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# UNCERTAINTY ANALYSIS FOR EFFICIENT FUEL ALLOCATION USING INFO-GAP THEORY

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Theory decision under uncertainty applies different methods to predict future events, physical measurements, which are already unknown. The paper shows one of the decision-making methods in conditions of severe uncertainty in application to a problem. Some aspects of the robustness and opportuneness functions of Info-Gap Theory are presented and used to evaluate the example of efficient fuel allocation. A general outline of decision analysis is presented, including its orientation and models.

Keywords: uncertainty, info-gap model, robustness function, opportuneness function

## 1. Introduction

There are different definitions of uncertainty but the most common is that it is a situation where the current state of knowledge is that the order or nature of things is unknown, the consequences, extent, or magnitude of circumstances, conditions, or events are unpredictable, and credible probabilities to possible outcomes cannot be assigned [6]. In decision-making theory there are many problems with limited information or conjecture. Along with the development of science and technology, new methods and tools dealing with uncertainty appear. Among these methods and theories: fuzzy mathematics by L.A. Zadeh (1960s) [11], interval arithmetic by R.E. Moore (1960s) [8], grey systems theory by J. Deng (1980s) [7], rough set theory by Z. Pawlak (1980s) [9], uncertainty mathematics by H. Bandemer (2005)

[2] can be mentioned. Each of these methods has its own interpretation and modeling of uncertainty and they are used to different problems. It is difficult to compare these methods, because there is no standardized method of verification. Hence some scientists introduce and suggest still new approaches and methods. One of them is Info-Gap Theory developed by Y. Ben-Haim (2001) [3, 4]. Info-Gap Theory is used to non-probabilistic problems where there is no information on probabilities and there are not any probability distributions. It concerns seeking for and determining optimal or possibly good problem solutions taking into account robustness and opportuneness models, which are in detail described in next section. The problem of efficient fuel allocation solved in this paper is a good example to show an assessment of this method applicability.

## 2. Description of Info-Gap Theory

Info-Gap Decision Theory is the quantitative assessments assist the decision maker to evaluate options and strategies in light of the analysis of uncertainties. The method is described in many papers, books and used in a range of applications [12]. The name of the method comes from the word info-gap. The definition of info-gap says that it is the disparity between what is known, and what needs to be known in order to make a comprehensive and reliable decision. The method consists of a main info-gap model of uncertainty. It is an unbounded family of nested sets that share a common structure. [3]. The structure of the sets in an info-gap model depends on information about the uncertainty. In general terms, the structure of an info-gap model of uncertainty is chosen to define the smallest or strictest family of sets whose elements are consistent with the prior information. This model measures how distant other values of the parameter are from the estimate. It can be represented as follows [4]:

$$U(\alpha,\tilde{u}) = \{u : |u - \tilde{u}| \le \alpha\}, \alpha \ge 0 \qquad (1)$$

where $\tilde{u}$ denotes the best estimate of an uncertain function *u*, while the fractional error from this estimate, $\alpha$, is unknown. Info-Gap Theory assumes that $\tilde{u}$ represents a poor guess at the true values of the parameters. At any level of uncertainty $\alpha$, the set $U(\alpha,\tilde{u})$ contains all functions *u* whose fractional deviation from $\tilde{u}$ is no greater than $\alpha$ [1]. The consequences of uncertainty can be both catastrophic failure or windfall success. Hence Ben-Haim introduced to his method two immunity functions: robustness and opportuneness functions. The first one assesses the immunity to failure, while the second one assesses the immunity of windfall. These functions express the basic knowledge and expectations, which a decision maker brings to the decision problem. Using robustness and opportuneness functions the preferences on the options can be formulated.

The robustness function can be expressed as the maximum value of the uncertainty parameter $\alpha$ of an info-gap model [1, 3]:

$$\widehat{\alpha}(q) = \max\{\alpha : \text{min requirements are always satisfied}\}$$

$$\widehat{\alpha}(q) = \max_{\alpha \geq 0}\{\alpha : \forall u \in U(\alpha, \widetilde{u})\} \tag{2}$$

where $q$ denotes a vector of decision variables such as choice of a model or its parameters. The robustness function involves maximization of the uncertainty, or the range of variation in a variable, parameter or model, at which decision $q$ would satisfy the performance at a tolerable level [3]. The opportuneness function can be expressed as the lowest horizon of uncertainty that is necessary for better than anticipated outcomes to be possible:

$$\widehat{\beta}(q) = \min\{\alpha : \text{sweeping success is possible}\}$$

$$\widehat{\beta}(q) = \min_{\alpha \geq 0}\{\alpha : \forall u \in U(\alpha, \widetilde{u})\} \tag{3}$$

where $q$ denotes a vector of decision variables such as choice of a model or its parameters. A small value of $\widehat{\beta}(q)$ reflects the opportune situation that great reward is possible even in the presence of little ambient uncertainty. If we use robustness and opportuneness function simultaneously and add decision maker's expectations we can get possibly good solution, what is presented in next section in the numerical example.

