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EXISTENCE OF THREE SOLUTIONS FOR IMPULSIVE NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS

Shapour Heidarkhani, Massimiliano Ferrara, Giuseppe Caristi, and Amjad Salari

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Abstract. In this work we present new criteria on the existence of three solutions for a class of impulsive nonlinear fractional boundary-value problems depending on two parameters. We use variational methods for smooth functionals defined on reflexive Banach spaces in order to achieve our results.

Keywords: fractional differential equation, impulsive condition, classical solution, variational methods, critical point theory.

Mathematics Subject Classification: 34A08, 34B37, 58E05, 58E30, 26A33.

1. INTRODUCTION

In this paper we consider the following impulsive nonlinear fractional boundary value problem

$$\begin{gathered} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right) + a(t)u(t) = \lambda f(t,u(t)) + h(u(t)), \quad t \neq t_j, \text{ a.e. } t\in[0,T], \\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_j) = \mu I_j(u(t_j)), \quad j=1,\dots,n, \\ u(0)=u(T)=0, \end{gathered} \tag{$D_{\lambda,\mu}$}$$

where $\alpha \in (1/2,1]$, $a \in C([0,T])$ and there are two positive constants a_1 and a_2 such that $0 < a_1 \leq a(t) \leq a_2$ for every $t \in [0,T]$, λ and μ are two non-negative real parameters, $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|h(x_1) - h(x_2)| \leq L|x_1 - x_2|$$

for every $x_1, x_2 \in \mathbb{R}$ and $h(0) = 0$, $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$,

$$\Delta \left({}_tD_T^{\alpha-1} \left({}_0^cD_t^\alpha u\right)\right)(t_j) = {}_tD_T^{\alpha-1} \left({}_0^cD_t^\alpha u\right)(t_j^+) - {}_tD_T^{\alpha-1} \left({}_0^cD_t^\alpha u\right)(t_j^-),$$

$${}_tD_T^{\alpha-1} \left({}_0^cD_t^\alpha u\right)(t_j^+) = \lim_{t\to t_j^+} \left({}_tD_T^{\alpha-1} \left({}_0^cD_t^\alpha u\right)(t)\right),$$

$${}_tD_T^{\alpha-1} \left({}_0^cD_t^\alpha u\right)(t_j^-) = \lim_{t\to t_j^-} \left({}_tD_T^{\alpha-1} \left({}_0^cD_t^\alpha u\right)(t)\right)$$

and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, n$ are continuous functions.

Fractional differential equations (FDEs) have gained importance due to their numerous applications in various fields of science and engineering, such as physics, biophysics, blood flow phenomena, aerodynamics, electro magnetic, fluid flow, diffusive transport akin to diffusion, chemistry, electron-analytical chemistry, electro dynamics of complex medium, polymer rheology, viscoelasticity, control, porous media, probability, electrical networks, biology, etc. For details, see [13, 17, 18, 28, 31, 33, 37, 39, 44]. Many researchers have studied the existence of solutions for nonlinear FDEs with different tools such as fixed-point theorems, the topological degree theory, and the method of upper and lower solutions, for instance, see [5, 6, 20].

Critical point theory has been very useful in determining the existence of solution for integer order differential equations with some boundary conditions, for example [11, 35, 38, 45]. But until now, there are few results on the solution to fractional boundary value problems which were established by the critical point theory, since it is often very difficult to establish a suitable space and variational functional for fractional boundary value problems. Recently, Jiao and Zhou in [29] by using the critical point theory investigated the fractional boundary-value problem

$$\begin{aligned} \frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{aligned} \tag{1.1}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$ respectively, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t, x)$ is the gradient of F at x. Also, Chen and Tang in [10] studied the existence and multiplicity of solutions for the fractional boundary value problem (1.1) where $F(t, \cdot)$ are superquadratic, asymptotically quadratic, and subquadratic, respectively. In particular, Bai in [1] by a local minimum theorem investigated the existence of at least one non-trivial solution to a nonlinear fractional boundary value problem. In [19] the authors using critical point theory, discussed the existence of multiple solutions of the system of fractional boundary value problems of the form (1.1). In fact, they found sufficient conditions under which the problem has at least two or infinitely many nontrivial solutions. In [2, 15, 21, 22, 32, 40–43] by using variational methods and critical point theory the existence of multiple solutions for fractional boundary value problems was investigated, and in [16] the authors exploited a critical point result for differentiable functionals in order to prove that a suitable class of one-dimensional fractional problems admits at least one non-trivial solution under an asymptotical behaviour of the nonlinear datum at zero. We also refer to the paper [27] in which using variational methods the existence of one weak solution for a class of fractional differential systems was investigated.

On the other hand, impulsive differential equations appear as the natural descriptions of the observed evolutionary phenomena of several real problems in biology, physics, engineering, etc. For example, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics and frequency modulated systems, do exhibit impulsive effects. For the general aspects and applications of impulsive differential equations, we refer the reader to [4, 7, 34, 36, 49]. The existence of multiple solutions of impulsive problems has been studied also using the variational methods and critical point theorems (see [14,23]). Both FDEs and impulsive differential equations have drawn intense attention from researchers in the last decades due to the numerous applications. The idea that combining these two classes of differential equations may yield an interesting and promising object of investigation, viz., impulsive FDEs, prompted numerous papers. For the recent developments in theory and applications of impulsive FDEs, we refer the reader to the papers [12, 51, 52] and the references therein. Impulsive problems for fractional equations have been treated by topological methods in [3,30]. In particular, in [8, 47] based on variational methods the existence and multiplicity of solutions for the problem $(D_{\lambda,\mu})$, in the case $h(x) = 0$ for all $x \in \mathbb{R}$ was studied.

We also refer the reader to [25, 26] in which using variational methods and critical point theory the existence of multiple solutions for impulsive fractional differential systems were discussed.

