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# SOLUTIONS OF THE NONLINEAR EVOLUTION PROBLEMS AND THEIR APPLICATIONS

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**Abstract:** In this article, a well-known technique, the variational iterative method with the Laplace transform, is used to solve nonlinear evolution problems of a simple pendulum and mass spring oscillator, which represents the duffing equation. In the variational iteration method (VIM), finding the Lagrange multiplier is an important step, and the variational theory is often used for this purpose. This paper shows how the Laplace transform can be used to find the multiplier in a simpler way. This method gives an easy approach for scientists and engineers who deal with a wide range of nonlinear problems. Duffing equation is solved by different analytic methods, but we tackle this for the first time to solve the duffing equation and the nonlinear oscillator by using the Laplace-based VIM. In the majority of cases, Laplace variational iteration method (LVIM) just needs one iteration to attain high accuracy of the answer for linearization anddiscretization, or intensive computational work is needed. The convergence criteria of this method are efficient as compared with the VIM. Comparing the analytical VIM by Laplace transform with MATLAB's built-in command Simulink that confirms the method's suitability for solving nonlinear evolution problems will be helpful. In future, we will be able to find the solution of highly nonlinear oscillators.

**Keywords:** Laplace variational iteration method, nonlinear problems, duffing equation, simple pendulum, mass and spring oscillator, Simulink

## 1. INTRODUCTION

The theory of nonlinear systems can be used to solve problems in economics, chemistry, astronomy, physiology of nerves, start of turbulence, control of heartbeats, electronic circuits, cryptography, secure communications and many other fields. In our modern world, the majority of systems are inherently nonlinear [1-2]. A collection of nonlinear equations known as a nonlinear system may be algebraic, differential, integral, fractional or a combination of these. A nonlinear system has been utilizedto describe a wide range of phenomena throughout the last few decades in the physical, social and life sciences. In natural phenomena, nonlinear dynamical systems, which describe changes in variables over time, are sometimes chaotic, unpredictable or illogical. Nonlinear oscillations play a significant role in nonlinear systems that arise in a variety of engineering applications and our daily lives.

There are many methods to solve nonlinear problems, such as the Homotopy perturbation method (HPM) [3], which is used to solve attachment oscillations that occur in nanotechnology. The nonlinear oscillation system was studied using the Akbari Ganji method [4]. The energy balance method (EBM) [5] investigated the behavior of CNTnano resonators. The He-Elzaki method [6] is used to study the biological population model. The variational iteration method (VIM)—the Pade method [7]—is applied to solve a nonlinear oscillator with cubic and restoring forces. He's multiple-scale method [8] solved nonlinear vibrations. He's parameter-expansion method [9] solved the oscillation of the mass connected to the elastic wire. The damping duffing equation is solved using the multistage differential transform method [10]. Nonlinear one-dimensional K-dV equation arising in plasma physics is solved by using the auxiliary equation mapping method[11].The Zakharov–Kuznetsov (ZK) equation is an isotropic nonlinear evolution equation; the stability analysis of two-dimensional ZK equation is derived by applying the extended direct algebraic technique[12]. The solution for geophysical Korteweg–de Vries equation (GKdVE) is found with the help of the Hirota bilinear method (HBM) [13].Marin et al. [14] solved mixed problems in thermoelasticity of type III for Cosserat media. The most famous model of nonlinear sciences namely (2 + 1)- dimensional nonlinear spin dynamics of Heisenberg ferromagnetic spin chains (HFSC) model for the evaluation of optical travelling waves by employing unified method (UM) [15]. The reciprocal impacts of Young moduli and mass density of the wave propagation behaviours of functionally graded (FG) nanobeams are investigated by Ebrahimi et al. [16].

In science and mathematics, a nonlinear system is the one where the change in output is not the same as the change in input. There is no well-established method for dealing with all types of nonlinear problems.The VIM was proposed in 1998 [17] and is widely used to solve a variety of nonlinear problems [18]. The major goal of this method is to build a correction function by using a general Lagrange multiplier that is properly chosen so that its correction solution is better than the initial trail function. A large number of results based on the VIM fail to explain it; in many situations, the integral of the correction function is convolution; as a result, a modification of the Laplace transform should be used. The Lagrange multiplier is a key part of the VIM. To do this, variational theory is used. The Lagrange multiplier is so much simpler to identify with the Laplace variational technique than with the variational theory [19].