## 3. Numerical example

The problem concerns the efficient fuel allocation for a rally vehicle. The car takes part in a multiday race and during one day it should be able to travel at least a distance $d_c$. Otherwise, the driver will be disqualified from the race. If the driver travels a longer distance than $d_{c,}$ the total distance to the end will be shorter and it will increase his chance to win. To ensure passing the required minimum route, the driver must take the appropriate amount of fuel $q$. The distance, which a vehicle can travel with quantity $q$ of fuel takes form [3]:

$$d(q,c) = \frac{q}{1+cq^2} \tag{4}$$

where $c$ is a coefficient of road difficulty.

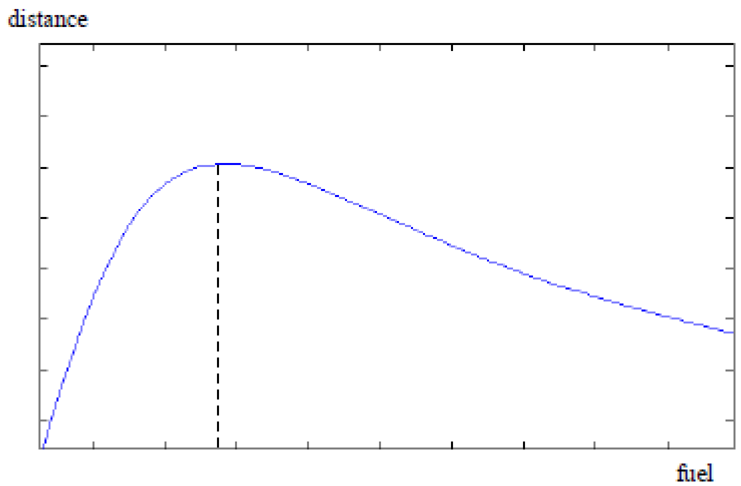


**Figure 1.** A qualitative distance depending on a fuel

It seems that the greater amount of fuel $q$ increases the chance of overcoming the minimum distance $d_c$. However, if the car takes less fuel, it is lighter (smaller total weight), can easier overcome hills and has smaller fuel consumption. For this reason, too much fuel $q$ does not guarantee to pass the minimum route $d_c$. The distance is not only dependent on a fuel but on a coefficient of road difficulty as well. The value of this coefficient is unknown and depends on many factors as weather, terrain, car's weight, etc. The only knowledge is that the more difficult road, the greater value of coefficient $c$.

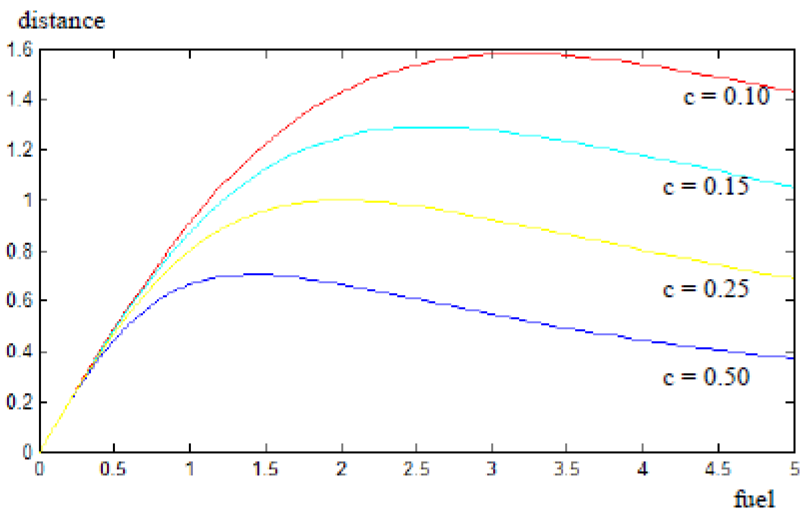


**Figure 2.** The distance $d(q, c)$ with different values of coefficient $c$

Fig. 2 shows that a vehicle can sometimes travel 0.8 units of distance or 1.4 units taking 2 units of fuel. It depends on the coefficient *c*. The left and right border of interval value of *c* is uncertain, so it is difficult to say in which interval [$c_{min}$, $c_{max}$] the value of *c* is contained. The lack of knowledge about value of *c* is an info-gap in this problem. The question is how to estimate this coefficient to travel the appropriate distance.

