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Random fuzzy continuous-time Markov chains

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Abstract

Continuous-time Markov chains is an important subclass in stochastic processes, which have facilitated many applications in business decisions, investment risk analysis, insurance policy making and reliability modeling. It should be fully aware that the existing continuous-time Markov chains theory is merely an ideology under which the random uncertainty governs the phenomena. However, the real world phenomena are often revealing the randomness and vagueness co-existence reality and thus the probabilistic continuous-time Markov chains modeling practices may be not adequate. In this paper, we define the random fuzzy continuous-time Markov chains, explore the related average chance distributions, and propose a scheme for the parameter estimation and a simulation scheme as well. It is expecting that a foundational work can be established for reliability modeling and risk analysis, particularly, repairable system modeling.

1. Introduction

It should be fully aware that vagueness is an intrinsic feature in today's diversified business environments, just as Carvalho and Machado [2] commented, "In a global market, companies must deal with a high rate of changes in business environment. … The parameters, variables and restrictions of the production system are inherently vagueness." Therefore the co-existence of random uncertainty and fuzzy uncertainty is inevitable reality of safety and reliability analysis and modelling.

It is obvious that probabilistic modeling is only a good approximation to real world problem when random uncertainty governs the phenomenon. Philosophically, if fuzziness and randomness both appear then probabilistic modeling may be questionable. Therefore, it is logical to develop appropriate models for modeling fuzziness and randomness co-existence.

Markov processes have been applied to large and complex system modeling and analysis in reliability literature, say, recent work of Kolowrocki, [14], [15], Love et al [18], Soszynska [21], and Tamura [22], etc.

It is also noticed that in recent years researchers on the repairable system modeling, particularly, in Asian reliability communities, proposed repair impact scenario models, which are assumed that the repair impacts to a repairable system may be classified into several states: no improvement, minor improvement, medium improvement, and major improvement, and thus utilize Kijima's age models [12] to estimate those repair effects on the system repair states for optimal maintenance policy decision making, see [3], [4], [13], [18], [19], [20], and [26]. However, less attention has been paid to the repair effect estimation, except a few authors, Guo and Love [6], [7], Lim and Lie [13], Yun et al [23], and etc.

In this paper, we will give a systematic treatment for the random fuzzy continuous-time Markov chains only in the mathematical sense (building models based on postulates and definitions) but also in the statistical sense (estimation and hypothesis testing based on sample data).

2. Probabilistic continuous-time Markov chains

Grimmett and Strizaker [5] and also Guo [9] describe continuous-time Markov chains by focusing the stochastic semigroup and the rate matrix.

Let $X=\{X_t, t\geq 0\}$ be a Markov chain with state space $\mathbb{S}=\{0,1,2,\cdots,N-1\}$. Further, let

$$p_{ij}(s,t)=\Pr\{X_t=j \mid X_s=i\} \tag{1}$$

be the transition probabilities. For the stationary Markov chain

$$p_{ij}(0,t-s)=p_{ij}(s,t),\ \forall s<t \tag{2}$$

Definition 1: (Grimmett and Strizaker [5]) A stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t, t\geq 0\}$, with $\mathrm{P}_t=\left(p_{ij}(t)\right)_{N\times N}$ satisfies the following properties:

(a) $\mathrm{P}_0=\mathrm{I}$, an $N\times N$ identity matrix;

(b) for $\forall t$, $0\leq p_{ij}(t)\leq 1$, $\sum_j p_{ij}(t)=1$;

(c) The Chapman-Kolmogorov equations, for any $s,t>0$, $\mathrm{P}_{t+s}=\mathrm{P}_t\mathrm{P}_s$.

A stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t, t\geq 0\}$ is standard if $\lim_{t\downarrow 0}\mathrm{P}_t=\mathrm{I}$.

The characterization of a stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t, t\geq 0\}$ can be stated as a theorem.

Theorem 1: For a standard stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t, t\geq 0\}$, the limit

$$\lim_{h\downarrow 0}\frac{p_{ii}(h)-1}{h}=-q_i\left(=q_{ii}\right) \tag{3}$$

exists (maybe $-\infty$), while the limit

$$\lim_{h\downarrow 0}\frac{p_{ij}(h)}{h}=q_{ij} \tag{4}$$

exists and is finite.

Guo [9] details the proof *Theorem 1*.

Definition 2: The matrix Q

$$\mathrm{Q}=\begin{bmatrix} -q_0 & q_{01} & \cdots & q_{0,N-1} \\ q_{10} & -q_{11} & \cdots & q_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N-1,1} & q_{N-1,2} & \cdots & -q_{N-1,N-1} \end{bmatrix} \tag{5}$$

where

$$\lim_{h\downarrow 0}\frac{p_{ij}(h)-\delta_{ij}}{h}=q_{ij} \tag{6}$$

with $\delta_{ij}=1,\ i=j$, *0* otherwise.