Nonlinearity arising in the nature, science and technology does not hand easily. A number of difficulties are faced during

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finding the solutions of nonlinear phenomena. The number of the above discussed exact solution methods has no capacity to deal all kind of nonlinear physical problems. Number of the computational methods used to solve the nonlinear dynamical problems [20-22] but analytical approaches are good as compared to the numerical approach because the physics of the problem understand easily in the case analytical and semi-analytical methods. Some nonlinear problems do not handle easily because the factor stability is much important. The identification of the Lagrange multiplier in the technique requires the knowledge of the variational theory, and the complex identification process might hinder applications of the method to practical problems. This paper suggests using the Laplace transform, which can be found in all math books, to make the process of identifying things easier. In this article, a Laplace variational iteration technique, combined with the VIM and the Laplace transform, presents a numerical solution to the duffing equation by using MATLAB's built-in command Simulink that confirms suitability for solving nonlinear evolution problems.

## 2. METHOD SUMMARY

Consider a general nonlinear oscillator equation is given as follows:

$$v''(t) + f(v) = 0 \tag{2.1}$$

Initial conditions are as follows:

$$v(0) = A, v'(0) = 0$$

Eq. (2.1) can be rewrite as follows:

$$v'' + \omega^2 v + h(v) = 0 \tag{2.2}$$

where $\omega$ is the frequency which can be calculated as follows:

$$h(v) = f(v) - \omega^2 v$$

The correctional functional for Eq. (2.2) is defined as follows for the VIM:

$$v_{n+1}(t) = v_n(t) + \int_0^t \lambda(t,\eta)[v_n''(\eta) + \omega^2 v_n(\eta) + h^{\smile}(u_n)]d\eta, \quad n = 0,1,2,.... \tag{2.3}$$

It is possible to derive the generic Lagrange multiplier λ from the immobile requirements of Eq. (3) with respect to $v_n$ by using variational theory [17]. The letter h stands for the restricted variant, and the number n after it means the nth approximation. Here, the Lagrange multiplier written is this form [22]:

$$\lambda = \lambda(t - \eta)$$

corrective functional employed in Eq. (2.3) is essentially the convolution; therefore, the Laplace transform is applied. Then, Laplace transform is applied to both sides of the Eq. (2.3). In this form, the correctional functional will be converted as follows:

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] + \mathcal{L}[\int_0^t \lambda(t-\eta)[v_n''(\eta) + \omega^2 v_n(\eta) + h^{\smile}(u_n)]d\eta \tag{2.4}$$

$$= \mathcal{L}[v_n(t)] + \mathcal{L}[\lambda(t) * (v_n''(\eta) + \omega^2 v_n(\eta) + h^{\smile}(u_n))]$$

$$= \mathcal{L}[v_n(t)] + \mathcal{L}[\lambda(t)]\mathcal{L}[(v_n''(\eta) + \omega^2 v_n(\eta) + h^{\smile}(u_n))]$$

$$= \mathcal{L}[v_n(t)] + \mathcal{L}[\lambda(t)][(s^2 + \omega^2)\mathcal{L}[v_n(t)] - sv_n(0) - v_n'(0) + \mathcal{L}[h^{\smile}(u_n)]] \tag{2.5}$$

The value of λ can be determined by taking Eq. (2.5) as a stationary one with respect to $v_n(t)$:

$$\frac{\delta}{\delta v_n}\mathcal{L}[v_{n+1}(t)] = \frac{\delta}{\delta v_n}\mathcal{L}[v_n(t)] + \frac{\delta}{\delta v_n}\mathcal{L}[\lambda(t)][(s^2 + \omega^2)\mathcal{L}[v_n(t)] - sv_n(0) - v_n'(0) + \mathcal{L}[h^{\smile}(u_n)]$$

$$\{1 + \mathcal{L}[\lambda(t)](s^2 + \omega^2)\}\frac{\delta}{\delta v_n}\mathcal{L}[v_n(t)] = 0$$