Motivated by the above works, in this paper we are interested to investigate the existence of at least three non-trivial classical solutions for $(D_{\lambda,\mu})$ for appropriate values of the parameters λ and μ belonging to real intervals. Our approach is variational methods and a three critical points theorem due to Ricceri [46].

Here, we state a special case of our main result.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $I_j : \mathbb{R} \to \mathbb{R}$ be a continuous function for $j = 1, \ldots, n$, and $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant L such that*

$$0 < L < (2\alpha - 1)\Gamma^2(\alpha), \quad h(0) = 0.$$

Assume that there exists a positive constant ϵ such that

$$\max\left\{ \limsup_{u\to 0} \frac{\int_0^u f(\xi)\mathrm{d}\xi}{|u|^2}, \limsup_{|u|\to\infty} \frac{\int_0^u f(\xi)\mathrm{d}\xi}{|u|^2} \right\} < \epsilon < \frac{\left(1 - \frac{L}{(2\alpha-1)\Gamma^2(\alpha)}\right) \int_0^1 \int_0^{\overline{w}(t)} f(\xi)\mathrm{d}\xi\mathrm{d}t}{A(\alpha)\delta^2 - 2\int_0^1 \int_0^{\overline{w}(t)} h(\xi)\mathrm{d}\xi\mathrm{d}t + \frac{2}{3}}.$$

where

$$\overline{w}(t) = \begin{cases} 4t, & \text{if } t \in [0, \frac{1}{4}), \\ 1, & \text{if } t \in [\frac{1}{4}, \frac{3}{4}], \\ 4(1-t), & \text{if } t \in (\frac{3}{4}, 1] \end{cases}$$

and

$$A(\alpha) := \frac{1}{\Gamma^2(1-\alpha)4^{2\alpha-1}} \frac{6\alpha^2 - 19\alpha + 16}{(1-\alpha)^2(2-\alpha)(3-2\alpha)}.$$

Then, for each compact interval $[c, d] \subset (\lambda_1'', \lambda_2'')$, *where*

$$\lambda_1'' = \inf \left\{ \frac{|u|_\alpha^2 - 2\int_0^1 \int_0^{\overline{w}(t)} h(\xi)\mathrm{d}\xi \mathrm{d}t}{2\int_0^1 \int_0^{u(t)} f(\xi)\mathrm{d}\xi \mathrm{d}t} : \ u \in C_0^\infty([0,1]), u(0) = u(1) = 0, \quad f(\xi) > 0 \text{ for all } \xi \in \mathbb{R} \right\}$$

and

$$\lambda_2'' = \left(\max \left\{ 0, \limsup_{|u| \to 0} \frac{2\int_0^1 \int_0^{u(t)} f(\xi)\mathrm{d}\xi \mathrm{d}t}{|u|_\alpha^2 - 2\int_0^1 \int_0^{\overline{w}(t)} h(\xi)\mathrm{d}\xi \mathrm{d}t}, \limsup_{|u|_\alpha \to +\infty} \frac{2\int_0^1 \int_0^{u(t)} f(\xi)\mathrm{d}\xi \mathrm{d}t}{|u|_\alpha^2 - 2\int_0^1 \int_0^{\overline{w}(t)} h(\xi)\mathrm{d}\xi \mathrm{d}t} \right\} \right)^{-1}$$

with

$$|u|_\alpha = \left(\Gamma^{-2}(1-\alpha) \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} u'(s)\mathrm{d}s \right) \mathrm{d}t + \int_0^1 |u(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}},$$

there exists $R > 0$ *with the following property: for every* $\lambda \in [c, d]$ *there exists* $\gamma > 0$ *such that, for each* $\mu \in [0, \gamma]$, *the problem*

$$\begin{gathered} {}_tD_1^\alpha \left({}_0^cD_t^\alpha u(t) \right) + u(t) = \lambda f(u(t)) + h(u(t)), \quad t \neq t_j, \ \text{a.e. } t \in [0,1], \\ \Delta \left({}_tD_1^{\alpha-1} \left({}_0^cD_t^\alpha u \right) \right)(t_j) = \mu I_j(u(t_j)), \quad j = 1, \ldots, n, \\ u(0) = u(1) = 0 \end{gathered}$$

has at least three classical solutions u_1, u_2 *and* u_3 *such that* $|u_i|_\alpha < R$, $i = 1, 2, 3$.

2. PRELIMINARIES

In this section, we will introduce some notations, definitions and preliminary facts which will be used throughout this paper.

Definition 2.1 ([31])**.** Let f be a function defined on $[a,b]$ and $\alpha > 0$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by

$$ {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s, \quad t\in[a,b], $$

$$ {}_tD_b^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\mathrm{d}s, \quad t\in[a,b], $$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2 ([31])**.** Let a, b be real numbers and denote by $AC([a,b])$ the space of absolutely continuous functions on $[a,b]$. For $0<\alpha\le 1$, $f\in AC([a,b])$ there are defined left and right Riemann-Liouville and Caputo fractional derivatives as follows:

$$ {}_aD_t^{\alpha}f(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}{}_aD_t^{\alpha-1}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-s)^{-\alpha}f(s)\mathrm{d}s, $$

$$ {}_tD_b^{\alpha}f(t) \equiv -\frac{\mathrm{d}}{\mathrm{d}t}{}_tD_b^{\alpha-1}f(t) = -\frac{1}{\Gamma(1-\alpha)}\int_t^b (s-t)^{-\alpha}f(s)\mathrm{d}s, $$

$$ {}_a^cD_t^{\alpha}f(t) \equiv {}^cD_{a+}^{\alpha}f(t) := {}_aD_t^{\alpha-1}f'(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}f'(s)\mathrm{d}s $$

and

$$ {}_t^cD_b^{\alpha}f(t) \equiv {}^cD_{b-}^{\alpha}f(t) := -{}_tD_b^{\alpha-1}f'(t) = -\frac{1}{\Gamma(1-\alpha)}\int_t^b (s-t)^{-\alpha}f'(s)\mathrm{d}s, $$

where $\Gamma(\alpha)$ is the gamma function. Note that when $\alpha = 1$, ${}_a^cD_t^1f(t) = f'(t)$ and ${}_t^cD_b^1f(t) = -f'(t)$

We have the following property of fractional integration.

