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STABILITY SWITCHES IN A LINEAR DIFFERENTIAL EQUATION WITH TWO DELAYS

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Abstract. This paper is devoted to the study of the effect of delays on the asymptotic stability of a linear differential equation with two delays

$$x'(t) = -ax(t) - bx(t-\tau) - cx(t-2\tau), \quad t \geq 0,$$

where a, b, and c are real numbers and $\tau > 0$. We establish some explicit conditions for the zero solution of the equation to be asymptotically stable. As a corollary, it is shown that the zero solution becomes unstable eventually after undergoing stability switches finite times when τ increases only if $c - a < 0$ and $\sqrt{-8c(c-a)} < |b| < a + c$. The explicit stability dependence on the changing τ is also described.

Keywords: delay differential equations, stability switches, two delays.

Mathematics Subject Classification: 34K06, 34K20, 34K25.

1. INTRODUCTION

For the last half century, great attention has been paid to delay differential equations which have significant background in physics, engineering, and biology to investigate some dynamical behaviors such as stability, periodic phenomenon, bifurcation, and chaos. If a delay differential equation is autonomous, then the stability of the trivial solution (i.e., the zero solution) of the linearized equation depends on the distribution of the roots of the associated characteristic equation; see, e.g., [7, 9]. If all roots of the characteristic equation are located in the left half-plane of the complex plane, that is, all characteristic roots have negative real parts, then the trivial solution is locally asymptotically stable; if there is some characteristic root in the right half-plane of the complex plane, that is, there is some characteristic root with positive real part, then the trivial solution is unstable. In particular, stability dependence on some of changing system parameters belongs among the key qualitative properties of studied dynamical systems.

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This study addresses the stability problem for a linear autonomous differential equation with two delays

$$x'(t) = -ax(t) - bx(t-\tau) - cx(t-\sigma), \quad t \geq 0, \tag{E_0}$$

where a, b, and c are real numbers and τ and σ are positive constants.

When $c = 0$, it is well known [3, 8, 12] that the zero solution of the equation

$$x'(t) = -ax(t) - bx(t-\tau), \quad t \geq 0, \tag{E_1}$$

is asymptotically stable if and only if

$$a + b > 0 \quad \text{and} \quad b^2 - a^2 \leq 0$$

or

$$a + b > 0, \quad b^2 - a^2 > 0, \quad \text{and} \quad \tau < \frac{1}{\sqrt{b^2 - a^2}} \arccos\left(-\frac{a}{b}\right).$$

When $a = 0$ and $b = c$, it is known [9, 17] that the zero solution of the equation

$$x'(t) = -bx(t-\tau) - bx(t-\sigma), \quad t \geq 0,$$

is asymptotically stable if and only if

$$0 < b(\tau + \sigma)\cos\left(\frac{\tau - \sigma}{\tau + \sigma}\frac{\pi}{2}\right) < \frac{\pi}{2}.$$

In the last three decades, a great deal of effort has been devoted to the study of the asymptotic stability of (E_0) with $bc \neq 0$. For example, those results can be found in [1, 2, 4–6, 10, 11, 15, 16, 18, 19] and the references cited therein. However, to our best knowledge, the explicit stability criteria for (E_0) have not yet been obtained. Despite the first-order scalar differential equation with delays, one of the reasons why stability analysis for (E_0) is very difficult is that *stability switches* with increasing τ may occur in (E_0); that is, when τ increases, the zero solution of (E_0) may change finite times from stability to instability to stability, and becomes unstable eventually. Notice that a restabilization with increasing τ is not possible in (E_1) with real coefficients. On the other hand, if a is a complex number, stability switches occur in (E_1) under certain conditions; see, e.g., [13, 14].

In this paper we investigate stability properties for (E_0) in the special case $\sigma = 2\tau$, namely,

$$x'(t) = -ax(t) - bx(t-\tau) - cx(t-2\tau), \quad t \geq 0. \tag{1.1}$$

Recently in [18], Yan and Shi discussed the asymptotic stability of (1.1) under $a > 0$, $b > 0$, and $c > 0$, and gave stability criteria for (1.1) which include stability switches. Our purpose is to present the explicit stability criteria for (1.1) completely, i.e., to establish new necessary and sufficient conditions for the zero solution of (1.1) to be asymptotically stable.

Throughout this paper, for brevity, let ω_+ and ω_- be constants defined as

$$\omega_+ = \sqrt{\frac{-2(a^2-c^2)+b^2+\sqrt{b^2(b^2+8c(c-a))}}{2}},$$
$$\omega_- = \sqrt{\frac{-2(a^2-c^2)+b^2-\sqrt{b^2(b^2+8c(c-a))}}{2}}.$$

Under suitable conditions, real numbers ω_+^2 and ω_-^2 exist and are positive roots of a quadratic equation; see Lemma 2.3. For $n \in \mathbb{Z}^+ := \{0,1,2,\ldots\}$, let τ_n^+, τ_n^-, σ_n^+, and σ_n^- be critical values of τ defined as

$$\tau_n^+ = \frac{1}{\omega_+}\left(\arccos\left(\frac{b(c-a)}{\omega_+^2+a^2-c^2}\right)+2n\pi\right),$$
$$\tau_n^- = \frac{1}{\omega_-}\left(\arccos\left(\frac{b(c-a)}{\omega_-^2+a^2-c^2}\right)+2n\pi\right),$$
$$\sigma_n^+ = \frac{1}{\omega_+}\left(-\arccos\left(\frac{b(c-a)}{\omega_+^2+a^2-c^2}\right)+2(n+1)\pi\right),$$
$$\sigma_n^- = \frac{1}{\omega_-}\left(-\arccos\left(\frac{b(c-a)}{\omega_-^2+a^2-c^2}\right)+2(n+1)\pi\right);$$

see Remark 2.6 and Lemmas 2.7 and 2.8 on the condition of positiveness of each constant above. Especially, when $n = 0$, we set $\tau_{n-1}^- = \sigma_{n-1}^- = 0$. In addition, we name the following two conditions (A_0) and (A_1) respectively:

$$b^2+8c(c-a) \neq 0 \quad \text{if } 2(a^2-c^2)-b^2<0, \tag{A_0}$$
$$a-b+c>0,\ \ 2(a^2-c^2)-b^2<0,\ \ \text{and}\ \ b^2+8c(c-a)>0. \tag{A_1}$$

Our main results are stated as follows:

Theorem 1.1. *Let $b > 0$. Suppose that (A_0) is satisfied. Then the zero solution of (1.1) is asymptotically stable if and only if condition $a+b+c>0$ holds and any one of the following five conditions is satisfied:*

$$b^2+8c(c-a)<0 \ \ \textit{or}\ \ \text{“}a-b+c\geq 0 \ \ \textit{and}\ \ 2(a^2-c^2)-b^2\geq 0\text{”}, \tag{1.2}$$
$$a-b+c<0 \ \ \textit{and}\ \ \tau<\tau_0^+, \tag{1.3}$$
$$a-b+c=0,\ \ 2(a^2-c^2)-b^2<0,\ \ \textit{and}\ \ \tau<\tau_0^+, \tag{1.4}$$
$$a-b+c>0,\ \ c-a\geq 0,\ \ \textit{and}\ \ \tau<\tau_0^+, \tag{1.5}$$
$$(A_1),\ \ c-a<0,\ \ \textit{and}\ \ \tau\in(0,\tau_0^+)\cup(\tau_0^-,\tau_1^+)\cup\ldots\cup(\tau_{k-1}^-,\tau_k^+). \tag{1.6}$$

Here k is a nonnegative integer given by

$$k = \left\lceil \frac{\omega_+\omega_-(\tau_1^+-\tau_0^-)}{2(\omega_+-\omega_-)\pi} \right\rceil, \tag{1.7}$$

where $\lceil\cdot\rceil$ denotes the ceiling function, namely, $\lceil x\rceil = \min\{r\in\mathbb{Z} \mid x\leq r\}$.

Theorem 1.2. *Let $b < 0$. Suppose that (A_0) is satisfied. Then the zero solution of* (1.1) *is asymptotically stable if and only if condition $a + b + c > 0$ holds and any one of the following four conditions is satisfied:*

$$b^2 + 8c(c-a) < 0 \quad \text{or} \quad \text{“}a - b + c \geq 0 \quad \text{and} \quad 2(a^2 - c^2) - b^2 \geq 0\,\text{”}, \tag{1.8}$$

$$c - a = 0 \quad \text{and} \quad \tau < \sigma_0^+, \tag{1.9}$$

$$c - a > 0 \quad \text{and} \quad \tau < \min\{\sigma_0^+, \tau_0^-\}, \tag{1.10}$$

$$(A_1), \quad c - a < 0, \quad \text{and} \quad \tau \in (0, \sigma_0^+) \cup (\sigma_0^-, \sigma_1^+) \cup \ldots \cup (\sigma_{\ell-1}^-, \sigma_\ell^+). \tag{1.11}$$

Here ℓ is a nonnegative integer given by

$$\ell = \left\lceil \frac{\omega_+\omega_-(\sigma_1^+ - \sigma_0^-)}{2(\omega_+ - \omega_-)\pi} \right\rceil. \tag{1.12}$$

Moreover, it is easily seen that conditions $a + b + c > 0$, (A_1), and $c - a < 0$ are reduced to

$$c - a < 0 \quad \text{and} \quad \sqrt{-8c(c-a)} < |b| < a + c. \tag{1.13}$$

By virtue of Theorems 1.1 and 1.2, one can obtain the following corollary.

Corollary 1.3. *Stability switches occur in* (1.1) *only if condition* (1.13) *is satisfied.*

To illustrate this, we take $a = 1$ and $c = 0.8$. Then condition (1.13) becomes

$$1.13137 \approx \sqrt{1.28} < |b| < 1.8.$$

Rewriting k defined by (1.7) as $k(b)$ and ℓ defined by (1.12) as $\ell(b)$, one can find

$$k(1.2) = 1, \quad k(1.14) = 3, \quad k(1.132) = 13, \quad k(1.1314) = 61,$$
$$\ell(-1.2) = 0, \quad \ell(-1.14) = 3, \quad \ell(-1.132) = 12, \quad \ell(-1.1314) = 60.$$

For example, in the case $a = 1$, $b = -1.14$, and $c = 0.8$, Theorem 1.2 indicates that the zero solution of (1.1) is asymptotically stable if and only if

$$\tau \in (0, \sigma_0^+) \cup (\sigma_0^-, \sigma_1^+) \cup (\sigma_1^-, \sigma_2^+) \cup (\sigma_2^-, \sigma_3^+),$$

where the critical values of σ_n^+ and σ_n^- are expressed numerically as

$$\sigma_0^+ \approx 8.274, \quad \sigma_1^+ \approx 18.609, \quad \sigma_2^+ \approx 28.944, \quad \sigma_3^+ \approx 39.279, \quad \sigma_4^+ \approx 49.614,$$
$$\sigma_0^- \approx 11.181, \quad \sigma_1^- \approx 24.892, \quad \sigma_2^- \approx 38.603, \quad \sigma_3^- \approx 52.314.$$

Remark 1.4. In case (A_0) is not satisfied, that is,

$$b^2 + 8c(c-a) = 0 \quad \text{and} \quad 2(a^2 - c^2) - b^2 < 0, \tag{1.14}$$

the explicit stability criteria for (1.1) have not yet been obtained; see Remark 2.10.

2. PROOF OF THEOREMS

The characteristic equation of (1.1) is expressed as

$$f(\lambda) := \lambda + a + be^{-\lambda\tau} + ce^{-2\lambda\tau} = 0. \tag{2.1}$$

Notice that (2.1) has the root 0 if and only if $a + b + c = 0$. Also, if $a + b + c < 0$, then (2.1) has at least one positive root because $f(0) < 0$ and $\lim_{\lambda\to+\infty} f(\lambda) = +\infty$. Hence, one can immediately obtain the following lemma.

Lemma 2.1. *If $a + b + c \leq 0$, then* (2.1) *has at least one nonnegative root.*

By virtue of Lemma 2.1, it suffices to examine the location of roots of (2.1) in the complex plane under $a + b + c > 0$. Notice that $f(\lambda)$ is an analytic function, and thus, the roots of (2.1) are continuously depending on τ. The next proposition presented by Corollary 2.4 in [15] plays an important role in our proof.

Proposition 2.2. *As τ varies, the sum of the multiplicities of roots of* (2.1) *in the open right half-plane can change only if a root appears on or crosses the imaginary axis.*

Therefore, we will investigate the existence of purely imaginary roots of (2.1) and the crossing of roots of (2.1) through the imaginary axis.