From above eq., we get the following equation:

$$\mathcal{L}[\lambda(t)] = \frac{1}{s^2 + \omega^2} \tag{2.6}$$

From the above calculation, we suppose that

$$\frac{\delta}{\delta v_n}\mathcal{L}[h^{\smile}(u_n)] = 0$$

Applying the inverse Laplace transform to the Eq. (2.6), we get the following equation:

$$\lambda(t) = -\frac{1}{\omega} sinwt$$

By using Eq. (2.4), the required formula becomes in this form:

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] - \frac{1}{\omega}\mathcal{L}\left[\int_0^t sinw(t-\eta)[v_n''(\eta) + \omega^2 v_n(\eta) + \tilde{h}(v_n)]\right]d\eta$$

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] - \frac{1}{\omega}\mathcal{L}[sinwt][v_n''(\eta) + \omega^2 v_n(\eta) + \tilde{h}(v_n)] \tag{2.7}$$

As a result, the Lagrange multiplier may be found considerably more quickly than with variational theory.

## 3. APPLICATIONS

Duffing's equation is described in Section 3 where it is provided an example of nonuniformity arising from the occurrence of secular terms. We will see how the asymptotic expansion of the solution of Duffing's equation can be rendered uniform by the laplace variational iteration method (LVIM)technique. In the following paper, we will see how it arises in the description of two different mechanical systems. It also governs certain electrical systems. It is an example of a class of nonlinear oscillators which we will study in some detail. The variable v can represent a variety of quantities such as an angle of oscillation, the deformation of an elastic system, a current or a voltage. The independent variable, t, is time.

This section thoroughly examines three distinct practical examples of nonlinear oscillators.

### 3.1. Mathematical modeling of the simple pendulum

Fig. (1.1) shows a mass M attached by a rod having length L at point A. The mass M oscillates in a vertical direction due to gravity. The length of the rod is connected to a fixed point, and its weight is negligible. The force of gravity acts on the mass and tension in the rod. The tangential component of the arc of the circle on which the mass moves is driven by the force — $Mg\sin\theta$.A particle travelling on a circle with fixed radius

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experiences tangential acceleration, which equals $L(\frac{d^2\theta}{dT^2})$ and Newton's second law holds the governing Eq. [23]:

$$ML\frac{d^2\theta}{dT^2} = -Mgsin\theta \tag{3.1.1}$$

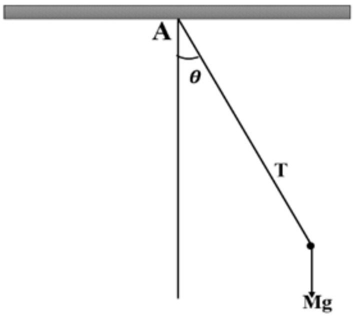


**Fig.1.** Motion of simple pendulum

Consider that the pendulum is lifted from static position when $\theta = \theta_0$ at T=0. The angle of oscillation is $v = \frac{\theta}{\theta_0}$, and the parameter related to the frequency is $\Omega = \sqrt{\frac{g}{L}}$; the governing equation can be written as follows:

$$\frac{d^2v}{dT^2} = -\Omega^2\frac{\sin(\theta_0 v)}{\theta_0} \tag{3.1.2}$$

with the initial conditions $v = 1$ and $\frac{dv}{dT} = 0$ when T=0

If $\theta_0$is very small, a truncated Maclaurin can be used to approximate the sine term. When two terms are included in the expansion, we get the following:

$$\frac{d^2v}{dT^2} = -\Omega^2(v - \frac{{\theta_0}^2v^3}{6}) \tag{3.1.3}$$

Developing nondimensional time $t = \Omega T$ and the parameter $\varepsilon = \frac{{\theta_0}^2}{6}$ results to the Duffing's equation which is non-linear in nature [23].