Proposition 2.3 ([31, 48])**.** *We have the following property of fractional integration*

$$ \int_a^b [{}_aD_t^{-\gamma}f(t)]g(t)\mathrm{d}t = \int_a^b [{}_tD_b^{-\gamma}g(t)]f(t)\mathrm{d}t, \quad \gamma>0, $$

provided that $f\in L^p([a,b],\mathbb{R}^N)$, $g\in L^q([a,b],\mathbb{R}^N)$ *and* $p\ge 1$, $q\ge 1$, $1/p+1/q\le 1+\gamma$ *or* $p\ne 1$, $q\ne 1$, $1/p+1/q = 1+\gamma$.

To construct appropriate function spaces and apply critical point theory to investigate the existence of solutions for problem $(D_{\lambda,\mu})$, we need the following basic notations and results which will be used in the proofs of our main results.

Let $0 < \alpha \leq 1$, $1 < p < \infty$ and $E_0^{\alpha,p}(0,T)$ be the Banach space, which is closure of $C_0^\infty([0,T])$ with respect to the norm

$$\|u\|^p_{E_0^{\alpha,p}(0,T)} = \|{}_0^cD_t^\alpha u(t)\|^p_{L^p(0,T)} + \|u\|^p_{L^p(0,T)}.$$

It is known that $E_0^{\alpha,p}(0,T)$ is a reflexive and separable Banach space (see [29, Proposition 3.1]). Denote for short $E_{0,T}^{\alpha,2} = E^\alpha$, and by $\|\cdot\|$ and $\|\cdot\|_\infty$ the norms in $L^2(0,T)$ and $C([0,T])$:

$$\|u\|^2 = \int_0^T |u(t)|^2 \mathrm{d}t, \quad u \in L^2(0,T),$$

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|, \quad u \in C([0,T]).$$

E^α is a Hilbert space with the inner product

$$(u,v)_\alpha = \int_0^T ({}_0^cD_t^\alpha u(t)\, {}_0^cD_t^\alpha v(t) + u(t)v(t))\mathrm{d}t$$

and the norm

$$\|u\|_\alpha^2 = \int_0^T (|{}_0^cD_t^\alpha u(t)|^2 + |u(t)|^2)\mathrm{d}t.$$

Note that if $a \in C([0,T])$ and there are two positive constants a_1 and a_2, such that $0 < a_1 \leq a(t) \leq a_2$, an equivalent norm in E^α is

$$\|u\|_{a,\alpha}^2 = \int_0^T (|{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t + a(t)|u(t)|^2)\mathrm{d}t.$$

Proposition 2.4 ([29])**.** *Let $0 < \alpha \leq 1$. For $u \in E^\alpha$, we have*

$$\|u\| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^cD_t^\alpha u\|. \tag{2.1}$$

Moreover, for $\frac{1}{2} < \alpha \leq 1$,

$$\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} \|{}_0^cD_t^\alpha u\|.$$

According to (2.1), we can consider E^α with respect to the norm

$$\|u\|_{0,\alpha} = \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 \mathrm{d}t \right)^{1/2} = \|{}_0^cD_t^\alpha u\|, \quad u \in E^\alpha$$

in the following analysis.

By Proposition 2.4, when $\alpha > 1/2$, for each $u \in E^\alpha$ we have

$$\|u\|_\infty \leq k\left(\int\limits_0^T |{}_0^cD_t^\alpha u(t)|^2 \mathrm{d}t\right)^{1/2} = k\|u\|_{0,\alpha} < k\|u\|_{a,\alpha}, \tag{2.2}$$

where

$$k = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}. \tag{2.3}$$

Definition 2.5. A function

$$u \in \left\{u \in AC([0,T]) \ : \ \int\limits_{t_j}^{t_{j+1}} (|{}_0^cD_t^\alpha u(t)|^2 + |u(t)|^2)\mathrm{d}t < \infty,\ j = 0,\ldots n\right\}$$

is said to be a classical solution of problem $(D_{\lambda,\mu})$ if u satisfies the equation

$${}_tD_T^\alpha\left({}_0^cD_t^\alpha u(t)\right) + a(t)u(t) = \lambda f(t,u(t)) + h(u(t)), \quad \text{a.e. } t \in [0,T]\backslash\{t_1,\ldots,t_n\},$$

the limits ${}_tD_T^{\alpha-1}\left({}_0^cD_t^\alpha u\right)(t_j^+)$ and ${}_tD_T^{\alpha-1}\left({}_0^cD_t^\alpha u\right)(t_j^-)$ exist and satisfy the impulsive condition $\Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^\alpha u\right)\right)(t_j) = \mu I_j(u(t_j))$ and boundary condition $u(0) = u(T) = 0$ holds.

Now we give the definition of weak solution for the problem $(D_{\lambda,\mu})$ as follows.