It is known that the driver took part in previous races twice and he defeated considered road once with good weather conditions having $q = 2$ and traveled distance $d = 1.6667$ and next time with bad weather conditions he took $q = 5$ and traveled distance $d = 0.6896$. On the basis of this information, *c* can be calculated by the following formula:

$$c = \frac{q-d}{dq^2} \tag{5}$$

Travel 1: $q_1 = 2$ AND $d_1 = 1.6667$ THEN $c_1 = 0.05$

Travel 2: $q_2 = 5$ AND $d_2 = 0.6896$ THEN $c_2 = 0.25$

Information $c_1$ and $c_2$ are the only information on the degree of coefficient *c* which we have. On this basis the estimated degree of coefficient *c* is calculated.

$$\tilde{c} = \frac{c_1 + c_2}{2} = 0.15 \tag{6}$$

The estimated value of $\tilde{c}$ is highly uncertain and based only on two experiment trips. The info-gap model of this problem is developed.

$$U(\alpha, \tilde{c}) = \{c : |c - \tilde{c}| \le \alpha\tilde{c}\}, \alpha \ge 0 \tag{7}$$

where $\alpha$ - is a coefficient of uncertainty of the *c* value. Since $\tilde{c}$ = 0.15, a more concrete form of info-gap model of uncertainty can be presented:

$$U(\alpha, \tilde{c}) = \{c : |c - 0.15| \le 0.15\alpha\}, \alpha \ge 0 \tag{8}$$

Interpretation of the model:

Value $|c - \tilde{c}|$ is an absolute error of the estimate of actual *c* coefficient value on the route during the journey.

Record $|c - \tilde{c}| \le \alpha\tilde{c}$ means that it is easier to make a big estimated error $\Delta\tilde{c}$ when the value of $\tilde{c}$ is greater. The minimal distance which has to be traveled is $d_c = 1.0$. The robustness function $\hat{\alpha}$ takes form:

$$\hat{\alpha}(c, d_c) = \max\{\alpha : \min_{c \in U(\alpha, \tilde{c})} d(q_i, c) \ge d_c\} \tag{9}$$

We want to determine the amount of fuel $q_i$, which enables car to travel at least a critical minimum distance $d_c (d \ge d_c)$. Robustness $\hat{\alpha}$ determines the maximum allowable error of estimate $\tilde{c}$. If the real value of the uncertainty $\alpha$ does not exceed it, the amount of fuel $q_i$ gives the possibility to travel at least $d_c$.

Robustness $\hat{\alpha}(q_i): c > \tilde{c}$ means that so difficult road and conditions during race were not suspected.
On the basis of (7) we get:

$$c - \tilde{c} \le \alpha\tilde{c}$$
$$c \le \tilde{c} + \alpha\tilde{c}$$
$$c \le (1+\alpha)\tilde{c}$$

$$\text{IF } c \le (1+\alpha)\tilde{c} \text{ THEN } d \ge \frac{q}{1+(1+\alpha)\tilde{c}q^2} \quad (10)$$

If the real uncertainty of $\alpha$ equals the uncertainty limit $\hat{\alpha}$ defined as the robustness, then the distance *d* will be equal $d_c$.

$$\text{IF } (\alpha = \hat{\alpha}) \text{ THEN } d = d_c = \frac{q}{1+(1+\alpha)\tilde{c}q^2} \quad (11)$$

The robustness $\hat{\alpha}$ where the vehicle travels at least distance $d_c$ takes form:

$$\hat{\alpha} = \frac{q - d_c}{\tilde{c}d_c q^2} - 1 \quad (12)$$

Let consider five options of taken fuel (only full fuel tanks can be taken):
$q_1 = 1$ and $q_2 = 2$ and $q_3 = 3$ and $q_4 = 4$ and $q_5 = 5$ [units].
For $q_1 = 1$ ($d_c = 1$, $\tilde{c} = 0.15$):

$$\hat{\alpha}_1 = \frac{q_1 - d_c}{\tilde{c}d_c q_1^2} - 1 = -1$$

Negative value of robustness $\hat{\alpha} < 0$ means that there is no robustness according Info-Gap Theory. The lowest value of fuel *q* where the robustness is nonnegative equals *q* = 1.5 units.
The maximum robustness has a decision where *q* = 2. To understand the meaning of $\alpha$, we can assume that if $\alpha = \hat{\alpha}$ then $c = \hat{c}$:

$$\hat{c}(\hat{\alpha}) = \hat{c} + \hat{\alpha}\tilde{c}$$
$$\Delta\hat{c}(\hat{\alpha}) = \hat{c} - \tilde{c} = \hat{\alpha}\tilde{c} \quad (13)$$