$$v'' + v - \varepsilon v^3 = 0 \tag{3.1.4}$$

With initial condition:

$$v(0) = 1, v'(0) = 0 \quad ,$$

The general nonoscillator form of the Eq. (1) is given as follows:

$$v'' + \omega^2 v + h(v) = 0$$

where $h(v) = -\omega^2 v - v'' - v + \varepsilon v^3$

The correctional functional is written as follows:

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] + \mathcal{L}[\int_0^t -\frac{1}{\omega}sinw(t-\eta)[v_n''(\eta) + \omega^2 v_n(\eta) + h\hat{}(u_n)]d\eta \tag{3.1.5}$$

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] - \frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[v_n'' + v_n - \varepsilon v_n^3]$$

putting $n = 0$ we have,

$$\mathcal{L}[v_1(t)] = \mathcal{L}[v_0(t)] - \frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[v_0'' + v_0 - \varepsilon v_0^3] \tag{3.1.6}$$

Assume the initial solution is $v_0(t) = Acoswt$,

$$[v_1\mathcal{L}(t)] = \mathcal{L}[Acoswt] - \frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[-A\omega^2coswt + Acoswt - \frac{\varepsilon A^3}{4}(cos3wt + 3coswt)$$

After some calculations, above expression is written as follows:

$$\mathcal{L}[v_1(t)] = \mathcal{L}[Acoswt] - \frac{1}{\omega}\left(-A\omega^2 + A - \frac{3\varepsilon A^3}{4}\right)\mathcal{L}[sinwt]\mathcal{L}[coswt] + \frac{\varepsilon A^3}{4\omega}\mathcal{L}[sinwt]\mathcal{L}[cos3wt] \tag{3.1.7}$$

Taking the inverse Laplace transform to the Eq. (3.1.7), we get the first-order approximate results in this form:

$$v_1(t) = Acoswt - \frac{1}{\omega}\left(-A\omega^2 + A - \frac{3\varepsilon A^3}{4}\right)\left(\frac{1}{2}tsinwt\right) + \frac{1}{\omega}(\frac{\varepsilon A^3}{4})(\frac{1}{8\omega}(coswt - cos3wt) \tag{3.1.8}$$

For no seculartermin Eq. (3.1.8), we have the following equation:

$$-\frac{1}{\omega}\left(-A\omega^2 + A - \frac{3\varepsilon A^3}{4}\right) = 0 \tag{3.1.9}$$

This results in the expression for the system's angular frequency.

$$\omega = \sqrt{1 - \frac{3\varepsilon A^2}{4}} \tag{3.1.10}$$

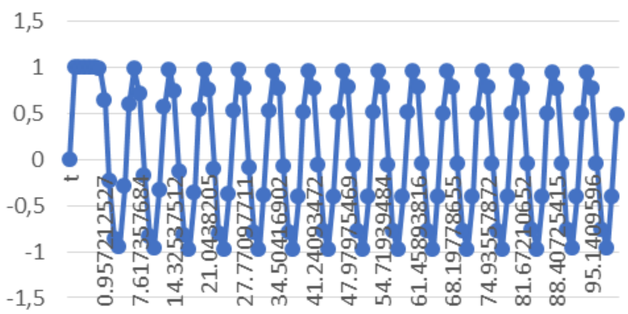


**Fig. 2.** Numerical results for simple pendulum obtained by Simulink

Mathematical modelling of the simple pendulum whose results are compared with MATLABbuilt-in command Simulinkis shown in Figs 2, 3, 5, 7, and 8. Model-based design is an approach that market-leading firms use to revolutionizedthe creation of complex systems. This approach involves the systematic use of models throughout the entire process with the assistance of built-in tools from MATLAB. Model-based systems engineering (MBSE) refers to the process of applying models in order to support the entire lifespan of a system. The development process can be bridged with Simulink from requirements and system architecture all the way to the precise component design, implementation and testing of the entire system. The solution pots for all discussed problems are also shown in Figures4and6.The graphical depiction of the general function is known as a sine wave. The sine wave can be recognised by its signature "W" shape, which indicates that it oscillates in a periodic and uniform manner both above and below 0. The sine function is a type of trigonometric function that maps the set of all nonnegative real numbers to the interval [–1,1]. This means that the sine function accepts as an input any non-negative

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real number and returns output as a value that falls somewhere in the range of [–1,1]. In the modelling of periodic occurrences and processes that adhere to recognizable cyclical patterns, the sine function and sine waves are two important building blocks.