Definition 2.6. A function $u \in E^\alpha$ is said to be a weak solution of the problem $(D_{\lambda,\mu})$, if for every $v \in E^\alpha$,

$$\int\limits_0^T \left[({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t)) + a(t)u(t)v(t)\right]\mathrm{d}t + \mu\sum_{j=1}^n I_j(u(t_j))v(t_j)$$
$$= \lambda\int\limits_0^T f(t,u(t))\,v(t)\mathrm{d}t + \int\limits_0^T h(u(t))v(t)\mathrm{d}t.$$

Lemma 2.7 ([8, Lemma 2.1])**.** *The function $u \in E^\alpha$ is a weak solution of $(D_{\lambda,\mu})$ if and only if u is a classical solution of $(D_{\lambda,\mu})$.*

Our main tool is Theorem 2.8 which has been obtained by Ricceri ([46, Theorem 2]). It is as follows:

If X is a real Banach space, denoted by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty}\Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

For example, if X is uniformly convex and $g : [0,+\infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then, by a classical result, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Theorem 2.8. *Let X be a separable and reflexive real Banach space; let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$
$$\sigma = \sup_{u\in\Phi^{-1}(]0,+\infty[)} \frac{J(u)}{\Phi(u)},$$

assume that $\rho < \sigma$. Then for each compact interval $[c, d] \subset (\frac{1}{\sigma}, \frac{1}{\rho})$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$), there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$,

$$\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$$

has at least three solutions in X whose norms are less than R.

We refer the reader to the papers [9, 24, 50] in which Theorem 2.8 was successfully employed to ensure the existence of at least three solutions for boundary value problems.

Corresponding to the functions f, h and I_j, $j = 1\ldots, n$, we introduce the functions $F : [0, T]\times\mathbb{R} \longrightarrow \mathbb{R}$, $H : \mathbb{R} \longrightarrow \mathbb{R}$ and $J_j : [0, T]\times\mathbb{R} \longrightarrow \mathbb{R}$, $j = 1, \ldots, n$, respectively, as follows:

$$F(t, \xi) := \int_0^\xi f(t, x)\mathrm{d}x \quad \text{for all } \xi \in \mathbb{R},$$
$$H(\xi) := \int_0^\xi h(x)\mathrm{d}x \quad \text{for all } \xi \in \mathbb{R}$$

and

$$J_j(\xi) := \int_0^\xi I_j(x)\mathrm{d}x \quad \text{for all } \xi \in \mathbb{R},\ j = 1, \ldots n.$$

Now for every $u \in E^\alpha$, we define

$$\Phi(u) := \frac{1}{2}\|u\|^2_{a,\alpha} - \int_0^T H(u(t))\mathrm{d}t, \tag{2.4}$$

$$J(u) = \int_0^T F(t, u(t))\mathrm{d}t \tag{2.5}$$

and

$$\Psi(u) = -\sum_{j=1}^{n} J_j(u(t_j)). \tag{2.6}$$

Standard arguments show that $\Phi - \mu\Psi - \lambda J$ is a Gâteaux differentiable functional whit Gâteaux derivative at the point $u \in E^\alpha$ given by

$$\begin{aligned}(\Phi' - \mu\Psi' - \lambda J')(u)(v) = &\int_0^T \left(({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t)) + a(t)u(t)v(t)\right) \mathrm{d}t \\ &+ \mu \sum_{j=1}^{n} I_j(u(t_j))v(t_j) - \int_0^T h(u(t))v(t)\mathrm{d}t \\ &- \lambda \int_0^T f(t, u(t))v(t)\mathrm{d}t\end{aligned}$$

for all $v \in X$ (see [8] for more details). Hence, a critical point of the functional $\Phi - \mu\Psi - \lambda J$, gives us a weak solution of $(D_{\lambda,\mu})$, and in view of Lemma 2.7 every weak solution of the problem $(D_{\lambda,\mu})$ is a classical one.

We suppose that the Lipschitz constant $L > 0$ of the function h satisfies $LTk^2 < 1$.

We need the following proposition in the proof of our main result.

Proposition 2.9. *Let $S : E^\alpha \longrightarrow (E^\alpha)^*$ be the operator defined by*

$$S(u)(v) = \int_0^T [({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t)) + a(t)u(t)v(t)]\, \mathrm{d}t - \int_0^T h(u(t))v(t)\mathrm{d}t$$

for every $u, v \in E^\alpha$. Then, S admits a continuous inverse on $(E^\alpha)^*$.

Proof. Recalling (2.2) we have

$$\begin{aligned}S(u)(u) &= \int_0^T \left(|{}_0^cD_t^\alpha u(t)|^2 + a(t)|u(t)|^2\right) \mathrm{d}t - \int_0^T h(u(t))u(t)\mathrm{d}t \\ &\geq \|u\|_{a,\alpha}^2 - L\int_0^T |u(t)|^2\mathrm{d}t \geq \left(1 - LTk^2\right) \|u\|_{a,\alpha}^2.\end{aligned}$$

Since $LTk^2 < 1$, this follows that S is coercive. Owing to our assumptions on the data, one has

$$\begin{aligned}\langle S(u) - S(v), u - v\rangle &= \int_0^T \left(|{}_0^cD_t^\alpha(u(t) - v(t))|^2 + a(t)|u(t) - v(t)|^2\right) \mathrm{d}t \\ &\quad - \int_0^T h(u(t) - v(t))(u(t) - v(t))\mathrm{d}t \geq (1 - LTk^2)\|u - v\|_{a,\alpha}^2 > 0\end{aligned}$$

for every $u, v \in E^\alpha$, which means that S is strictly monotone. Moreover, since E^α is reflexive, for $u_n \to u$ strongly in E^α as $n \to +\infty$, one has $S(u_n) \to S(u)$ weakly in $(E^\alpha)^*$ as $n \to +\infty$. Hence, S is demicontinuous, so by [53, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is continuous. Indeed, let e_n be a sequence of $(E^\alpha)^*$ such that $e_n \to e$ strongly in $(E^\alpha)^*$ as $n \to +\infty$. Let u_n and u in E^α be such that $S^{-1}(e_n) = u_n$ and $S^{-1}(e) = u$. Taking in to account that S is coercive, one has that the sequence u_n is bounded in the reflexive space E^α. For a suitable subsequence, we have $u_n \to \hat{u}$ weakly in E^α as $n \to +\infty$, which concludes

$$\langle S(u_n) - S(u), u_n - \hat{u}\rangle = \langle e_n - e, u_n - \hat{u}\rangle = 0.$$