Let $\pm i\omega$ be a pair of purely imaginary roots of (2.1) with $\omega > 0$. Then $f(i\omega) = 0$, or

$$(i\omega + a)e^{i\omega\tau} + b + ce^{-i\omega\tau} = 0. \tag{2.2}$$

Separating the real and imaginary parts of (2.2), one can obtain

$$\begin{cases} (a + c)\cos\omega\tau - \omega\sin\omega\tau + b = 0, \\ \omega\cos\omega\tau + (a - c)\sin\omega\tau = 0, \end{cases}$$

namely,

$$\begin{cases} (\omega^2 + a^2 - c^2)\cos\omega\tau = b(c - a), \\ (\omega^2 + a^2 - c^2)\sin\omega\tau = b\omega. \end{cases} \tag{2.3}$$

In case $\omega^2 + a^2 - c^2 = 0$, relation (2.3) gives $b\omega = 0$, which is a contradiction. So $\omega^2 + a^2 - c^2 \neq 0$, and hence, $f(i\omega) = 0$ is equivalent to

$$\cos\omega\tau = \frac{b(c - a)}{\omega^2 + a^2 - c^2}, \quad \sin\omega\tau = \frac{b\omega}{\omega^2 + a^2 - c^2}. \tag{2.4}$$

From equality $\cos^2\omega\tau + \sin^2\omega\tau = 1$, we find

$$(\omega^2 + a^2 - c^2)^2 = b^2(\omega^2 + (a - c)^2). \tag{2.5}$$

An easy calculation shows that (2.5) is reduced to

$$\omega^4 + (2(a^2 - c^2) - b^2)\omega^2 + (a + b + c)(a - b + c)(a - c)^2 = 0,$$

which, by $z = \omega^2$, becomes

$$z^2 + (2(a^2 - c^2) - b^2)z + (a+b+c)(a-b+c)(a-c)^2 = 0. \tag{2.6}$$

From the quadratic formula of (2.6), it follows that

$$z = \frac{-2(a^2 - c^2) + b^2 \pm \sqrt{D}}{2} \ (= \omega_\pm^2),$$

where D is the discriminant defined by

$$\begin{aligned} D &:= (2(a^2 - c^2) - b^2)^2 - 4(a+b+c)(a-b+c)(a-c)^2 \\ &= b^2(b^2 + 8c(c-a)). \end{aligned}$$

Notice that

$$\omega_+^2 + a^2 - c^2 = \frac{b^2 + \sqrt{D}}{2}, \quad \omega_-^2 + a^2 - c^2 = \frac{b^2 - \sqrt{D}}{2} = -\frac{4c(c-a)}{b^2 + \sqrt{D}}. \tag{2.7}$$

The next lemma classifies the existence of positive roots of (2.6).

Lemma 2.3. *Let $b \neq 0$ and $a + b + c > 0$. Then the following statements hold:*

(I) *Equation (2.6) has no positive roots if and only if $b^2 + 8c(c-a) < 0$ or "$a - b + c \geq 0$ and $2(a^2 - c^2) - b^2 \geq 0$".*
(IIa) *Equation (2.6) has one positive root ω_+^2 and one negative root ω_-^2 if and only if $a - b + c < 0$.*
(IIb) *Equation (2.6) has one positive root ω_+^2 and the root 0 if and only if $a = c$ or "$a - b + c = 0$ and $2(a^2 - c^2) - b^2 < 0$".*
(IIc) *Equation (2.6) has double positive roots ω_+^2 if and only if $b^2 + 8c(c-a) = 0$ and $2(a^2 - c^2) - b^2 < 0$.*
(IIIa) *Equation (2.6) has two distinct positive roots ω_+^2, ω_-^2 with $\omega_-^2 + a^2 - c^2 < 0$ if and only if (A_1) and $c - a > 0$.*
(IIIb) *Equation (2.6) has two distinct positive roots ω_+^2, ω_-^2 with $\omega_-^2 + a^2 - c^2 > 0$ if and only if (A_1) and $c - a < 0$.*

Proof. Let p and q be real numbers. The existence of positive roots of the quadratic equation $x^2 + px + q = 0$ can be classified as follows:

(i) The equation has no positive roots if and only if $\Delta := p^2 - 4q < 0$ or "$p \geq 0$ and $q \geq 0$".
(iia) The equation has one positive root and one negative root if and only if $q < 0$.
(iib) The equation has one positive root and the root 0 if and only if $q = 0$ and $p < 0$.
(iic) The equation has double positive roots if and only if $\Delta = 0$ and $p < 0$.
(iii) The equation has two distinct positive roots if and only if $\Delta > 0$, $p < 0$, and $q > 0$.

By applying this classification to (2.6), we can immediately obtain assertions of (I), (IIa), (IIb), (IIc) in Lemma 2.3, and

(III) Equation (2.6) has two distinct positive roots ω_+^2, ω_-^2 if and only if (A_1) is satisfied.

Therefore, if inequality $c > 0$ is valid under conditions $b \neq 0$, $a+b+c > 0$, and (A_1), then relation (2.7) yields assertions (IIIa) and (IIIb) in Lemma 2.3. To this end, suppose that $c \leq 0$. Inequality $a+b+c > 0$ with $a-b+c > 0$ in (A_1) implies $a > |b| - c > 0$. From this, we observe that

$$a^2 > (|b|-c)^2 = b^2 - 2|b|c + c^2 > \frac{b^2}{2} + c^2,$$

which contradicts $2(a^2-c^2) - b^2 < 0$ in (A_1). This completes the proof. □

Remark 2.4. Lemma 2.3 indicates that there are six possible cases in analyzing purely imaginary roots of (2.1) as follows:

(a) $b^2 + 8c(c-a) < 0$ or "$a-b+c \geq 0$ and $2(a^2-c^2) - b^2 \geq 0$".
(b) $a-b+c < 0$.
(c) $a = c$ or "$a-b+c = 0$ and $2(a^2-c^2) - b^2 < 0$".
(d) $b^2 + 8c(c-a) = 0$ and $2(a^2-c^2) - b^2 < 0$.
(e) (A_1) and $c - a > 0$.
(f) (A_1) and $c - a < 0$.

Remark 2.5. If $b > 0$ and $a+b+c > 0$, Case (e) is equivalent to the case $a-b+c > 0$ and $c-a > 0$ because $2(a^2-c^2)-b^2 < 0$ and $b^2+8c(c-a) > 0$ in (A_1) are always satisfied under $c(c-a) > 0$. On the other hand, if $b < 0$ and $a+b+c > 0$, Case (e) is equivalent to the case $c-a > 0$ because condition (A_1) is always satisfied under $c(c-a) > 0$.

Remark 2.6. In the case $b \neq 0$, inequality

$$\left| \frac{b(c-a)}{\omega_\pm^2 + a^2 - c^2} \right| < 1 \tag{2.8}$$

is valid for $\omega_+ > 0$ and $\omega_- > 0$. In fact, suppose that

$$|b(c-a)| \geq |\omega_\pm^2 + a^2 - c^2|.$$

Recall that ω_+^2 and ω_-^2 are positive roots of (2.6) and satisfy (2.5). Then we have

$$b^2(c-a)^2 \geq (\omega_\pm^2 + a^2 - c^2)^2 = b^2(\omega_\pm^2 + (a-c)^2),$$

or, $0 \geq b^2\omega_\pm^2$, which is a contradiction.

When $b > 0$, we establish the following result on purely imaginary roots of (2.1).