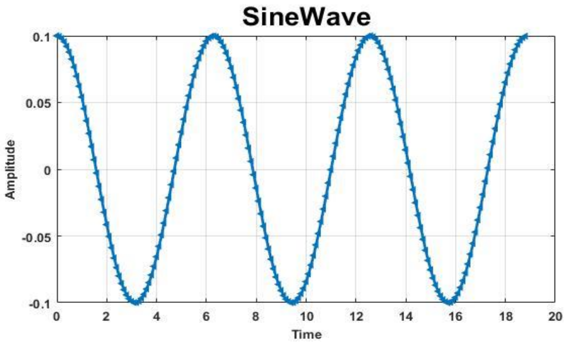


**Fig.3.** Sinewave plot for the solution of the Duffing equation of simple pendulum against $\varepsilon = 0.18, A = 0.1$

### 3.2. Mathematical modelling of a mass and spring oscillator

Suppose a mass M is attached to the fixed ends A and B. Each spring's natural length is L, and its spring constant is Λ.The distance between A and B is 2H, where H>L. The mass is released from AB by a perpendicular distance $X_0$. The mass swings along a path perpendicular to the point AB. The spring forces F fluctuate linearly with the extension while gravity is ignored [23].

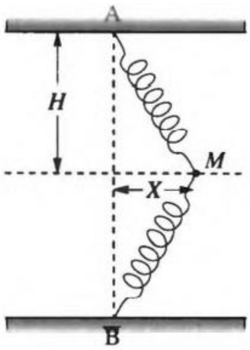


**Fig. 4.** Motion of mass and Spring oscillator

$$F = \Lambda\left(\frac{\sqrt{H^2+X^2}-L}{L}\right) \quad (3.2.1)$$

Newton's seconds law holds.

$$M\frac{d^2X}{dT^2} = -2F\cos\theta \text{ and } \cos\theta = \frac{X}{\sqrt{H^2+X^2}}$$

$$M\frac{d^2X}{dT^2} = -\frac{2\Lambda}{L}X\left(1-\frac{L}{\sqrt{H^2+X^2}}\right) \quad (3.2.2)$$

When the distance $X$ is very small as compared with $H$, then inverse square root term can be changed by truncated Maclaurin expansion.

When two terms are established in the expression, we have the following equation:

$$\frac{d^2X}{dT^2} == -\frac{2\Lambda}{L}X\left[1-\frac{L}{H}\left(1-\frac{1}{L}\frac{X^2}{H^2}\right)\right] \quad (3.2.3)$$

Similarly, the nondimensional displacement $v = \frac{X}{X_0}$ and the frequency parameter $\Omega = \sqrt{2\Lambda(H-L)MLH}$ the govern the equation;then, we have the following equation:

$$\frac{d^2v}{dT^2} = -\Omega^2\left(v + \frac{L}{2(H-L)H^2}V^3\right) \quad (3.2.4)$$

Then, for non-dimensional time $t = \Omega T$ and the parameter $\varepsilon = \frac{LX_0{}^2}{2(H-L)H_0{}^2}$, we get the duffing Eq. [23]:

$$\frac{d^2v}{dt^2} + v + \varepsilon v^3 = 0 \quad (3.2.5)$$

With initial conditions:

$$v(0) = 1\,, \qquad v'(0) = 0$$

We use the VIM by Laplace transform to solve the Eq. (3.2.5). To obtain the correctional functional, the general nonoscillator form of the Eq. (3.2.5) is given as follows:

$$v'' + \omega^2 v + h(v) = 0$$

where $h(v) = -\omega^2 v - v'' - v - \varepsilon v^3$

The correctional functional is written as follows:

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] + \mathcal{L}\left[\int_0^t -\frac{1}{\omega}sinw(t-\eta)[v_n''(\eta)+\omega^2 v_n(\eta) + h\check{}(u_n)]d\eta\right.$$