Note that if $u_n \to \hat{u}$ weakly in E^α as $n \to +\infty$ and $S(u_n) \to S(\hat{u})$ strongly in $(E^\alpha)^*$ as $n \to +\infty$, one has $u_n \to \hat{u}$ strongly in E^α as $n \to +\infty$, and since S is continuous, we have $u_n \to \hat{u}$ weakly in E^α as $n \to +\infty$ and $S(u_n) \to S(\hat{u}) = S(u)$ strongly in $(E^\alpha)^*$ as $n \to +\infty$. Hence, taking into account that S is an injection, we have $u = \hat{u}$. □

3. MAIN RESULTS

Put

$$\lambda_1 = \inf\left\{\frac{\|u\|_{a,\alpha}^2 - 2\int_0^T H(u(t))\mathrm{d}t}{2\int_0^T F(t, u(t))\mathrm{d}t} : u \in E^\alpha, \int_0^T F(t, u(t))\mathrm{d}t > 0\right\}$$

and

$$\lambda_2 = \left(\max\left\{0, \limsup_{|u|\to 0} \frac{2\int_0^T F(t, u(t))\mathrm{d}t}{\|u\|_{a,\alpha}^2 - 2\int_0^T H(u(t))\mathrm{d}t}, \limsup_{\|u\|_{a,\alpha}\to+\infty} \frac{2\int_0^T F(t, u(t))\mathrm{d}t}{\|u\|_{a,\alpha}^2 - 2\int_0^T H(u(t))\mathrm{d}t}\right\}\right)^{-1}.$$

We formulate our main result as follows.

Theorem 3.1. *Suppose the following conditions hold:*

($\mathcal{A}_1$) *there exists a constant* $\epsilon > 0$ *such that*

$$\max\left\{\limsup_{u\to 0}\frac{\max_{t\in[0,T]}F(t,u)}{|u|^2}, \limsup_{|u|\to+\infty}\frac{\max_{t\in[0,T]}F(t,u)}{|u|^2}\right\} < \epsilon;$$

($\mathcal{A}_2$) *there exists a function* $w \in E^\alpha$ *such that* $\|w\|_{a,\alpha}^2 - 2\int_0^T H(w(t))\mathrm{d}t \neq 0$ *and*

$$Tk^2\epsilon < \frac{(1-LTk^2)\int_0^T F(t,w(t))\mathrm{d}t}{\|w\|_{a,\alpha}^2 - 2\int_0^T H(w(t))\mathrm{d}t}.$$

Then, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$*, there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$*, the problem* $(D_{\lambda,\mu})$ *has at least three classical solutions whose norms in* E^α *are less than* R.

Proof. Take $X = E^\alpha$. Clearly, X is a separable and uniformly convex Banach space. Let the functionals Φ, J and Ψ be as given in (2.4), (2.5) and (2.6), respectively. The functional Φ is a C^1 functional, and due to Proposition 2.9 its derivative admits a continuous inverse on X^*. Moreover, by the sequentially weakly lower semicontinuity of $\|u\|_{a,\alpha}$, Φ is sequentially weakly lower semicontinuous in X. On the other hand, since

$$\Phi(u) \geq \frac{1-LTk^2}{2}\|u\|_{a,\alpha}^2 \tag{3.1}$$

for every $u \in X$, Φ is coercive. Moreover, Φ is bounded on each bounded subset of X. Indeed, let M be a bounded subset of X. That is, there exists a constant $c > 0$ such that $\|u\|_{a,\alpha} \leq c$ for each $u \in M$. Then, we have

$$|\Phi(u)| \leq \frac{1+LTk^2}{2}c^2.$$

Furthermore, $\Phi \in \mathcal{W}_X$. Indeed, let sequence $\{u_n\} \subset X$, $u_n \rightharpoonup u \in X$ and $\liminf_{n\to\infty}\Phi(u_n) \leq \Phi(u)$. Since h is continuous, one has

$$\liminf_{n\to\infty}\frac{\|u_n\|_{a,\alpha}^2}{2} \leq \frac{\|u\|_{a,\alpha}^2}{2}.$$

Thus, $\{u_n\}$ has a subsequence converging strongly to u. Therefore, $\Phi \in \mathcal{W}_X$. The functionals J and Ψ are two C^1 functionals with compact derivatives. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$.
In view of ($\mathcal{A}_1$), there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(t,u) \leq \epsilon|u|^2, \tag{3.2}$$

for every $t \in [0,T]$ and every u with $|u| \in [0,\tau_1) \cup (\tau_2,+\infty)$. Since f is L^1-Carathéodory, $F(t,u)$ is bounded on $t \in [0,T]$, $u \in \mathbb{R}$ with $|u| \in [\tau_1,\tau_2]$; we can choose $\eta > 0$ and $\nu > 2$ such that

$$F(t,u) \le \epsilon|u|^2 + \eta|u|^\nu,$$

for all $(t,u) \in [0,T] \times \mathbb{R}$. So, by (2.2), we have

$$J(u) \le Tk^2\epsilon\|u\|^2_{a,\alpha} + Tk^\nu\eta\|u\|^\nu_{a,\alpha}, \tag{3.3}$$

for all $u \in X$, with k given in (2.3). Hence, from (3.1) and (3.3) we have

$$\limsup_{|u|\to 0} \frac{J(u)}{\Phi(u)} \le \frac{2Tk^2\epsilon}{1-LTk^2}. \tag{3.4}$$

Moreover, by using (3.2), for each $u \in X \setminus \{0\}$, we obtain

$$\begin{aligned}
\frac{J(u)}{\Phi(u)} &= \frac{\int\limits_{|u|\le\tau_2} F(t,u(t))\mathrm{d}t}{\Phi(u)} + \frac{\int\limits_{|u|>\tau_2} F(t,u(t))\mathrm{d}t}{\Phi(u)} \\
&\le \frac{T\sup_{t\in[0,T],|u|\in[0,\tau_2]} F(t,u)}{\Phi(u)} + \frac{Tk^2\epsilon\|u\|^2_{a,\alpha}}{\Phi(u)} \\
&\le \frac{2T\sup_{t\in[0,T],|u|\in[0,\tau_2]} F(t,u)}{(1-LTk^2)\|u\|^2_{a,\alpha}} + \frac{2Tk^2\epsilon}{1-LTk^2}.
\end{aligned}$$