Lemma 1. In a matrix Q,

$$q_i=\sum_{j=1, j\neq i}^{N-1} q_{ij},\ i,j=1,2,\cdots,N-1 \tag{7}$$

The rate matrix Q characterizes the movements of the continuous-time Markov chain $X=\{X_t, t\geq 0\}$. The following theorem reveals such a fundamental fact.

Theorem 2: If the process $X=\{X_t, t\geq 0\}$ is currently holds at state i, it holds in state i during an exponentially distributed time with parameter q_i, independently of how the process reached state i and how long it gets there. Furthermore, The process $X=\{X_t, t\geq 0\}$ leaves state i, and moves to state j with probability q_{ij}/q_i $(i\neq j)$.

Theorem 3. A standard stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t, t\geq 0\}$ satisfies Kolmogorov equations:

$$\begin{aligned} \frac{d}{dt}\mathrm{P}_t &= \mathrm{P}_t\mathrm{Q} \text{ (Forward)} \\ \frac{d}{dt}\mathrm{P}_t &= \mathrm{Q}\mathrm{P}_t \text{ (Backward)} \end{aligned} \tag{8}$$

Corollary 1. A standard stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t, t\geq 0\}$ satisfies

$$\mathrm{P}_t=e^{\mathrm{Q}t} \tag{9}$$

where matrix

$$e^{\mathrm{Q}t}=\sum_{i=0}^{\infty}\frac{1}{i!}(\mathrm{Q}t)^i \tag{10}$$

It is well-established fact that every entry of P_t, $p_{ij}(t)$ can be expressed by a linear combination of

$e^{\rho_l t}$ with appropriate coefficient $c(l)$, where ρ_l is the l^{th} eigenvalue of Q or an appropriate minor matrix of Q, i.e.,

$$p_{ij}(t)=\sum_{l=0}^{N-1}c(l)e^{\rho_l t} \tag{11}$$

Example 1: Two-state continuous-time Markov chain. Let the rate matrix

$$\mathrm{Q}=\begin{bmatrix}-\nu & \nu\\ \lambda & -\lambda\end{bmatrix} \tag{12}$$

The eigenvalues are $(\rho_1,\rho_2)=(0,-(\nu+\lambda))$, thus

$$\mathrm{P}_t=\begin{bmatrix}\frac{\lambda}{\lambda+\nu}+\frac{\nu}{\lambda+\nu}e^{-(\lambda+\nu)t} & \frac{\nu}{\lambda+\nu}-\frac{\nu}{\lambda+\nu}e^{-(\lambda+\nu)t}\\ \frac{\lambda}{\lambda+\nu}-\frac{\lambda}{\lambda+\nu}e^{-(\lambda+\nu)t} & \frac{\nu}{\lambda+\nu}+\frac{\lambda}{\lambda+\nu}e^{-(\lambda+\nu)t}\end{bmatrix} \tag{13}$$

which confirms the formality of Equation (11).

3. Foundation of random fuzzy processes

Without a solid understanding of the intrinsic feature of random fuzzy processes, there is no base for exploring the modelling of random fuzzy continuous-time Markov chains. Liu's [16], [17] hybrid variable theory established on the axiomatic credibility measure and probability measure foundations provides the mathematical foundation.

Guo, Nyirenda, and Guo [11] give a systematic review on random fuzzy variable theory. In order to shorten the current paper, we only keep necessary contents for notational clarity. For details, please see Section 2, in [10] or directly Liu's books [16], [17].

First let us review the credibilistic fuzzy variable theory. Let Θ be a nonempty set, and $\mathfrak{P}(\Theta)$ the power set on Θ.

Definition 3: Any set function $\mathrm{Cr}:\mathfrak{P}(\Theta)\to[0,1]$ satisfies Liu's four Axioms [12], [13] is called a credibility measure. The triple $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ is called the credibility measure space.

Definition 4: A fuzzy variable ξ is a measurable mapping, i.e., $\xi:(\Theta,\mathfrak{P}(\Theta))\to(\mathbb{R},\mathfrak{B}(\mathbb{R}))$.

A fuzzy variable is not a fuzzy set in the sense of Zadeh's fuzzy theory [20], [21], in which a fuzzy set is defined by a membership function.

Definition 5: (Liu [16, 17]) The credibility distribution $\Lambda:\mathbb{R}\to[0,1]$ of a fuzzy variable ξ on $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ is

$$\Lambda(x)=\mathrm{Cr}\{\theta\in\Theta \,|\, \xi(\theta)\le x\} \tag{14}$$

Liu [13], [14] defines a random fuzzy variable as a mapping from the credibility space $(\Theta,2^{\Theta},\mathrm{Cr})$ to a set of random variables.

Definition 6: A random fuzzy variable, denoted as $\xi=\{X_{\beta(\theta)},\theta\in\Theta\}$, is a collection of random variables X_β defined on the common probability space $(\Omega,\mathfrak{A},\mathrm{Pr})$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta,2^{\Theta},\mathrm{Cr})$.