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)]\frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[v_n'' + v_n + \varepsilon v_n^3] \quad (3.2.6)$$

For an approximate solution, put $n = 0$;then, we have the following equation:

Suppose the initial condition is $v_0(t) = Acoswt$, the Eq. (3.2.7) is in this form:

$$\mathcal{L}[v_1(t)] = \mathcal{L}[v_0(t)] - \frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[v_0'' + v_0 + \varepsilon v_0^3] \quad (3.2.7)$$

$$\mathcal{L}[v_1(t)] = \mathcal{L}[Acoswt] - \frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[-A\omega^2 coswt + Acoswt + \varepsilon A^3 cos^3 wt]$$

$$\mathcal{L}[v_1(t)] = \mathcal{L}[Acoswt] - \frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[-A\omega^2 coswt + Acoswt + \frac{\varepsilon A^3}{4}(cos3wt + 3coswt)$$

After some calculations, we get the following expressions as follows:

$$\mathcal{L}[v_1(t)] = \mathcal{L}[Acoswt] - \frac{1}{\omega}\left(-A\omega^2 + A + \frac{3\varepsilon A^3}{4}\right)\mathcal{L}[sinwt]\mathcal{L}[coswt] - \frac{\varepsilon A^3}{4\omega}\mathcal{L}[sinwt]\mathcal{L}[cos3wt] \quad (3.2.8)$$

Applying the inverse Laplace transform to the Eq. (3.2.8), we get the first-order approximate results in this form:

For no seculartem in Eq. (3.2.9), we have the following equation:

$$v_1(t) = Acoswt - \frac{1}{\omega}\left(-A\omega^2 + A + \frac{3\varepsilon A^3}{4}\right)\left(\frac{1}{2}tsinwt\right) - \frac{1}{\omega}\left(\frac{\varepsilon A^3}{4}\right)\left(\frac{1}{8\omega}\right)(coswt - cos3wt) \quad (3.2.9)$$

$$-\frac{1}{\omega}\left(-A\omega^2 + \frac{3\varepsilon A^3}{4}\right) = 0 \quad (3.2.10)$$

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This results in the expression for the system's angular frequency.

$$\omega = \sqrt{1+\frac{3\varepsilon A^2}{4}} \qquad (3.2.11)$$

The expression for angular frequency of the second problem in Eq. (3.2.11) is exactly the same Tables 1 and 2 as obtained by the EBM [27] in Eq. (4.4)

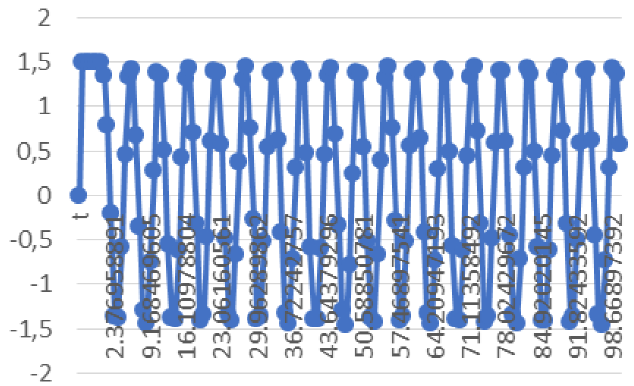


**Fig.5.** Numerical results for Mass and spring oscillator obtained by Simulink A1

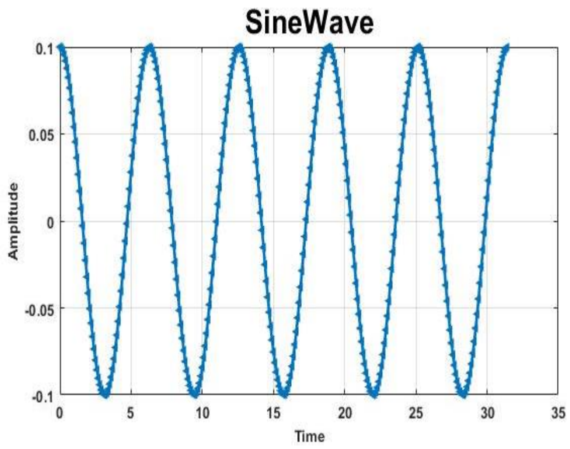


**Fig.6.** Sinewave plot for the solution of the Duffing equation of Mass and Spring Oscillator against $\varepsilon = 0.18, A = 0.1$