Lemma 2.7. *Let $b > 0$ and $a + b + c > 0$. Suppose that $i\omega$ is a root of (2.1) with $\omega > 0$. Then the positive values of ω and τ are given as follows:*

(i) *If $b^2 + 8c(c-a) > 0$ or "$a - b + c \geq 0$ and $2(a^2 - c^2) - b^2 \geq 0$", then (2.1) has no purely imaginary roots for $\tau > 0$.*
(ii) *If $a - b + c < 0$, then $(\omega, \tau) = (\omega_+, \tau_n^+)$ for $n \in \mathbb{Z}^+$.*
(iii) *If $a = c$ or "$a - b + c = 0$ and $2(a^2 - c^2) - b^2 < 0$", then $(\omega, \tau) = (\omega_+, \tau_n^+)$ for $n \in \mathbb{Z}^+$.*
(iv) *If $b^2 + 8c(c-a) = 0$ and $2(a^2 - c^2) - b^2 < 0$, then $(\omega, \tau) = (\omega_+, \tau_n^+)$ for $n \in \mathbb{Z}^+$.*
(v) *If $a-b+c > 0$ and $c-a > 0$, then $(\omega, \tau) = (\omega_+, \tau_n^+), (\omega_-, \sigma_n^-)$ with $\omega_-^2 + a^2 - c^2 < 0$ for $n \in \mathbb{Z}^+$.*
(vi) *If (A_1) and $c - a < 0$, then $(\omega, \tau) = (\omega_+, \tau_n^+), (\omega_-, \tau_n^-)$ with $\omega_-^2 + a^2 - c^2 > 0$ for $n \in \mathbb{Z}^+$.*

Conversely, if $\omega_+ > 0$ and $\tau = \tau_n^+$ for $n \in \mathbb{Z}^+$, then $i\omega_+$ is a root of (2.1); if $\omega_- > 0$ and $\tau = \tau_n^-$ (resp. $\tau = \sigma_n^-$) with $c - a < 0$ (resp. $c - a > 0$) for $n \in \mathbb{Z}^+$, then $i\omega_-$ is a root of (2.1).

Proof. We recall that $f(i\omega) = 0$ is equivalent to relation (2.4). Let $i\omega$ be a root of (2.1) with $\omega > 0$. Notice that ω^2 is a positive root of (2.6). Our argument is based on six cases in Remark 2.4 under $b > 0$ and $a + b + c > 0$.

Case (a). Lemma 2.3 (I) asserts that statement (i) in Lemma 2.7 is verified.

Cases (b), (c), (d), (e), and (f). By Lemma 2.3 (IIa), (IIb), (IIc), (IIIa), and (IIIb), we have $\omega = \omega_+$. Observe that $\omega_+^2 + a^2 - c^2 > 0$ by (2.7). Thus, (2.4) and $\sin \omega_+ \tau > 0$ lead to

$$\omega_+ \tau = \arccos\left(\frac{b(c-a)}{\omega_+^2 + a^2 - c^2}\right) + 2n\pi, \quad n \in \mathbb{Z}^+,$$

which implies $(\omega, \tau) = (\omega_+, \tau_n^+)$ for $n \in \mathbb{Z}^+$.

Case (e). As stated in Remark 2.5, this case is equivalent to the case $a - b + c > 0$ and $c - a > 0$. By Lemma 2.3 (IIIa), we find $\omega = \omega_-$ with $\omega_-^2 + a^2 - c^2 < 0$. Hence, (2.4) and $\sin(-\omega_- \tau) > 0$ give

$$-\omega_- \tau = \arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right) - 2(n+1)\pi, \quad n \in \mathbb{Z}^+,$$

which yields $(\omega, \tau) = (\omega_-, \sigma_n^-)$ for $n \in \mathbb{Z}^+$.

Case (f). By Lemma 2.3 (IIIb), we find $\omega = \omega_-$ with $\omega_-^2 + a^2 - c^2 > 0$. Therefore, (2.4) and $\sin(\omega_- \tau) > 0$ lead to

$$\omega_- \tau = \arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right) + 2n\pi, \quad n \in \mathbb{Z}^+,$$

which implies $(\omega, \tau) = (\omega_-, \tau_n^-)$ for $n \in \mathbb{Z}^+$.

Conversely, suppose $\omega_+ > 0$ and $\tau = \tau_n^+$ for $n \in \mathbb{Z}^+$. Then $\omega_+^2 + a^2 - c^2 > 0$ by (2.7), and the definition of τ_n^+ and (2.8) give

$$\cos \omega_+ \tau_n^+ = \frac{b(c-a)}{\omega_+^2 + a^2 - c^2}, \quad \sin \omega_+ \tau_n^+ = \sin\left(\arccos\left(\frac{b(c-a)}{\omega_+^2 + a^2 - c^2}\right)\right) > 0.$$

By (2.5), we have

$$\sin \omega_+ \tau_n^+ = \sqrt{1 - \cos^2 \omega_+ \tau_n^+} = \frac{\sqrt{(\omega_+^2 + a^2 - c^2)^2 - b^2(c-a)^2}}{\omega_+^2 + a^2 - c^2}$$
$$= \frac{\sqrt{b^2(\omega_+^2 + (a-c)^2) - b^2(a-c)^2}}{\omega_+^2 + a^2 - c^2} = \frac{b\omega_+}{\omega_+^2 + a^2 - c^2}.$$

These facts mean that (2.4) is valid in $(\omega, \tau) = (\omega_+, \tau_n^+)$; i.e., $f(i\omega_+) = 0$.

Suppose $\omega_- > 0$ and $\tau = \tau_n^-$ with $c - a < 0$ for $n \in \mathbb{Z}^+$. Then $\omega_-^2 + a^2 - c^2 > 0$ by Lemma 2.3 (IIIb), and the definition of τ_n^- and (2.8) lead to

$$\cos \omega_- \tau_n^- = \frac{b(c-a)}{\omega_-^2 + a^2 - c^2}, \quad \sin \omega_- \tau_n^- = \sin\left(\arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right)\right) > 0.$$

By (2.5), we have

$$\sin \omega_- \tau_n^- = \sqrt{1 - \left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right)^2} = \frac{b\omega_-}{\omega_-^2 + a^2 - c^2}.$$

These facts imply that (2.4) is verified in $(\omega, \tau) = (\omega_-, \tau_n^-)$; i.e., $f(i\omega_-) = 0$.

Suppose $\omega_- > 0$ and $\tau = \sigma_n^-$ with $c - a > 0$ for $n \in \mathbb{Z}^+$. Then $\omega_-^2 + a^2 - c^2 < 0$ by Lemma 2.3 (IIIa), and the definition of σ_n^- and (2.8) give

$$\cos \omega_- \sigma_n^- = \frac{b(c-a)}{\omega_-^2 + a^2 - c^2}, \quad \sin \omega_- \sigma_n^- = \sin\left(-\arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right)\right) < 0.$$

By (2.5), we obtain

$$\sin \omega_- \sigma_n^- = -\sqrt{1 - \left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right)^2} = \frac{b\omega_-}{\omega_-^2 + a^2 - c^2}.$$

These facts mean that (2.4) is valid in $(\omega, \tau) = (\omega_-, \sigma_n^-)$; i.e., $f(i\omega_-) = 0$. □

When $b < 0$, we establish the following result on purely imaginary roots of (2.1).