Definition 7: (Liu [16], [17]) Let ξ be a random fuzzy variable, then the average chance measure denoted by $\mathrm{ch}\{\cdot\}$, of a random fuzzy event $\{\xi\le x\}$, is

$$\mathrm{ch}\{\xi\le x\}=\int_0^1 \mathrm{Cr}\{\theta\in\Theta|\mathrm{Pr}\{\xi(\theta)\le x\}\ge\alpha\}\mathrm{d}\alpha \tag{15}$$

Then function $\Psi(\cdot)$ is called as average chance distribution if and only if

$$\Psi(x)=\mathrm{ch}\{\xi\le x\} \tag{16}$$

Definition 8: A random fuzzy process is a family of random fuzzy variables defined on the common Product measure space $(\Theta,2^{\Theta},\mathrm{Cr})\times(\Omega,\mathfrak{A},\mathrm{Pr})$, denoted by $\xi=\{\xi_t,t\in\mathbb{T}\}$, where $\mathbb{T}$ is called as the index set.

Theorem 4: Let ζ be a fuzzy variable defined on the credibility space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ and τ be a random variable defined on the probability space $(\Omega,\mathfrak{A}(\Omega),P)$, then

(1) Let $\oplus$ be an arithmetic operator, which can be "$+$", "$-$", "$\times$" or "$\div$" operation, such that $\zeta\oplus\tau$ maps from $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ to a collection of random variables on $(\Omega,\mathfrak{A}(\Omega),P)$, denoted by ξ. Then ξ is a random fuzzy variable defined on hybrid product space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})\times(\Omega,\mathfrak{A}(\Omega),P)$.

(2) Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous, such that $f(\zeta,\tau)$ maps from $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ to a collection of random

variables on $(\Omega,\mathfrak{A}(\Omega),P)$, denoted by ξ. Then $\xi=f(\zeta,\tau)$ is a random fuzzy variable defined on hybrid product space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})\times(\Omega,\mathfrak{A}(\Omega),P)$.

(3) Let $F(x;\theta)$ be the probability distribution of random variable τ with parameter θ (possible vector-valued), then $F(x;\zeta)$ defines a random fuzzy variable ξ on the hybrid product space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})\times(\Omega,\mathfrak{A}(\Omega),P)$.

Note that the *Theorem 4* merely repeats facts stated in Liu's books, [16], [17].

4. Stationary random fuzzy continuous-time Markov chain

Let $X=\{X_t,t\geq 0\}$ be a Markov process with a standard stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t,t\geq 0\}$ having a fuzzy rate matrix Q defined on credibility space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ with credibility distribution function matrix $\Lambda=(\Lambda_{ij})_{N\times N}$. Then by a direct application of *Theorem 4*, item (3), a random fuzzy continuous-time Markov chain can be obtained.

Definition 9: A process is called as random fuzzy continuous-time Markov chain $\xi=\{\xi_t,t\geq 0\}$ taking values in set $\mathbb{S}=\{0,1,2,\cdots,N-1\}$, if

(a) $\xi=\{\xi_t,t\geq 0\}$ satisfies Markov property:

$$\begin{aligned}&\Pr\{\xi_t=j\,|\,\xi_{t_1}=i_1,\xi_{t_2}=i_2,\cdots,\xi_s=i\}\\&=\Pr\{\xi_t=j\,|\,\xi_s=i\}\end{aligned}\tag{17}$$

for all $t_1<t_2<\cdots<s<t$ and any $i_1,i_2,\cdots,i,j\in\mathbb{S}$.

(b) the stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t,t\geq 0\}$ is standard;

(c) if the fuzzy rate matrix

$$\mathrm{Q}=(q_{ij})_{N\times N}=\lim_{t\downarrow 0}\frac{\mathrm{P}_t-\mathrm{I}}{t}\tag{18}$$

is defined on credibility space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ with credibility distribution function matrix $\Lambda=(\Lambda_{ij})_{N\times N}$.

It is obvious that in *Definition 9* for a given value of matrix $\mathrm{Q}=\mathrm{Q}_0$, $\xi=\{\xi_t,t\geq 0\}$ is a probabilistic continuous-time Markov chain. However, if Q is a fuzzy matrix, then for any given time t, the count ξ_t is a random fuzzy variable according to *Theorem 5*. Therefore, *Definition 9* defines a stationary random fuzzy Poisson process.

Theorem 5: If the process $\xi=\{\xi_t,t\geq 0\}$ is currently holds at state i, it holds in state i during an exponentially distributed time with fuzzy parameter q_i, independently of how the process reached state i and how long it gets there. Furthermore, The process $\xi=\{\xi_t,t\geq 0\}$ leaves state i, and moves to state j with a fuzzy probability q_{ij}/q_i $(i\neq j)$.