So, we get

$$\limsup_{\|u\|_{a,\alpha}\to\infty} \frac{J(u)}{\Phi(u)} \le \frac{2Tk^2\epsilon}{1-LTk^2}. \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\rho = \max\left\{0, \limsup_{\|u\|_{a,\alpha}\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to 0} \frac{J(u)}{\Phi(u)}\right\} \le \frac{2Tk^2\epsilon}{1-LTk^2}. \tag{3.6}$$

Assumption $(\mathcal{A}_2)$ in conjunction with (3.6) yields

$$\begin{aligned}
\sigma &= \sup_{u\in\Phi^{-1}(0,+\infty)} \frac{J(u)}{\Phi(u)} = \sup_{X\setminus\{0\}} \frac{J(u)}{\Phi(u)} \\
&\ge \frac{\int_0^T F(t,w(t))\mathrm{d}t}{\Phi(w(t))} = \frac{2\int_0^T F(t,w(t))\mathrm{d}t}{\|w\|^2_{a,\alpha} - 2\int_0^T H(w(t))\mathrm{d}t} > \frac{2Tk^2\epsilon}{1-LTk^2} \ge \rho.
\end{aligned}$$

Thus, all the hypotheses of Theorem 2.8 are satisfied. Clearly, $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = \frac{1}{\rho}$. Then, for each compact interval $[c,d] \subset (\lambda_1,\lambda_2)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the problem $(D_{\lambda,\mu})$ has at least three classical solutions whose norms in E^α are less than R. □

The another announced application of Theorem 1.1 reads as follows:

Theorem 3.2. *Let*

$$\max\left\{\limsup_{u\to 0}\frac{\max_{t\in[0,T]}F(t,u)}{|u|^2},\limsup_{|u|\to+\infty}\frac{\max_{t\in[0,T]}F(t,u)}{|u|^2}\right\}\le 0 \tag{3.7}$$

and

$$\sup_{u\in E^\alpha}\frac{\int_0^T F(t,u(t))\mathrm{d}t}{\|u\|_{a,\alpha}^2-2\int_0^T H(u(t))\mathrm{d}t}>0. \tag{3.8}$$

Then for each compact interval $[c,d]\subset(\lambda_1,+\infty)$ there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ there exists $\gamma>0$ such that, for each $\mu\in[0,\gamma]$, the problem $(D_{\lambda,\mu})$ has at least three classical solutions whose norms in E^α are less than R.

Proof. In view of (3.7), there exist an arbitrary $\epsilon>0$ and τ_1,τ_2 with $0<\tau_1<\tau_2$ such that

$$F(t,u)\le\epsilon|u|^2,$$

for every $t\in[0,T]$ and every u with $|u|\in[0,\tau_1)\cup(\tau_2,+\infty)$. Since f is L^1-Carathéodory, $F(t,u)$ is bounded on $t\in[0,T]$, $u\in\mathbb{R}$ with $|u|\in[\tau_1,\tau_2]$; we can choose $\eta>0$ and $\nu>2$ such that

$$F(t,u)\le\epsilon|u|^2+\eta|u|^\nu,$$

for all $(t,u)\in[0,T]\times\mathbb{R}$. So, by the same process in proof of Theorem 3.1 we have relations (3.4) and (3.5). Since ϵ is arbitrary, (3.4) and (3.5) gives

$$\max\left\{\limsup_{\|u\|_{a,\alpha}\to+\infty}\frac{J(u)}{\Phi(u)},\limsup_{u\to 0}\frac{J(u)}{\Phi(u)}\right\}\le 0.$$

Then, with the notation of Theorem 1.1, we have $\rho=0$. By (3.8), we also have $\sigma>0$. Thus, all the hypotheses of Theorem 1.1 are satisfied. Clearly, $\lambda_1=\frac{1}{\sigma}$. □

Remark 3.3. In Assumption $(\mathcal{A}_2)$ if we choose

$$w(t)=w^\star(t)=\begin{cases}\frac{\delta}{mT}t, & \text{if } t\in[0,mT),\\ \delta, & \text{if } t\in[mT,(1-m)T],\\ \frac{\delta}{mT}(T-t), & \text{if } t\in((1-m)T,T],\end{cases} \tag{3.9}$$

where $0 < m < \frac{1}{2}$ and $\delta > 0$. Clearly $w^\star(0) = w^\star(T) = 0$ and $w^\star \in L^2[0,T]$. A direct calculation shows that

$$
\begin{aligned}
&|{}_0^cD_t^\alpha w^\star(t)| \\
&= \frac{1}{\Gamma(1-\alpha)} \begin{cases} \frac{\delta}{mT}\frac{t^{1-\alpha}}{1-\alpha}, & \text{if } t \in [0, mT), \\ \frac{\delta}{mT}\frac{(mT)^{1-\alpha}}{1-\alpha}, & \text{if } t \in [mT, (1-m)T], \\ \frac{\delta}{mT}\frac{1}{1-\alpha}\Big((mT)^{1-\alpha} - (t-(1-m)T)^{1-\alpha}\Big), & \text{if } t \in ((1-m)T, T], \end{cases}
\end{aligned}
$$

so that

$$
\int_0^T |{}_0^cD_t^\alpha w^\star(t)|^2 \mathrm{d}t = \frac{(mT)^{1-2\alpha}}{\Gamma^2(1-\alpha)} \frac{2(m^{-1}-1)\alpha^2 + (9-7m^{-1})\alpha + 6m^{-1} - 8}{(1-\alpha)^2(2-\alpha)(3-2\alpha)}\delta^2.
$$