**Table 1.** The values of error $\Delta\hat{c}(\hat{\alpha})$ of $\tilde{c}$ for each robustness $\hat{\alpha}$

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $q_i$ | 1.5 | 2 | 3 | 4 | 5 |
| $\hat{\alpha}_i$ | 0 | 0.667 | 0.481 | 0.250 | 0.067 |
| $\Delta\hat{c}(\hat{\alpha}_i)$ | 0 | 0.100 | 0.072 | 0.038 | 0.010 |

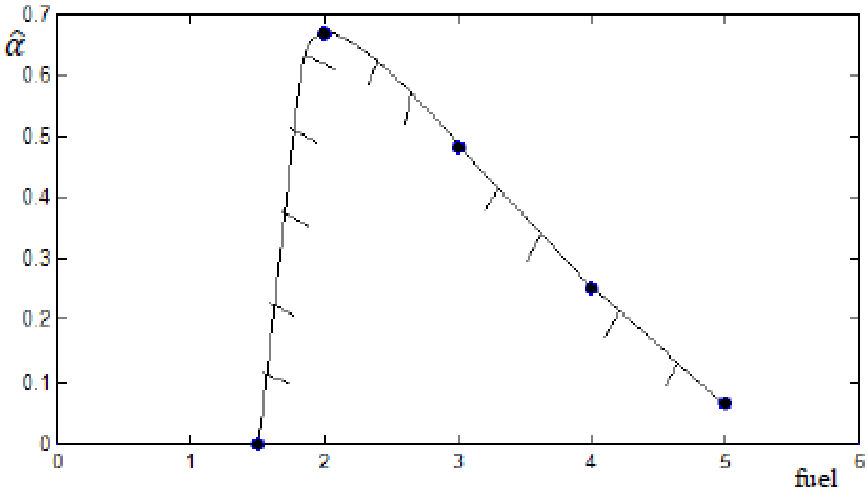


**Figure 3.** The robustness $\hat{\alpha}_i$ for $i$ = 1-5

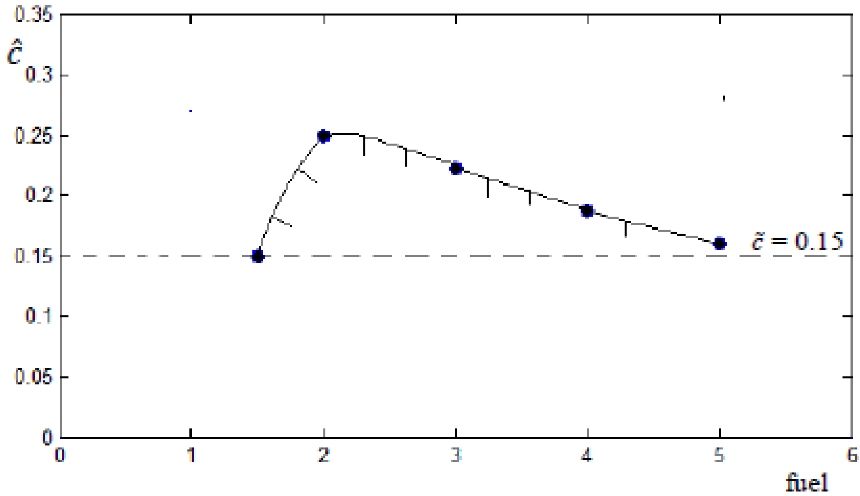


**Figure 4.** The relationship between $\hat{c}$ and fuel for robustness function

Fig. 4. shows that if $q$ = 2, the error of estimate value $\tilde{c}$ will be 0.100 what represents 66.7% of $\tilde{c} = 0.15$ and the distance will be $d \geq d_c$. Other decisions about taking a different amount of fuel do not have such a great robustness, so they are more risky.