Proof: A straightforward application of *Definition 9* and *Theorem 2.*

Corollary 2: If q_{ij} $(i\neq j)$, $i,j=0,1,\cdots,N-1$, follow piecewise linear credibility distributions

$$\Lambda_{ij}(x)=\begin{cases}0 & x<a_{ij}\\ \dfrac{x-a_{ij}}{2(b_i-a_{ij})} & a_{ij}\leq x<b_{ij}\\ \dfrac{x+c_{ij}-2b_{ij}}{2(c_{ij}-b_{ij})} & b_{ij}\leq x<c_{ij}\\ 1 & x\geq c_{ij}\end{cases},\ i\neq j\tag{19}$$

The holding times, denoted by T_i, $i=0,1,\cdots,N-1$, are independent random fuzzy exponential variable with fuzzy parameter $q_i=\sum_j q_{ij}$ following a piecewise linear credibility distribution

$$\Lambda_i(x)=\begin{cases}0 & x<a_i\\ \dfrac{x-a_i}{2(b_i-a_i)} & a_i\leq x<b_i\\ \dfrac{x+c_i-2b_i}{2(c_i-b_i)} & b_i\leq x<c_i\\ 1 & x\geq c_i\end{cases},\tag{20}$$

where

$$\begin{cases}a_i=\displaystyle\sum_{j=1,j\neq i}^{N-1}a_{ij}\\ b_i=\displaystyle\sum_{j=1,j\neq i}^{N-1}b_{ij}\\ c_i=\displaystyle\sum_{j=1,j\neq i}^{N-1}c_{ij}\end{cases}\tag{21}$$

and thus the average chance distributions are

$$\Psi_i(t) = \int_0^1 \mathrm{Cr}\{\theta : q_i(\theta) \geq -\ln(1-\alpha)/t\} \mathrm{d}\alpha = 1 + \frac{e^{-b_i t} - e^{-a_i t}}{2(b_i - a_i)t} + \frac{e^{-c_i t} - e^{-b_i t}}{2(b_i - c_i)t} \tag{22}$$

Proof: Note that

$$\Pr\{T(q_i) \leq t\} = 1 - e^{-q_i t} \tag{23}$$

Therefore event $\{\theta : \Pr\{T(q_i(\theta)) \leq t\} \geq \alpha\}$ is a fuzzy event and is equivalent to the fuzzy event $\{\theta : q_i(\theta) \geq -\ln(1-\alpha)/t\}$. As a critical toward the derivation of the average chance distribution, it is necessary to calculate the credibility measure for fuzzy event $\{\theta : q_i(\theta) \geq -\ln(1-\alpha)/t\}$, i.e., obtain the expression for

$$\mathrm{Cr}\{\theta : q_i(\theta) \geq -\ln(1-\alpha)/t\} \tag{24}$$

Recall that for the credibilistic fuzzy variable, $q_i = \sum_{j \neq i} q_{ij}$, the credibility measure takes the form

$$\mathrm{Cr}\{\theta : q_i(\theta) > x\} = \begin{cases} 1 & x < a_i \\ \dfrac{2b_i - a_i - x}{2(b_i - a_i)} & a_i \leq x < b_i \\ \dfrac{c_i - x}{2(c_i - b_i)} & b_i \leq x < c_i \\ 0 & x \geq c_i \end{cases}, \tag{25}$$

Accordingly, the range for integration with α can be determined as shown in *Table 1*. Recall that the expression of $x = -\ln(1-\alpha)/t$ appears in Equation (25), which facilitates the link between intermediate variable α and average chance measure.
The average chance distribution for the exponentially distributed random fuzzy lifetime is then derived by splitting the integration into five terms according to the range of α and the corresponding mathematical expression for the credibility measure $\mathrm{Cr}\{\theta : q_i(\theta) \geq -\ln(1-\alpha)/t\}$, which is detailed in the following *Table 1*.

Table 1. Range analysis for α

x	α and credibility measure expression	
$-\infty < x \leq a$	Range for α	$0 \leq \alpha \leq 1 - e^{-at}$
	$\mathrm{Cr}\{\lambda(\theta) \geq -\ln(1-\alpha)/t\}$	1
$a < x \leq b$	Range for α	$1 - e^{-at} < \alpha \leq 1 - e^{-bt}$
	$\mathrm{Cr}\{\lambda(\theta) \geq -\ln(1-\alpha)/t\}$	$1 - (x-a)/(2(b-a))$
$b < x \leq c$	Range for α	$1 - e^{-bt} < \alpha \leq 1 - e^{-ct}$
	$\mathrm{Cr}\{\lambda(\theta) \geq -\ln(1-\alpha)/t\}$	$(c-x)/2(c-b)$
$d < x < +\infty$	Range for α	$1 - e^{-dt} < \alpha \leq 1$
	$\mathrm{Cr}\{\lambda(\theta) \geq -\ln(1-\alpha)/t\}$	0