Thus, $w^\star \in E^\alpha$, and

$$
\begin{aligned}
\Phi(w^\star) &= \frac{1}{2}\|w^\star\|_{a,\alpha}^2 - \int_0^T H(w^\star(t))\mathrm{d}t \\
&= \frac{1}{2}\Big(A(\alpha,m)\delta^2 + \int_0^T a(t)|w^\star(t)|^2\mathrm{d}t\Big) - \int_0^T H(w^\star(t))\mathrm{d}t,
\end{aligned}
$$

where

$$
A(\alpha,m) := \frac{(mT)^{1-2\alpha}}{\Gamma^2(1-\alpha)} \frac{2(m^{-1}-1)\alpha^2 + (9-7m^{-1})\alpha + 6m^{-1} - 8}{(1-\alpha)^2(2-\alpha)(3-2\alpha)}.
$$

Then, Assumption $(\mathcal{A}_2)$ becomes to the following form:

$(\mathcal{A}_2')$ there exists a positive constant δ such that

$$
A(\alpha,m)\delta^2 + \int_0^T a(t)|w^\star(t)|^2\mathrm{d}t \neq \int_0^T H(w^\star(t))\mathrm{d}t
$$

and

$$
Tk^2\epsilon < \frac{\int_0^T F(t, w^\star(t))\mathrm{d}t}{A(\alpha,m)\delta^2 + \int_0^T a(t)|w^\star(t)|^2\mathrm{d}t - 2\int_0^T H(w^\star(t))\mathrm{d}t}.
$$

Now, we point out some results in which the function f has separated variables. To be precise, consider the following problem

$$\begin{aligned} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right) + a(t)u(t) &= \lambda\theta(t)f(u(t)) + h(u(t)), \quad t \neq t_j,\ \text{a.e. } t \in [0,T], \\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_j) &= \mu I_j(u(t_j)), \quad j = 1,\dots n, \\ u(0) &= u(T) = 0, \end{aligned} \tag{$D^{\theta}_{\lambda,\mu}$}$$

where $\theta : [0,T] \to \mathbb{R}$ is a non-zero function such that $\theta \in L^1([0,T])$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Set $f(t,x) = \theta(t)f(x)$ for every $(t,x) \in [0,T] \times \mathbb{R}$ and put $F(\xi) = \int_0^{\xi} f(x)\mathrm{d}x$ for every $\xi \in \mathbb{R}$. The following existence results are consequences of Theorem 3.1.

Theorem 3.4. *Assume that*

($\mathcal{A}_1'$) *there exists a constant $\epsilon > 0$ such that*

$$T \sup_{t\in[0,T]} \theta(t). \max\left\{\limsup_{u\to 0} \frac{F(u)}{|u|^2}, \limsup_{|u|\to\infty} \frac{F(u)}{|u|^2}\right\} < \epsilon,$$

($\mathcal{A}_2''$) *there exists a positive constant δ such that*

$$Tk^2\epsilon < \frac{\left(1 - LTk^2\right)\int_0^T \theta(t)F(w^\star(t))\mathrm{d}t}{A(\alpha,m)\delta^2 + \int_0^T a(t)|w^\star(t)|^2\mathrm{d}t - 2\int_0^T H(w^\star(t))\mathrm{d}t},$$

where $w^\star$ is given by (3.9).

Then, for each compact interval $[c,d] \subset (\lambda_3, \lambda_4)$ where λ_3 and λ_4 are the same as λ_1 and λ_2, but $\int_0^T F(t,u(t))dt$ replaced by $\int_0^T \theta(t)F(u(t))\mathrm{d}t$, respectively, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the problem $(D^{\theta}_{\lambda,\mu})$ has at least three classical solutions whose norms in E^{α} are less than R.

Remark 3.5. Theorem 1.1 immediately follows from Theorem 3.4.

Theorem 3.6. *Assume that there exists a positive constant δ such that*

$$A(\alpha,m)\delta^2 + \int_0^T a(t)|w^\star(t)|^2\mathrm{d}t - \int_0^T H(w^\star(t))\mathrm{d}t > 0 \text{ and } \int_0^T \theta(t)F(w^\star(t))\mathrm{d}t > 0 \tag{3.10}$$

where $w^\star$ is given by (3.9). *Moreover, suppose that*

$$\limsup_{u\to 0} \frac{f(u)}{|u|} = \limsup_{|u|\to\infty} \frac{f(u)}{|u|} = 0. \tag{3.11}$$

Then, for each compact interval $[c,d] \subset (\lambda_3,\infty)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the problem $(D^{\theta}_{\lambda,\mu})$ has at least three classical solutions whose norms in E^{α} are less than R.

Proof. We easily observe that from (3.11) the assumption $(\mathcal{A}_1')$ is satisfied for every $\epsilon > 0$. Moreover, using (3.10), by choosing $\epsilon > 0$ small enough one can drive the assumption $(\mathcal{A}_2'')$. Hence, the conclusion follows from Theorem 3.4. □

Now, we exhibit an example in which the hypotheses of Theorem 3.6 are satisfied.

