row) and $\xi = 10$ (bottom row)

The velocity solution can be found from the original equation

$$\frac{\partial p}{\partial \tau} = -\gamma\frac{\partial u_z}{\partial \xi} - \frac{\gamma-2}{2}u_z \tag{87}$$

using the factorization method:

$$u_z = e^{\frac{(\gamma-2)\xi}{(2\gamma)}}\left[\int\limits_0^{\xi}\frac{e^{-\frac{(\gamma-2)\xi}{(2\gamma)}}}{-\gamma}\frac{\partial p}{\partial \tau}d\xi + C\right] \tag{88}$$

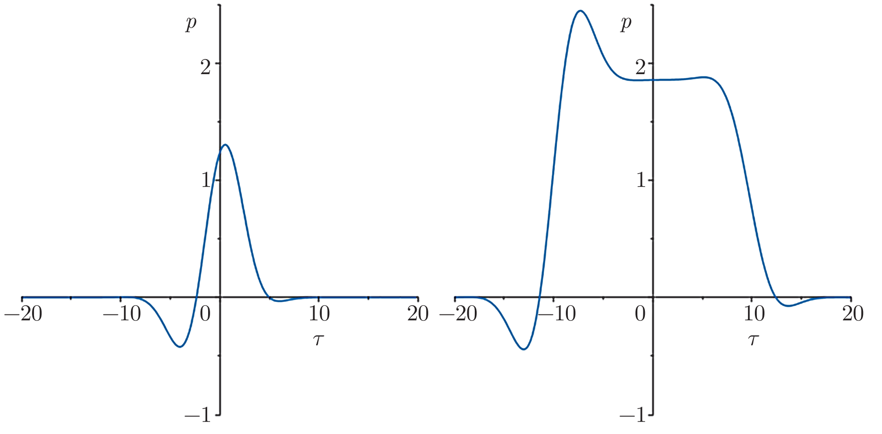

Figure 4. p at $\xi = 1$ (left) and $\xi = 10$ (right)

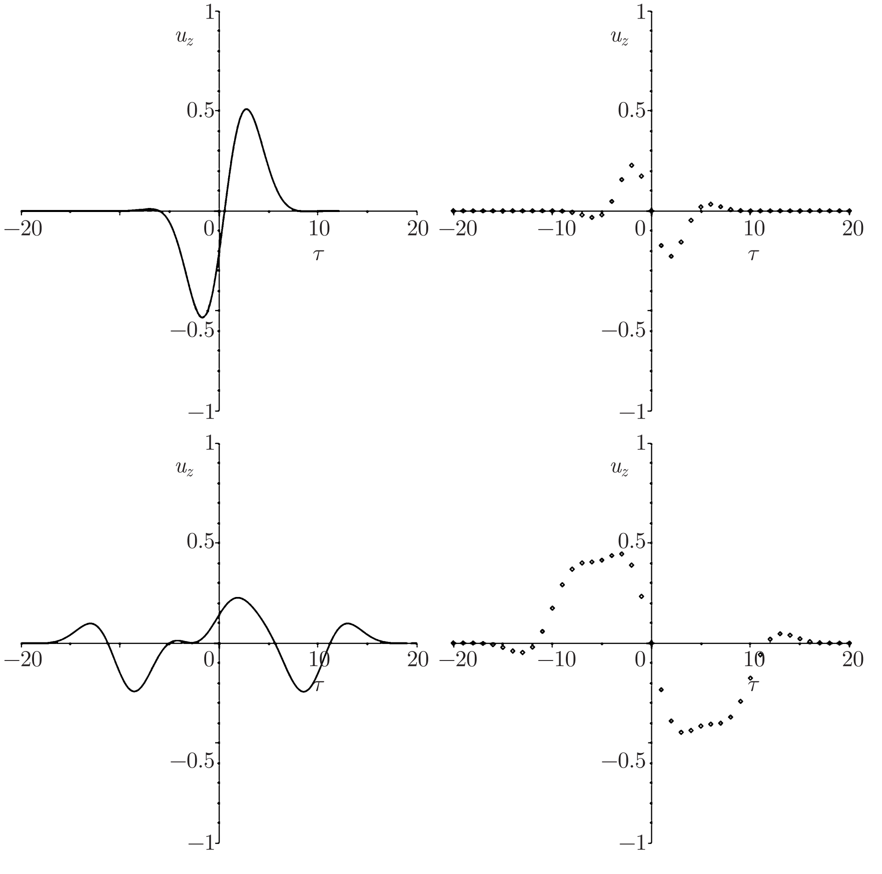

Figure 5. u_{z1+2}, u_{z3} at $\xi = 1$ (top row) and $\xi = 10$ (bottom row)

7. The ionospheric effect

The ionospheric effect is related to the movement of a neutral gas and is determined by its velocity.

The acoustic wave propagation is entering ionosphere acts ions. The electrons, being more light particles, follow the neutral particles but perform a circular motion and drift along the magnetic field lines due to the Lorentz force which is orthogonal to the velocity and magnetic field. The problem of the AGW ionosphere effect description has been studied for many years. In [12, 13] a simple formula for the electron concentration dynamics is derived, its coordinate dependence is calculated as a solution of the diffusion equation, parameterized by the velocity profile as a coefficient.

Due to the exponential growth of the acoustic wave amplitude with the increasing altitude above sea level, even small disturbances (for example, for speeds of the order of 25 cm/s) at sea level increase at altitudes of the ionosphere (about 80 km) approximately 800 times, which gives a speed of 200 m/s.

8. Conclusion

We see the application of a projecting operator that will specify the only direction of propagation as a direct development of the statement of the problem having in mind the boundary regime propagation. We also understand that some details in the ionosphere effect evaluation should be performed and published.

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