Then the exponential random fuzzy lifetime has an average chance distribution function:

$$\Psi(t) = \int_0^1 \mathrm{Cr}\{\theta : \lambda(\theta) \geq -\ln(1-\alpha)/t\} \mathrm{d}\alpha = 1 + \frac{e^{-bt} - e^{-at}}{2(b-a)t} + \frac{e^{-dt} - e^{-ct}}{2(d-c)t}$$

and the average chance density is

$$\psi(\mathrm{t}) = \frac{e^{-at} - e^{-bt}}{2(b-a)t^2} + \frac{be^{-bt} - ae^{-at}}{2(b-a)t} + \frac{e^{-ct} - e^{-dt}}{2(d-c)t^2} + \frac{ce^{-ct} - de^{-dt}}{2(d-c)t} \tag{26}$$

This concludes the proof.
Similar to the probabilistic reliability theory, we define a reliability function or survival function for a random fuzzy lifetime and accordingly name it as the average chance reliability function, which is defined accordingly as

$$\overline{\Psi}(t) = 1 - \Psi(t) \tag{27}$$

Then, for exponential random fuzzy lifetime, its average chance reliability function is

$$\overline{\Psi}(t) = \frac{e^{-at} - e^{-bt}}{2(b-a)t} + \frac{e^{-ct} - e^{-dt}}{2(d-c)t} \tag{28}$$

Remark 1: The average chance distributions of jump probabilities q_{ij}/q_j do not have closed forms, which require the application of Zadeh's [25] extension theorem. However, the values of fuzzy probability q_{ij}/q_j fall in intervals

$$\left[\min\left(\frac{a_{ij}}{\sum_{j\neq i}a_{ij}},\frac{c_{ij}}{\sum_{j\neq i}c_{ij}}\right),\max\left(\frac{a_{ij}}{\sum_{j\neq i}a_{ij}},\frac{c_{ij}}{\sum_{j\neq i}c_{ij}}\right)\right] \quad (29)$$

which will help the explorations on the process $\xi=\{\xi_t,t\geq 0\}$.

5. Non-stationary random fuzzy continuous-time Markov chain

The probabilistic non-stationary continuous-time Markov chain is an extension to the stationary one except the rate matrix is function of time, i.e., time-dependent. Therefore, a non-stationary random fuzzy continuous-time Markov chain can be defined as follows.

Definition 10: A process is called as random fuzzy continuous-time non-stationary Markov chain $\xi=\{\xi_t,t\geq 0\}$ taking values in state space $\mathbb{S}=\{0,1,2,\cdots,N-1\}$, if

(a) $\xi=\{\xi_t,t\geq 0\}$ satisfies Markov property:

$$\begin{aligned}&\Pr\{\xi_t=j\mid\xi_{t_1}=i_1,\xi_{t_2}=i_2,\cdots,\xi_s=i\}\\&=\Pr\{\xi_t=j\mid\xi_s=i\}\end{aligned} \quad (30)$$

for all $t_1<t_2<\cdots<s<t$ and any $i_1,i_2,\cdots,i,j\in\mathbb{S}$.

(b) for $\forall s<t$, $p_{ij}(s,t)=\Pr\{\xi_t=j\mid\xi_s=i\}$, the transitional probabilities satisfy

(i) for a small time-increment h, $\xi=\{\xi_t,t\geq 0\}$ moves from state i to state j with (fuzzy) probability:

$$p_{ij}(t,t+h)=q_{ij}(t)h+o(h)\;\; h\downarrow 0(i\neq j) \quad (31)$$

(ii) for a small time-increment h, $\xi=\{\xi_t,t\geq 0\}$ remaining in state i with (fuzzy) probability:

$$p_{ii}(t,t+h)=1-q_i(t)h+o(h)\;\; h\downarrow 0 \quad (32)$$

where rate functions

$$q_i(t)=\sum_{j=0,j\neq i}^{N-1}q_{ij}(t),\; i=0,1,\cdots,N-1 \quad (33)$$

(c) The parameters of rate functions, i.e., the entries of the fuzzy rate matrix $\mathrm{Q}(t)=(q_{ij}(t))_{N\times N}$ are credibilistic fuzzy variables defined on the common credibility measure space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$.

Theorem 5. If the process $\xi=\{\xi_t,t\geq 0\}$ is currently holds at state i, it holds in state i during an exponentially distributed time with fuzzy parameter $q_i(t)$, independently of how the process reached state i and how long it gets there. Furthermore, The process $\xi=\{\xi_t,t\geq 0\}$ leaves state i, and moves to state j with a fuzzy probability $q_{ij}(t)/q_i(t)$ $(i\neq j)$.