Lemma 2.8. *Let $b < 0$ and $a + b + c > 0$. Suppose that $i\omega$ is a root of (2.1) with $\omega > 0$. Then the positive values of ω and τ are given as follows:*

- (i) *If $b^2 + 8c(c-a) > 0$ or "$a - b + c \geq 0$ and $2(a^2 - c^2) - b^2 \geq 0$", then (2.1) has no purely imaginary roots for $\tau > 0$.*
- (ii) *If $a = c$, then $(\omega, \tau) = (\omega_+, \sigma_n^+)$ for $n \in \mathbb{Z}^+$.*
- (iii) *If $b^2 + 8c(c-a) = 0$ and $2(a^2 - c^2) - b^2 < 0$, then $(\omega, \tau) = (\omega_+, \sigma_n^+)$ for $n \in \mathbb{Z}^+$.*
- (iv) *If $c - a > 0$, then $(\omega, \tau) = (\omega_+, \sigma_n^+), (\omega_-, \tau_n^-)$ with $\omega_-^2 + a^2 - c^2 < 0$ for $n \in \mathbb{Z}^+$.*
- (v) *If (A_1) and $c - a < 0$, then $(\omega, \tau) = (\omega_+, \sigma_n^+), (\omega_-, \sigma_n^-)$ with $\omega_-^2 + a^2 - c^2 > 0$ for $n \in \mathbb{Z}^+$.*

Conversely, if $\omega_+ > 0$ *and* $\tau = \sigma_n^+$ *for* $n \in \mathbb{Z}^+$*, then* $i\omega_+$ *is a root of* (2.1)*; if* $\omega_- > 0$ *and* $\tau = \tau_n^-$ *(resp.* $\tau = \sigma_n^-$*) with* $c - a > 0$ *(resp.* $c - a < 0$*) for* $n \in \mathbb{Z}^+$*, then* $i\omega_-$ *is a root of* (2.1)*.*

Proof. We recall that $f(i\omega) = 0$ is equivalent to relation (2.4). Let $i\omega$ be a root of (2.1) with $\omega > 0$. Notice that ω^2 is a positive root of (2.6). Our argument is based on six cases in Remark 2.4 under $b < 0$ and $a + b + c > 0$. For convenience, divide Case (c) into Case (c1): $a = c$ and Case (c2): $a - b + c = 0$ and $2(a^2 - c^2) - b^2 < 0$. Notice that Case (b) and Case (c2) cannot occur because $a - b + c > a + b + c > 0$ by $b < 0$.

Case (a). Lemma 2.3 (I) asserts that statement (i) in Lemma 2.8 is verified.

Cases (c1), (d), (e), and (f). By Lemma 2.3 (IIb), (IIc), (IIIa), and (IIIb), we have $\omega = \omega_+$. Observe that $\omega_+^2 + a^2 - c^2 > 0$ by (2.7). Thus, (2.4) and $\sin(-\omega_+\tau) > 0$ lead to

$$-\omega_+\tau = \arccos\left(\frac{b(c-a)}{\omega_+^2 + a^2 - c^2}\right) - 2(n+1)\pi, \quad n \in \mathbb{Z}^+,$$

which yields $(\omega, \tau) = (\omega_+, \sigma_n^+)$ for $n \in \mathbb{Z}^+$.

Case (e). As stated in Remark 2.5, this case is equivalent to the case $c - a > 0$. By Lemma 2.3 (IIIa), we find $\omega = \omega_-$ with $\omega_-^2 + a^2 - c^2 < 0$. Hence, (2.4) and $\sin\omega_-\tau > 0$ give

$$\omega_-\tau = \arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right) + 2n\pi, \quad n \in \mathbb{Z}^+,$$

which implies $(\omega, \tau) = (\omega_-, \tau_n^-)$ for $n \in \mathbb{Z}^+$.

Case (f). By Lemma 2.3 (IIIb), we find $\omega = \omega_-$ with $\omega_-^2 + a^2 - c^2 > 0$. Therefore, (2.4) and $\sin(-\omega_+\tau) > 0$ lead to

$$-\omega_-\tau = \arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right) - 2(n+1)\pi, \quad n \in \mathbb{Z}^+,$$

which yields $(\omega, \tau) = (\omega_-, \sigma_n^-)$ for $n \in \mathbb{Z}^+$.

The rest of the proof can be carried out in the same way as the proof of Lemma 2.7, so it will be omitted. □

Next, we will examine how one pair of complex roots of (2.1) crosses through the imaginary axis as τ increases.

Lemma 2.9. *Let* $b \neq 0$ *and* $a + b + c > 0$*. Then the following statements hold:*

(i) *The roots* $\pm i\omega_+$ *enter the right half-plane as* τ *increases from* τ_n^+ $(b > 0)$ *or* σ_n^+ $(b < 0)$ *for* $n \in \mathbb{Z}^+$*.*
(ii) *If* (A_1) *and* $c - a > 0$*, then the roots* $\pm i\omega_-$ *enter the right half-plane as* τ *increases from* σ_n^- $(b > 0)$ *or* τ_n^- $(b < 0)$ *for* $n \in \mathbb{Z}^+$*.*
(iii) *If* (A_1) *and* $c - a < 0$*, then the roots* $\pm i\omega_-$ *enter the left half-plane as* τ *increases from* τ_n^- $(b > 0)$ *or* σ_n^- $(b < 0)$ *for* $n \in \mathbb{Z}^+$*.*

Proof. Take the derivative of λ with respect to τ on (2.1) to obtain

$$\frac{d\lambda}{d\tau} - b\left(\lambda e^{-\lambda\tau} + \tau e^{-\lambda\tau}\frac{d\lambda}{d\tau}\right) - c\left(2\lambda e^{-2\lambda\tau} + 2\tau e^{-2\lambda\tau}\frac{d\lambda}{d\tau}\right) = 0,$$

namely,

$$\begin{aligned}\frac{d\lambda}{d\tau} &= \frac{b\lambda e^{-\lambda\tau} + 2c\lambda e^{-2\lambda\tau}}{1 - b\tau e^{-\lambda\tau} - 2c\tau e^{-2\lambda\tau}}\\ &= \frac{b\lambda e^{-\lambda\tau} + 2\lambda(-\lambda - a - be^{-\lambda\tau})}{1 - b\tau e^{-\lambda\tau} - 2\tau(-\lambda - a - be^{-\lambda\tau})}\\ &= \frac{-2\lambda^2 - 2a\lambda - b\lambda e^{-\lambda\tau}}{1 + 2\lambda\tau + 2a\tau + b\tau e^{-\lambda\tau}}.\end{aligned}$$