As it was said in section two, uncertain variations may be either adverse or favorable. The uncertainty in this problem will be adverse if the real value of $c$ will be higher than $\tilde{c}$. Otherwise, it will be a windfall gain, because a vehicle will travel further than expected and it is called opportuneness situation.

$$c \le \tilde{c}$$

$$\tilde{c}(1-\alpha) \le \tilde{c}$$

The driver will be satisfied if he manages to overcome the distance $d_w = 1.4$, so the sweeping success will be if he travels further than $d_w$.

$$d \ge d_w$$

The opportuneness $\hat{\beta}$ is the least level of uncertainty $\alpha$ which must be tolerated in order to travel a distance equals $d_w$:

$$d = d_w = \frac{q}{1+\tilde{c}(1-\hat{\beta})q^2} \tag{14}$$

The opportuneness $\hat{\beta}$ takes form:

$$\hat{\beta}(q) = 1 - \frac{q - d_w}{\tilde{c}\, d_w q^2} \tag{15}$$

$$\text{IF } (\alpha \ge \hat{\beta}) \text{ THEN } (d \ge d_w)$$

Let consider the opportuneness $\hat{\beta}$ for each of five options of fuel $q$ and calculate the error of estimation $\tilde{c}\hat{\beta} = \tilde{c} - \hat{c} = \hat{\Delta}\tilde{c}$ .

**Table 2.** The values of error $\hat{\Delta}\tilde{c}$ for each opportuneness $\hat{\beta}$

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $q_i$ | 1.4 | 2 | 3 | 4 | 5 |
| $\hat{\beta}_i$ | 1 | 0.400 | 0.154 | 0.226 | 0.314 |
| $\hat{\Delta}\tilde{c}$ | 0 | 0.100 | 0.072 | 0.038 | 0.010 |

As shown the greatest opportunity to travel the distance $d \ge d_c$ gives a decision about taking 3 units of fuel because it has the lowest value of opportuneness $\hat{\beta}$ . In this case, if the real value of $c$ is 15.4% lower than $\tilde{c} = 0.15$, travelling distance $d_w = 1.4$ will be guaranteed. These all results are presented in Fig. 5 and Fig. 6.