Corollary 3: The probability distribution of holding times given the current state $\xi_{w_{l-1}}=x_{l-1}\in\mathbb{S}$,

$$\begin{aligned}&\Pr\{W_l-w_{l-1}>t,\xi_{w_{l-1}}=x_{l-1}\}=\\&\exp\left(-\left(m_{x_{l-1}}(w_{l-1}+t)\right)-m_{x_{l-1}}(w_{l-1}+t)\right)\end{aligned} \quad (34)$$

where

$$m_i(t)=\int_0^t q_i(u)du \quad (35)$$

is called the i^{th} integrated rate function.

Example 2: Assume a linear rate function:

$$q_{ij}(t)=\beta_{0,ij}+\beta_{1,ij}t,\;(j\neq i),\beta_{0,ij}>0,\;\beta_{1,ij}>0 \quad (36)$$

Further, we assume that β_0 and β_1 both have piecewise linear credibility distribution:

$$\Lambda_{ij}^{(k)}(x)=\begin{cases}0 & x<a_{ij}^{(k)}\\ \dfrac{x-a_{ij}^{(k)}}{2\left(b_{ij}^{(k)}-a_{ij}^{(k)}\right)} & a_{ij}^{(k)}\leq x<b_{ij}^{(k)}\\ \dfrac{x+c_{ij}^{(k)}-2b_{ij}^{(k)}}{2\left(c_{ij}^{(k)}-b_{ij}^{(k)}\right)} & b_{ij}^{(k)}\leq x<c_{ij}^{(k)}\\ 1 & x\geq c_{ij}^{(k)}\end{cases},\; k=0,1 \quad (37)$$

Then the diagonal entries $q_i(t)$, $i=0,1,\cdots,N-1$, have credibility distributions

$$\Lambda_i(x)=\begin{cases}0 & x<a_i\\ \dfrac{x-a_i}{2(b_i-a_i)} & a_i\leq x<b_i\\ \dfrac{x+c_i-2b_i}{2(c_i-b_i)} & b_i\leq x<c_i\\ 1 & x\geq c_i\end{cases},\; i=0,1,\cdots,N-1 \quad (38)$$

where

$$\begin{cases} a_i = \sum_{j=0}^{N-1}\left(a_{ij}^{(0)} + a_{ij}^{(1)} t\right) \\ b_i = \sum_{j=0}^{N-1}\left(b_{ij}^{(0)} + b_{ij}^{(1)} t\right) \\ c_i = \sum_{j=0}^{N-1}\left(c_{ij}^{(0)} + c_{ij}^{(1)} t\right) \end{cases} \quad (39)$$

The integrated diagonal entries of $\mathrm{Q}(t)$:

$$m_i(t) = \beta_{0,i} t + \beta_{1,i} t^2 \quad (40)$$

will have a credibility distributions:

$$\Lambda_{m_i(t)}(y) = \begin{cases} 0 & y < A_i \\ \dfrac{y - A_i}{2(B_i - A_i)} & A_i \le y < B_i \\ \dfrac{y + C_i - 2B_i}{2(C_i - B_i)} & B_i \le y < C_i \\ 1 & y \ge C_i \end{cases} \quad (41)$$

Where

$$\begin{cases} A_i = \sum_{j=0}^{N-1}\left(a_{ij}^{(0)} t + a_{ij}^{(1)} t^2\right) \\ B_i = \sum_{j=0}^{N-1}\left(b_{ij}^{(0)} t + b_{ij}^{(1)} t^2\right) \\ C_i = \sum_{j=0}^{N-1}\left(c_{ij}^{(0)} t + c_{ij}^{(1)} t^2\right) \end{cases} \quad (42)$$

In general, the credibility distribution of the integrated intensity function $m(t)$, it is necessary to apply Zadeh's [25] extension principle, denoted as $\Lambda_{m_i(t)}$, but for the piecewise linear credibility distribution case, the mathematical arguments are relatively simple.
Now let us derive the average chance distribution for the first holding times at i^{th} state (the initial state).

$$\Psi_T(t) = \int_0^1 \mathrm{Cr}\left(\theta : \Pr\left\{T_1(\theta) \le t\right\} \ge \alpha\right) d\alpha$$

Note that for the first arrival time,

$$\begin{aligned} &\left\{\theta : \Pr\left\{T_1(\theta) \le t\right\} \ge \alpha\right\} \\ &= \left\{\theta : 1 - \exp\left(-\int_0^t (\beta_0 + \beta_1 u)\, du\right) \ge \alpha\right\} \\ &= \left\{\theta : 1 - e^{-m(t)} \ge \alpha\right\} \\ &= \left\{\theta : m(t) \ge -\ln(1-\alpha)\right\} \end{aligned}$$

Therefore, the average chance distribution for T_1, the first holding at state *i*, is

$$\begin{aligned} &\Psi_{T_i}(t) \\ &= \int_0^1 \mathrm{Cr}\left(\theta : \Pr\left\{T_1(\theta) \le t\right\} \ge \alpha\right) d\alpha \\ &= \int_0^1 \mathrm{Cr}\left(\theta : m(t) \ge -\ln(1-\alpha)\right) d\alpha \end{aligned}$$