This implies that

$$\begin{aligned}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=\pm i\omega} &= \frac{2\omega^2 \mp 2ia\omega \mp ib\omega(\cos\omega\tau \mp i\sin\omega\tau)}{1 + 2a\tau \pm 2i\omega\tau + b\tau(\cos\omega\tau \mp i\sin\omega\tau)}\\ &= \frac{2\omega^2 - b\omega\sin\omega\tau \mp i(2a\omega + b\omega\cos\omega\tau)}{1 + 2a\tau + b\tau\cos\omega\tau \pm i(2\omega\tau - b\tau\sin\omega\tau)}.\end{aligned}$$

Therefore, by (2.4), we find

$$\begin{aligned}&\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega} = \operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=-i\omega}\\ &= \frac{(2\omega^2 - b\omega\sin\omega\tau)(1 + 2a\tau + b\tau\cos\omega\tau) - (2a\omega + b\omega\cos\omega\tau)(2\omega\tau - b\tau\sin\omega\tau)}{(1 + 2a\tau + b\tau\cos\omega\tau)^2 + (2\omega\tau - b\tau\sin\omega\tau)^2}\\ &= \frac{2\omega^2 - b\omega\sin\omega\tau}{(1 + 2a\tau + b\tau\cos\omega\tau)^2 + (2\omega\tau - b\tau\sin\omega\tau)^2}\\ &= \frac{\{2(\omega^2 + a^2 - c^2) - b^2\}\omega^2}{(\omega^2 + a^2 - c^2)\{(1 + 2a\tau + b\tau\cos\omega\tau)^2 + (2\omega\tau - b\tau\sin\omega\tau)^2\}}.\end{aligned}$$

From this and (2.7), it follows that

$$\begin{aligned}\operatorname{sgn}\left(\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega_\pm}\right) &= \operatorname{sgn}\big(\{2(\omega_\pm^2 + a^2 - c^2) - b^2\}(\omega_\pm^2 + a^2 - c^2)\big)\\ &= \operatorname{sgn}\big(\{(b^2 \pm \sqrt{D}) - b^2\}(\omega_\pm^2 + a^2 - c^2)\big)\\ &= \operatorname{sgn}\big(\pm\sqrt{D}\,(\omega_\pm^2 + a^2 - c^2)\big).\end{aligned} \tag{2.9}$$

By (2.7) again, we have $\omega_+^2 + a^2 - c^2 > 0$, and relation (2.9) leads to

$$\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega_+} > 0 \quad \text{at } \tau = \tau_n^+ \ (b > 0) \text{ or } \tau = \sigma_n^+ \ (b < 0) \text{ for } n \in \mathbb{Z}^+.$$

If (A_1) and $c-a>0$, then $\omega_-^2+a^2-c^2<0$ by Lemma 2.3 (IIIa), and relation (2.9) gives

$$\operatorname{Re}\frac{d\lambda}{d\tau}\bigg|_{\lambda=i\omega_-}>0 \quad \text{at } \tau=\sigma_n^- \ (b>0) \text{ or } \tau=\tau_n^- \ (b<0) \text{ for } n\in\mathbb{Z}^+.$$

If (A_1) and $c-a<0$, then $\omega_-^2+a^2-c^2>0$ by Lemma 2.3 (IIIb), and relation (2.9) gives

$$\operatorname{Re}\frac{d\lambda}{d\tau}\bigg|_{\lambda=i\omega_-}<0 \quad \text{at } \tau=\tau_n^- \ (b>0) \text{ or } \tau=\sigma_n^- \ (b<0) \text{ for } n\in\mathbb{Z}^+.$$

These facts above imply the assertions in Lemma 2.9. □

Remark 2.10. In case (1.14) is satisfied, we find $\omega_+=\omega_->0$ and

$$\operatorname{Re}\frac{d\lambda}{d\tau}\bigg|_{\lambda=i\omega_+}=0 \quad \text{at } \tau=\tau_n^+=\tau_n^- \ (b>0) \text{ or } \tau=\sigma_n^+=\sigma_n^- \ (b<0) \text{ for } n\in\mathbb{Z}^+.$$

Unfortunately, we cannot determine whether the roots $\pm i\omega_+$ enter the right half-plane or the left half-plane as τ increases from $\tau_n^+=\tau_n^-$ $(b>0)$ or $\sigma_n^+=\sigma_n^-$ $(b<0)$.

Now we can prove Theorems 1.1 and 1.2. For simplicity, let $N(\tau)$ be the number of roots of (2.1) at τ including multiplicity with positive real parts.

Proof of Theorem 1.1. By virtue of Lemma 2.1, it suffices to prove that all roots of (2.1) have negative real parts if and only if any one of (1.2), (1.3), (1.4), (1.5), and (1.6) holds under assumptions $b>0$, (A_0), and $a+b+c>0$. We observe that $N(0)=0$ because (2.1) has the only root $-(a+b+c)$ for $\tau=0$, and hence, $N(\tau)=0$ for sufficiently small $\tau>0$ by the continuity of the roots with respect to τ.

Let $i\omega$ be a root of (2.1) with $\omega>0$. Our argument is divided into six cases in Remark 2.4 under $b>0$ and $a+b+c>0$.

Case (a). Lemma 2.7 (i) indicates that (2.1) has no purely imaginary roots. From Proposition 2.2, we conclude that $N(\tau)=0$ for $\tau>0$.

Case (b). Lemma 2.7 (ii) asserts that the values of ω and τ are given by

$$(\omega,\tau)=(\omega_+,\tau_n^+), \quad n\in\mathbb{Z}^+.$$

For $\tau>0$ and $n\in\mathbb{Z}^+$, let $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ be one pair of complex roots of (2.1) with $\lambda_{1,n}(\tau_n^+)=i\omega_+$ and $\lambda_{2,n}(\tau_n^+)=-i\omega_+$. Lemma 2.9 (i) shows that as τ increases from τ_n^+, the roots $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ enter the right half-plane and cannot enter the left half-plane across the imaginary axis. Taking into consideration that $\tau_n^+<\tau_{n+1}^+$ for $n\in\mathbb{Z}^+$, we conclude that $N(\tau)=0$ for $0<\tau<\tau_0^+$ and $N(\tau)\geq 2$ for $\tau>\tau_0^+$.

Case (c). Lemma 2.7 (iii) asserts that the values of ω and τ are given by

$$(\omega,\tau)=(\omega_+,\tau_n^+), \quad n\in\mathbb{Z}^+.$$

An argument similar to Case (b) yields $N(\tau)=0$ for $0<\tau<\tau_0^+$ and $N(\tau)\geq 2$ for $\tau>\tau_0^+$.

Case (d). This case cannot occur by assumption (A_0).