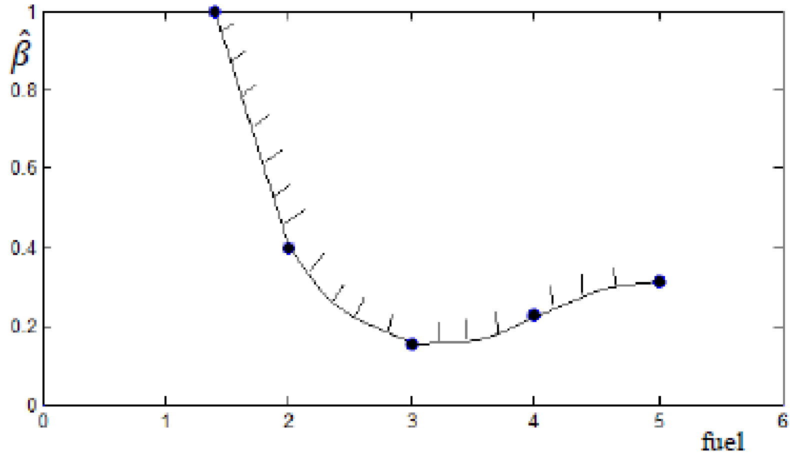


**Figure 5.** The opportuneness $\hat{\beta}_i$ for $i$ = 1–5

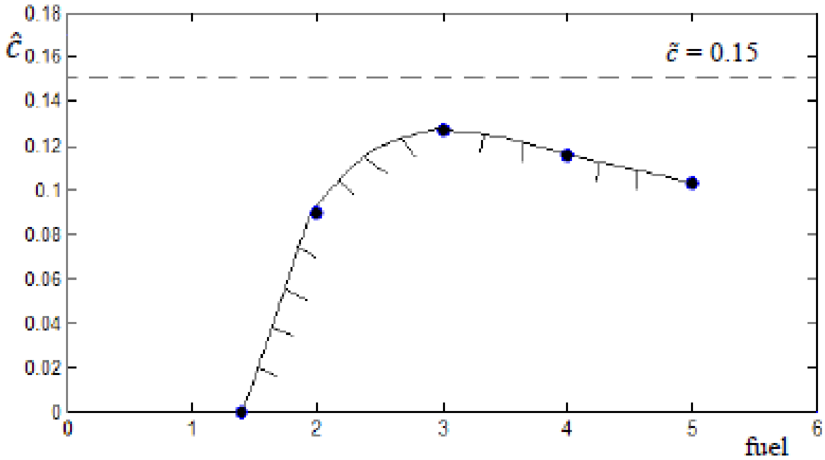


**Figure 6.** The relationship between $\hat{c}$ and fuel for opportuneness function

Fig.6. shows that the best chance of passing a long distance gives decision $q = 3$ units of fuel. However, taking into account the calculation of robustness $\hat{\alpha}$, it shows that better decision is $q = 2$ units of fuel. The relationship $\hat{\alpha}(q)$ and $\hat{\beta}(q)$ is presented in Fig. 7.

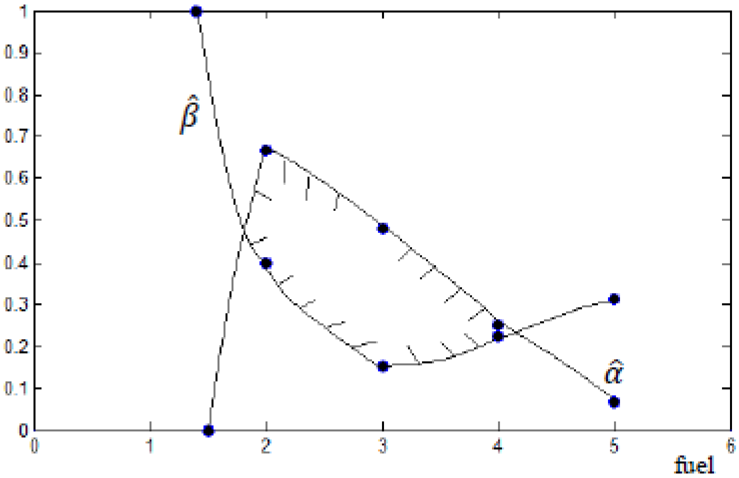


**Figure 7.** The robustness $\hat{\alpha}(q)$ and opportuneness $\hat{\beta}(q)$ curves

Fig. 7 shows that two decisions $q$ = 2 and $q$ = 3 compete with each other. Decision $q$ = 2 has a higher robustness $\hat{\alpha}_2$ but also a higher opportuneness $\hat{\beta}_2$. The question is which decision is better. The answer depends on decision maker's risk-taking propensity [5]. To determine the optimal decision, a criterion including risk-taking propensity should be developed. The proposed criterion has the following equation:

$K(q_i)$ = robustness (1 – risk-taking propensity) – opportuneness (risk-taking propensity)

$$K(q_i) = (1-r)\hat{\alpha}(q_i) - r\hat{\beta}(q_i) \quad (16)$$

Assume that risk-taking propensity in this problem is low and it is $r$ = 0.3, where $r \in [0; 1]$:

$$K(q_i) = 0.7\hat{\alpha}(q_i) - 0.3\hat{\beta}(q_i)$$

It means that a driver, who is a decision maker, prefers to have guarantee to travel the minimum required distance than to travel longer than satisfied distance $d_w$. With these assumptions, we can calculate the criterion for decisions $q_2$ = 2 and $q_3$ = 3:

$$K(2) = 0.7 \cdot 0.667 - 0.3 \cdot 0.400 = 0.052$$

$$K(3) = 0.7 \cdot 0.400 - 0.3 \cdot 0.153 = 0.043$$

As a result we get that with a low risk-taking propensity it is better to take $q$ = 2 units of fuel. If the risk-taking propensity would be higher, for example $r$ = 0.6, it would be better to take $q$ = 3 units of fuel and the results would be as following:

$$K(2) = 0.4 \cdot 0.667 - 0.6 \cdot 0.400 = 0.027$$
$$K(3) = 0.4 \cdot 0.400 - 0.6 \cdot 0.153 = 0.068$$

It shows how important risk-taking propensity of decision maker is.

## 4. Conclusions

Decision-making is a field used in all science areas. People want to understand and manage the gap between what they know and what they could know in order to make an appropriate decision. Info-Gap Theory is a method for supporting problems with severe uncertainty. It solves decision making problems without any probability distributions of uncertain variables or membership functions, which most of all classical decision making methods require. The paper presents the general outline of this method and its models. The numerical example of efficient fuel allocation with one uncertain variable is analyzed. The solved problem shows how to use robustness and opportuneness functions simultaneously and how important is decision maker's risk-taking propensity to make a final decision.

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