It is noticed that $y = -\ln(1-\alpha)$, therefore,

$$\mathrm{Cr}\left\{m(t) > y\right\} = \begin{cases} 1 & y < A_i \\ \dfrac{2B_i - 2 - y}{2(B_i - A_i)} & A_i \le y < B_i \\ \dfrac{C_i - y}{2(C_i - B_i)} & B_i \le y < C_i \\ 0 & y \ge C_i \end{cases}$$

Hence,

$$\begin{aligned} &\Psi_{T_i}(t) = \\ &1 - e^{-m(A_i)} + \frac{2B_i - A_i - 1}{2(B_i - A_i)}\left(e^{-m(A_i)} - e^{-m(B_i)}\right) \\ &+ \frac{1}{2(B_i - A_i)}\left(-m(B_i) e^{-m(B_i)} + m(A_i) e^{-m(A_i)}\right) \\ &+ \frac{C_i - 1}{2(C_i - B_i)}\left(e^{-m(B_i)} - e^{-m(C_i)}\right) \\ &+ \frac{1}{2(C_i - B_i)}\left(-m(C_i) e^{-m(C_i)} + m(B_i) e^{-m(B_i)}\right) \end{aligned}$$

4. A parameter estimation scheme

The parameter estimation is in nature an estimation problem of credibility distribution from fuzzy observations. Guo and Guo [10] recently proposed a maximally compatible random variable to a credibilistic fuzzy variable and thus the fuzzy estimation problem is converted into estimating the distribution function of the maximally compatible random variable. The following scheme is for estimating the piecewise linear credibility distribution.

Definition 11: Let X be a random variable defined in $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$ such that

$$\mu^c = \mathrm{Cr}\circ\xi^{-1} = \mu = P\circ X^{-1} \tag{43}$$

Then X is called a maximally compatible to fuzzy variable ξ.

In other words, random variable X can take all the possible real-values the fuzzy variable ξ may take with and the distribution of X , $F_X(r)$ equals the credibility distribution of ξ , $\Lambda_\xi(r)$ for all $r\in\mathbb{R}$.

It is aware that the induced measure $\mu^c = \mathrm{Cr}\circ\xi^{-1}$ and measure $\mu = P\circ X^{-1}$ are defined on the same measurable space $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$. Furthermore, we notice that the pre-image $\xi^{-1}(B)\in\mathfrak{P}(\Theta)$, but, the pre-image $X^{-1}(B)\in\mathfrak{A}(\Theta)\subset\mathfrak{P}(\Theta)$, which implies that for the same Borel set $B\in\mathfrak{B}(\mathbb{R})$, the pre-images under fuzzy variable ξ and random variable are not the same. It is expected that

$$\{\theta\in\Theta : X(\theta)\le r\}\subseteq\{\theta\in\Theta : \xi(\theta)\le r\}$$

but

$$\Pr\{\theta\in\Theta : X(\theta)\le r\} = \mathrm{Cr}\{\theta\in\Theta : \xi(\theta)\le r\}$$

The statistical estimation scheme for parameters (a,b,c) of the credibility distribution based on fuzzy observations $\{x_1,x_2,\cdots,x_n\}$ can be stated as:

Estimation Scheme 1.

Step 1: Rank fuzzy observations $\{x_1,x_2,\cdots,x_n\}$ to obtain "order" statistics $\{x_{(1)},x_{(2)},\cdots,x_{(n)}\}$ in ascending order;

Step 2: Set $\hat{a}=x_{(1)}$ and $\hat{c}=x_{(n)}$;

Step 3: Set a tentative estimator for b ,

$$\hat{b}_e = \frac{4\overline{x}_n - x_{(1)} - x_{(n)}}{2} \tag{44}$$

where

$$\overline{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{45}$$

Step 4: Identify $x_{(i_0)}$ from $\{x_{(1)},x_{(2)},\cdots,x_{(n)}\}$ such that $x_{(i_0)}\le\hat{b}_e < x_{(i_1)}$ and $1<i_0<i_1$, then we may see $\{x_{(1)},x_{(2)},\cdots,x_{(i_0)}\}$ as a set of order statistics from uniform [a,b]. Hence the "sufficient" statistic for parameter b is $x_{(i_0)}$.

Then $(\hat{a},\hat{b},\hat{c}) = (x_{(1)},x_{(i_0)},x_{(n)})$ is the parameter estimator for the piecewise linear credibility distribution.

$$\hat{\Lambda}(x)=\begin{cases} 0 & x<\hat{a} \\ \dfrac{x-\hat{a}}{2(\hat{b}-\hat{a})} & \hat{a}\le x<\hat{b} \\ \dfrac{x+\hat{c}-2\hat{b}}{2(\hat{c}-\hat{b})} & \hat{b}\le x<\hat{c} \\ 1 & x\ge\hat{c} \end{cases} \tag{46}$$

The next issue is how to extract the information on matrix rate Q in stationary random fuzzy the continuous-time Markov chain. Basawa and Rao [1] developed maximum likehood procedure for estimating the entries q_{ij} in Q. Interested readers may check Chapter 5 and 8 of [1].