Case (e). As stated in Remark 2.5, this case is equivalent to the case $a - b + c > 0$ and $c - a > 0$. Lemma 2.7 (v) asserts that the values of ω and τ are given by

$$(\omega, \tau) = (\omega_+, \tau_n^+), (\omega_-, \sigma_n^-), \quad n \in \mathbb{Z}^+.$$

For $\tau > 0$ and $n \in \mathbb{Z}^+$, let $\lambda_{1,n}(\tau)$, $\lambda_{2,n}(\tau)$, $\lambda_{3,n}(\tau)$, and $\lambda_{4,n}(\tau)$ be two pairs of complex roots of (2.1) with $\lambda_{1,n}(\tau_n^+) = i\omega_+$, $\lambda_{2,n}(\tau_n^+) = -i\omega_+$, $\lambda_{3,n}(\sigma_n^-) = i\omega_-$, and $\lambda_{4,n}(\sigma_n^-) = -i\omega_-$. Lemma 2.9 (i) (resp. Lemma 2.9 (ii)) shows that as τ increases from τ_n^+ (resp. σ_n^-), the roots $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ (resp. $\lambda_{3,n}(\tau)$ and $\lambda_{4,n}(\tau)$) enter the right half-plane and cannot enter the left half-plane across the imaginary axis. Taking into consideration that $\tau_n^+ < \tau_{n+1}^+$ and $\sigma_n^- < \sigma_{n+1}^-$ for $n \in \mathbb{Z}^+$, we have

$$\min\{\tau_n^+, \sigma_n^- \mid n \in \mathbb{Z}^+\} = \min\{\tau_0^+, \sigma_0^-\}.$$

Moreover, by (2.8), we obtain

$$0 < \arccos\left(\frac{b(c-a)}{\omega_+^2 + a^2 - c^2}\right) < \pi < 2\pi - \arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right) < 2\pi,$$

which implies that

$$\tau_0^+ = \frac{1}{\omega_+} \arccos\left(\frac{b(c-a)}{\omega_+^2 + a^2 - c^2}\right) < \frac{1}{\omega_-}\left(2\pi - \arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right)\right) = \sigma_0^-.$$

Therefore, we conclude that $N(\tau) = 0$ for $0 < \tau < \tau_0^+$ and $N(\tau) \geq 2$ for $\tau > \tau_0^+$.

Case (f). Lemma 2.7 (vi) asserts that the values of ω and τ are given by

$$(\omega, \tau) = (\omega_+, \tau_n^+), (\omega_-, \tau_n^-), \quad n \in \mathbb{Z}^+.$$

For $\tau > 0$ and $n \in \mathbb{Z}^+$, let $\lambda_{1,n}(\tau)$, $\lambda_{2,n}(\tau)$, $\lambda_{3,n}(\tau)$, and $\lambda_{4,n}(\tau)$ be two pairs of complex roots of (2.1) with $\lambda_{1,n}(\tau_n^+) = i\omega_+$, $\lambda_{2,n}(\tau_n^+) = -i\omega_+$, $\lambda_{3,n}(\tau_n^-) = i\omega_-$, and $\lambda_{4,n}(\tau_n^-) = -i\omega_-$. Lemma 2.9 (i) shows that $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ enter the right half-plane as τ increases from τ_n^+. On the other hand, Lemma 2.9 (iii) shows that $\lambda_{3,n}(\tau)$ and $\lambda_{4,n}(\tau)$ enter the left half-plane as τ increases from τ_n^-. These facts mean that $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ may cross the imaginary axis from right to left as τ increases from τ_n^+. Therefore, we will investigate the order of τ_n^+ and τ_n^-. By (2.8), $0 < \omega_- < \omega_+$, and $b(c-a) < 0$, we observe that

$$0 > \frac{b(c-a)}{\omega_+^2 + a^2 - c^2} > \frac{b(c-a)}{\omega_-^2 + a^2 - c^2} > -1,$$

which yields

$$0 < \arccos\left(\frac{b(c-a)}{\omega_+^2 + a^2 - c^2}\right) < \arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right).$$

Hence, the definition of τ_n^+ and τ_n^- leads to

$$\tau_n^+ < \tau_n^-, \quad n \in \mathbb{Z}^+. \tag{2.10}$$

In addition, from the evaluation

$$\tau_{n+1}^+ - \tau_n^+ = \frac{2\pi}{\omega_+} < \frac{2\pi}{\omega_-} = \tau_{n+1}^- - \tau_n^-, \quad n \in \mathbb{Z}^+,$$

there exists a positive integer $m = m(n)$ such that

$$\tau_n^- < \tau_m^+ < \tau_{n+1}^-. \tag{2.11}$$

Moreover, the number k defined by (1.7) is the smallest nonnegative integer satisfying $\tau_{k+1}^+ < \tau_k^-$ because

$$\tau_k^- - \tau_{k+1}^+ = \tau_0^- + \frac{2k\pi}{\omega_-} - \left(\tau_1^+ + \frac{2k\pi}{\omega_+}\right) > 0$$

is equivalent to

$$k > \frac{\omega_+\omega_-(\tau_1^+ - \tau_0^-)}{2(\omega_+ - \omega_-)\pi}.$$

Therefore, by (2.10), we obtain

$$0 < \tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^- < \tau_2^+ < \ldots < \tau_{k-1}^- < \tau_k^+ < \tau_{k+1}^+ < \tau_k^- < \ldots$$

From this and (2.11), it follows that

$$\begin{cases} N(\tau) = 0 & \text{if } \tau \in (0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup (\tau_1^-, \tau_2^+) \cup \ldots \cup (\tau_{k-1}^-, \tau_k^+), \\ N(\tau) = 2 & \text{if } \tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \ldots \cup (\tau_{k-1}^+, \tau_{k-1}^-), \\ N(\tau) \geq 2 & \text{if } \tau \in (\tau_k^+, \infty), \end{cases}$$

which, together with Lemma 2.7 again, implies the conclusion as desired. The proof is now complete. □

Proof of Theorem 1.2. By virtue of Lemma 2.1, it suffices to prove that all roots of (2.1) have negative real parts if and only if any one of (1.8), (1.9), (1.10), and (1.11) holds under assumptions $b < 0$, (A_0), and $a + b + c > 0$. We observe that $N(0) = 0$ because (2.1) has the only root $-(a + b + c)$ for $\tau = 0$, and hence, $N(\tau) = 0$ for sufficiently small $\tau > 0$ by the continuity of the roots with respect to τ.

Let $i\omega$ be a root of (2.1) with $\omega > 0$. Our argument is divided into six cases in Remark 2.4 under $b < 0$ and $a + b + c > 0$.

Case (a). Lemma 2.8 (i) indicates that (2.1) has no purely imaginary roots. From Proposition 2.2, we conclude that $N(\tau) = 0$ for $\tau > 0$.

Case (b). This case cannot occur because $a - b + c > a + b + c > 0$.