Example 3.7. Let $\alpha = 0.8$, $T = 1$, $n = 2$, $t_1 = \frac{1}{3}$ and $t_2 = \frac{2}{3}$. Let $a(t) = 2 + \cos t$, $\theta(t) = \frac{2+t^2}{1+t^2}$ for all $t \in [0, 1]$. Moreover, let

$$f(x) = \begin{cases} x \sin x, & \text{if } x < 0, \\ \sin^2 x, & \text{if } x \geq 0 \end{cases}$$

and

$$h(x) = \frac{1}{2} \sin x \qquad \text{for all } x \in \mathbb{R}.$$

Accordingly $k = 1.1089$ and $L = \frac{1}{2}$. Thus we have $1 > LTk^2 = 0.6148$. Let $I_1(x) = e^x$ and $I_2(x) = e^{-x}$ for all $x \in \mathbb{R}$. Now by choosing $m = \frac{1}{3}$ and $\delta = 1$ we have

$$w^\star(t) = \begin{cases} 3t, & \text{if } t \in [0, \frac{1}{3}), \\ 1, & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ 3(1-t), & \text{if } t \in (\frac{2}{3}, 1]. \end{cases}$$

Thus $w^\star(t) \geq 0$ for all $t \in [0, 1]$, $A(\alpha, m) = A(0.8, \frac{1}{3}) = \frac{8110\sqrt[5]{27}}{21\Gamma^2(0.2)} > 0$ and $\int_0^1 H(w^\star(t))\mathrm{d}t = -0.05$. So,

$$A(\alpha, m) + \int_0^1 a(t)|w^\star(t)|^2 \mathrm{d}t - \int_0^1 H(w^\star(t))\mathrm{d}t > 0.$$

Also we have

$$\begin{aligned} \int_0^1 \theta(t)F(w^\star(t))\mathrm{d}t &= \int_0^1 \frac{2+t^2}{1+t^2} F(w^\star(t))\mathrm{d}t \geq \int_0^1 F(w^\star(t))\mathrm{d}t \\ &= \int_0^1 \int_0^{w^\star(t)} \sin^2 x \mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_0^1 \Big(w^\star(t) - \frac{1}{2}\sin(2w^\star(t))\Big)\mathrm{d}t > 0 \end{aligned}$$

and

$$\lim_{u \to 0} \frac{f(u)}{|u|} = \lim_{u \to +\infty} \frac{f(u)}{|u|} = 0.$$

Hence, by applying Theorem 3.6, for each compact interval $[c, d] \subset (0, \infty)$, there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$, the problem

$$
\begin{aligned}
{}_tD_1^{0.8}\left({}_0^cD_t^{0.8}u(t)\right) + (2+\sin t)u(t) = \lambda\frac{2+t^2}{1+t^2}f(u(t)) + \frac{1}{2}\sin(u(t)), \quad t \neq \frac{1}{3},\, t \neq \frac{2}{3} \\
\text{a.e. } t \in [0,1], \\
\Delta\left({}_tD_1^{-0.2}\left({}_0^cD_t^{0.8}u\right)\right)\left(\frac{1}{3}\right) = \mu e^{u(\frac{1}{3})}, \quad \Delta\left({}_tD_1^{-0.2}\left({}_0^cD_t^{0.8}u\right)\right)\left(\frac{2}{3}\right) = \mu e^{-u(\frac{2}{3})}, \\
u(0) = u(1) = 0
\end{aligned}
$$

has at least three classical solutions whose norms in $E^{0.8}$ are less than R.

Finally, by choosing $I_j(x) = 0$ for every $x \in \mathbb{R}$, $j = 1, \ldots, n$, $a(t) = 1$ for all $t \in [0, T]$ and $h(x) = 0$ for all $x \in \mathbb{R}$ we have the following existence result as a consequence of Theorem 3.4:

Theorem 3.8. *Assume that $(\mathcal{A}_1')$ holds and*

$(\mathcal{A}_2''')$ *there exists a positive constant δ such that $A(\alpha, m) + \left(1 - \frac{4m}{3}\right)T \neq 0$, and*

$$
Tk^2\epsilon < \frac{\int_0^T \theta(t)F(w^\star(t))\mathrm{d}t}{\left[A(m,\alpha) + \left(1 - \frac{4m}{3}\right)T\right]\delta^2},
$$

where $w^\star$ is given by (3.9).

Then, for each compact interval $[c, d] \subset (\lambda_5, \lambda_6)$, where

$$
\lambda_5 = \inf\left\{\frac{\|u\|_\alpha^2}{2\int_0^T \theta(t)F(u(t))\mathrm{d}t} : u \in E^\alpha, \int_0^T \theta(t)F(u(t))\mathrm{d}t > 0\right\}
$$

and

$$
\lambda_6 = \left(\max\left\{0, \limsup_{\|u\|_\alpha\to\infty}\frac{2\int_0^T \theta(t)F(u(t))\mathrm{d}t}{\|u\|_\alpha^2}, \limsup_{|u|\to 0}\frac{2\int_0^T \theta(t)F(u(t))\mathrm{d}t}{\|u\|_\alpha^2}\right\}\right)^{-1},
$$

there exists $R > 0$ with the following property: for every $\lambda \in [c, d] \subset (\lambda_5, \lambda_6)$ there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$, the problem

$$
\begin{aligned}
{}_tD_T^\alpha\left({}_0^cD_t^\alpha u(t)\right) + u(t) = \lambda\theta(t)f(u(t)), \quad \text{a.e. } t \in [0,T], \\
u(0) = u(T) = 0
\end{aligned}
$$

has at least three classical solutions whose norms in E^α are less than R.

Remark 3.9. It is worth to mention that in the present paper, unlike the papers [8,47] in which the existence of multiple solutions for impulsive fractional boundary-value problems under continuity condition and a growth condition on impulses have been discussed, the only condition on the impulses is the continuity condition.

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Shapour Heidarkhani
s.heidarkhani@razi.ac.ir

Razi University
Faculty of Sciences
Department of Mathematics
67149 Kermanshah, Iran

Massimiliano Ferrara
massimiliano.ferrara@unirc.it

University Mediterranea of Reggio Calabria
Department of Law and Economics
Via dei Bianchi, 2 - 89131 Reggio Calabria, Italy

Giuseppe Caristi
gcaristi@unime.it

University of Messina
Department of Economics
Via dei Verdi, 75, Messina, Italy

Amjad Salari
amjads45@yahoo.com

Razi University
Faculty of Sciences
Department of Mathematics
67149 Kermanshah, Iran

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