It is noticed that for a given random fuzzy continuous-time Markov chain $\xi=\{\xi_t, t\ge t\}$, if we fix the fuzzy rate matrix at a given value Q_0, then $\xi=\{\xi_t, t\ge t\}$ becomes a probabilistic continuous Markov chain, Form the sample of the process: $K_\tau=\{N_\tau, X(0), W_1, X(W_1+), W_2,\cdots,W_{N_\tau} X(W_{N_\tau}+)\}$, which is sufficient. Then an MLE estimator for Q_0 , denoted as $\hat{\mathrm{Q}}_0$ is obtained. Repeat the sampling procedure from the random fuzzy continuous-time Markov chain as many times as possible, say, m times, then the fuzzy rate matrix "observation" sequence is

$$\{\hat{\mathrm{Q}}_1,\hat{\mathrm{Q}}_2,\cdots,\hat{\mathrm{Q}}_m\} = \{(\hat{q}_{ij}^{(1)}),(\hat{q}_{ij}^{(2)}),\cdots,(\hat{q}_{ij}^{(m)})\} \tag{47}$$

Apply the *Estimation Scheme 1* to the estimated observations $(i,j)^{th}$ entry of rate matrix Q $\{\hat{q}_{ij}^{(1)},\hat{q}_{ij}^{(2)},\cdots,\hat{q}_{ij}^{(m)}\}$, then the piecewise linear credibility distribution shown in Equation (54) for q_{ij} .

For the non-stationary random fuzzy continuous-time Markov chain, the parameters specifying the rate matrix $Q(t;\underline{\beta})$, may use maximum likelihood estimation procedure for estimating the parameters for defining fuzzy parameters $\underline{\beta}$. Therefore the idea is similar to that of stationary case but the credibility distribution treatments involved may be very complicated since Zadeh's extension principle [25] must be applied. mean measure involves two linear piecewise credibility distributions for fuzzy parameters β_0 and β_1 respectively.

5. A simulation scheme

Simulation of a random fuzzy continuous-time Markov chain is intrinsically two-stage procedure: a fuzzy parameter simulation for generating realizations $\left\{\left(q_{ij}^{(1)}\right),\left(q_{ij}^{(2)}\right),\cdots,\left(q_{ij}^{(m)}\right)\right\}$ from an matrix of credibility distribution functions $\left(\Lambda_{ij}\right)$ and then for each realization of $\left(q_{ij}\right)$, a probabilistic continuous-time Markov chain is simulated. Repeat this procedure until all the $\left(q_{ij}\right)$ realizations are used.

As to the fuzzy parameter simulation, we utilize the maximally compatible random variable to a fuzzy variable concept and the inverse transformation of the probability distribution function approach for generating fuzzy variable realizations. An algorithm is stated as follows:

Simulation scheme 1:

Step 1: Simulating uniform random variable uniform[0,1], and denote the simple random sample as $\{u_1,u_2,\cdots,u_n\}$;

Step 2: Set $\Lambda(x_i)=u_i,\ (k=1,2,\cdots,n)$;

Step 3: Set $x_i,\ (i=1,2,\cdots,n)$:

$$x_i=\begin{cases} a+2(b-a)u_i & \text{if } 0\le u_i\le 0.5 \\ 2b-c+2(c-b)u_i & \text{if } 0.5\le u_i\le 1 \end{cases} \tag{48}$$

Then $\{x_1,x_2,\cdots,x_n\}$ is a sample from the fuzzy variable ξ with a piecewise linear credibility distribution Λ.

Step 4: Repeat **Step 1** to **Step 3**, until m realizations of fuzzy rate matrix $\{Q_1,Q_2,\cdots,Q_m\}$ are obtained.

Step 5: For each rate matrix, say, Q_i, simulate a probabilistic continuous-time Markov chain, until m set of realizations of random fuzzy continuous-time Markov chain are obtained.

It should be mentioned that simulating a probabilistic continuous-time Markov chain is well-established in the literature.

6. Conclusion

In this paper, we give a systematic treatment of random fuzzy continuous-time Markov chains not only from the stationary one and then non-stationary one, but also a parameter estimation scheme as well as a simulation scheme is proposed. In this way, the foundation for the random fuzzy continuous-time Markov chains is formed although in its early stage. The applications to reliability engineering fields and the risk analysis now can extend from random uncertainty only cases to randomness and fuzziness co-existence cases. It is expecting that this development will help the reliability and risk analysis researchers as well as reliability analysts and engineers.

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