Case (c). The case $a - b + c = 0$ and $2(a^2 - c^2) - b^2 < 0$ cannot occur because $a - b + c > a + b + c > 0$. We consider the case $a = c$. Lemma 2.8 (ii) asserts that the values of ω and τ are given by

$$(\omega, \tau) = (\omega_+, \tau_n^+), \quad n \in \mathbb{Z}^+.$$

For $\tau > 0$ and $n \in \mathbb{Z}^+$, let $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ be one pair of complex roots of (2.1) with $\lambda_{1,n}(\tau_n^+) = i\omega_+$ and $\lambda_{2,n}(\tau_n^+) = -i\omega_+$. Lemma 2.9 (i) shows that as τ increases from τ_n^+, the roots $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ enter the right half-plane and cannot enter the left half-plane across the imaginary axis. Taking into consideration that $\tau_n^+ < \tau_{n+1}^+$ for $n \in \mathbb{Z}^+$, we conclude that $N(\tau) = 0$ for $0 < \tau < \tau_0^+$ and $N(\tau) \geq 2$ for $\tau > \tau_0^+$.

Case (d). This case cannot occur by assumption (A_0).

Case (e). As stated in Remark 2.5, this case is equivalent to the case $c - a > 0$. Lemma 2.8 (iv) asserts that the values of ω and τ are given by

$$(\omega, \tau) = (\omega_+, \sigma_n^+),\ (\omega_-, \tau_n^-), \quad n \in \mathbb{Z}^+.$$

For $\tau > 0$ and $n \in \mathbb{Z}^+$, let $\lambda_{1,n}(\tau)$, $\lambda_{2,n}(\tau)$, $\lambda_{3,n}(\tau)$, and $\lambda_{4,n}(\tau)$ be two pairs of complex roots of (2.1) with $\lambda_{1,n}(\sigma_n^+) = i\omega_+$, $\lambda_{2,n}(\sigma_n^+) = -i\omega_+$, $\lambda_{3,n}(\tau_n^-) = i\omega_-$, and $\lambda_{4,n}(\tau_n^-) = -i\omega_-$. Lemma 2.9 (i) (resp. Lemma 2.9 (ii)) shows that as τ increases from σ_n^+ (resp. τ_n^-), the roots $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ (resp. $\lambda_{3,n}(\tau)$ and $\lambda_{4,n}(\tau)$) enter the right half-plane and cannot enter the left half-plane across the imaginary axis. Taking into consideration that $\sigma_n^+ < \sigma_{n+1}^+$ and $\tau_n^- < \tau_{n+1}^-$ for $n \in \mathbb{Z}^+$, we have

$$\min\{\sigma_n^+, \tau_n^- \mid n \in \mathbb{Z}^+\} = \min\{\sigma_0^+, \tau_0^-\}.$$

Therefore, we conclude that $N(\tau) = 0$ for $0 < \tau < \min\{\sigma_0^+, \tau_0^-\}$ and $N(\tau) \geq 2$ for $\tau > \min\{\sigma_0^+, \tau_0^-\}$.

Case (f). Lemma 2.8 (v) asserts that the values of ω and τ are given by

$$(\omega, \tau) = (\omega_+, \sigma_n^+),\ (\omega_-, \sigma_n^-), \quad n \in \mathbb{Z}^+.$$

For $\tau > 0$ and $n \in \mathbb{Z}^+$, let $\lambda_{1,n}(\tau)$, $\lambda_{2,n}(\tau)$, $\lambda_{3,n}(\tau)$, and $\lambda_{4,n}(\tau)$ be two pairs of complex roots of (2.1) with $\lambda_{1,n}(\sigma_n^+) = i\omega_+$, $\lambda_{2,n}(\sigma_n^+) = -i\omega_+$, $\lambda_{3,n}(\sigma_n^-) = i\omega_-$, and $\lambda_{4,n}(\sigma_n^-) = -i\omega_-$. Lemma 2.9 (i) shows that $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ enter the right half-plane as τ increases from σ_n^+. On the other hand, Lemma 2.9 (iii) shows that $\lambda_{3,n}(\tau)$ and $\lambda_{4,n}(\tau)$ enter the left half-plane as τ increases from σ_n^-. These facts mean that $\lambda_{1,n}(\tau)$ and $\lambda_{2,n}(\tau)$ may cross the imaginary axis from right to left as τ increases from σ_n^+. Therefore, we will investigate the order of σ_n^+ and σ_n^-. By (2.8), $0 < \omega_- < \omega_+$, and $b(c-a) > 0$, we observe that

$$0 < \frac{b(c-a)}{\omega_+^2 + a^2 - c^2} < \frac{b(c-a)}{\omega_-^2 + a^2 - c^2} < 1,$$

which yields

$$-\arccos\left(\frac{b(c-a)}{\omega_+^2 + a^2 - c^2}\right) < -\arccos\left(\frac{b(c-a)}{\omega_-^2 + a^2 - c^2}\right).$$

Hence, the definition of σ_n^+ and σ_n^- leads to

$$\sigma_n^+ < \sigma_n^-, \quad n \in \mathbb{Z}^+. \tag{2.12}$$

In addition, from the evaluation

$$\sigma_{n+1}^+ - \sigma_n^+ = \frac{2\pi}{\omega_+} < \frac{2\pi}{\omega_-} = \sigma_{n+1}^- - \sigma_n^-, \quad n \in \mathbb{Z}^+,$$

there exists a positive integer $m = m(n)$ such that

$$\sigma_n^- < \sigma_m^+ < \sigma_{n+1}^-. \tag{2.13}$$

Moreover, the number ℓ defined by (1.12) is the smallest nonnegative integer satisfying $\sigma_{\ell+1}^+ < \sigma_\ell^-$ because

$$\sigma_\ell^- - \sigma_{\ell+1}^+ = \sigma_0^- + \frac{2\ell\pi}{\omega_-} - \left(\sigma_1^+ + \frac{2\ell\pi}{\omega_+}\right) > 0$$

is equivalent to

$$\ell > \frac{\omega_+\omega_-(\sigma_1^+ - \sigma_0^-)}{2(\omega_+ - \omega_-)\pi}.$$

Therefore, by (2.12), we obtain

$$0 < \sigma_0^+ < \sigma_0^- < \sigma_1^+ < \sigma_1^- < \sigma_2^+ < \ldots < \sigma_{\ell-1}^- < \sigma_\ell^+ < \sigma_{\ell+1}^+ < \sigma_\ell^- < \ldots.$$

From this and (2.13), it follows that

$$\begin{cases} N(\tau) = 0 & \text{if } \tau \in (0, \sigma_0^+) \cup (\sigma_0^-, \sigma_1^+) \cup (\sigma_1^-, \sigma_2^+) \cup \ldots \cup (\sigma_{\ell-1}^-, \sigma_\ell^+), \\ N(\tau) = 2 & \text{if } \tau \in (\sigma_0^+, \sigma_0^-) \cup (\sigma_1^+, \sigma_1^-) \cup \ldots \cup (\sigma_{\ell-1}^+, \sigma_{\ell-1}^-), \\ N(\tau) \geq 2 & \text{if } \tau \in (\sigma_\ell^+, \infty), \end{cases}$$

which, together with Lemma 2.8 again, yields the conclusion as desired. The proof is now complete. □

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