**Tab. 1.** Comparison of obtained results of different kinds of analytical approaches for Eq. (3.2.5)

| $\varepsilon A^2$ | $\omega_{FEBM-C}$[30] | $\omega_{SHBM}$[32] | $\omega_{REBM}$[27] | $\omega_{LVIM}$ |
|---|---|---|---|---|
| 0.5 | 1.1619<br>0.76% | 1.1707<br>0.001% | 1.1707<br>0.001% | 1.1707<br>0.001% |
| 1 | 1.3038 | 1.3178 | 1.3178 | 1.3178 |
| | 1.06% | 0.0043% | 0.0043% | 0.0043% |
| 5 | 2.1213 | 2.1509 | 2.1507 | 2.1507 |
| | 1.35% | 0.0262% | 0.015% | 0.015% |
| 10 | 2.8284 | 2.8678 | 2.8672 | 2.8672 |
| | 1.33% | 0.0419% | 0.0217% | 0.0217% |
| 100 | 8.4261 | 8.5390 | 8.5360 | 8.5360 |
| | 1.25% | 0.0643% | 0.0287% | 0.0287% |
| 1000 | 26.476 | 26.8289 | 26.8187 | 26.8187 |
| | 1.24% | 0.0681% | 0.03011% | 0.03011% |
| 5000 | 59.169 | 59.9563 | 59.9337 | 59.9337 |
| | 1.24% | 0.06783% | 0.0301% | 0.0301% |

### 3.3. Mathematical modelling of the cubic quintic order duffing equation

Consider a fifth-order nonlinear duffing equation in this form [24-26]:

$$v'' + av + bv^3 + cv^5 = 0 \qquad (3.3.1)$$

With initial condition,

$$v(0) = 1\,, \qquad v'(0) = 0$$

To obtain the correctional functional, the general nonlinear oscillator form of the Eq. (3.3.1) is given as follows:

$$v'' + \omega^2 v + h(v) = 0$$

where $h(v) = -\omega^2 v - v'' - av - bv^3 - cv^5$

The correctional functional of Eq. (3.3.1) can be rewrite as follows:

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] + \mathcal{L}[\int_0^t -\frac{1}{\omega} sinw(t-\eta)[v_n''(\eta) + \omega^2 v_n(\eta) + h\tilde{}(u_n)]d\eta \qquad (3.3.2)$$

Putting n=0 in Eq. (3.3.2), we get the following equation:

$$\mathcal{L}[v_1(t)] = \mathcal{L}[v_0(t)] - \frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[v_0'' + av_0 + bv_0^3 + cv_0^5] \qquad (3.3.3)$$

Let us consider the initial condition is $v_0(t) = Acoswt$, then Eq. (3.3.3) in this form:

$$\mathcal{L}[v_1(t)] = \mathcal{L}[Acoswt] - \frac{1}{\omega}\mathcal{L}[sinwt]\mathcal{L}[-A\omega^2 coswt + Aacoswt + A^3bcos^3wt + A^5ccos^5wt]$$

After some modification, the above expression can be written as follows:

$$\mathcal{L}[v_1(t)] = \mathcal{L}[Acoswt] - \frac{1}{\omega}\left(-A\omega^2 + Aa + \frac{3bA^3}{4} + \frac{5cA^5}{8}\right)\mathcal{L}[sinwt]\mathcal{L}[coswt] - \frac{1}{\omega}\left(\frac{5cA^5}{16} + \frac{1bA^3}{4}\right)\mathcal{L}[sinwt]\mathcal{L}[cos3wt] - \frac{1}{\omega}\left(\frac{A^5c}{16}\right)\mathcal{L}[sinwt]\mathcal{L}[cos5wt] \qquad (3.3.4)$$

Taking the inverse Laplace transform on Eq. (3.3.4), we obtain the first-order approximate results as follows:

For no secularterm in Eq. (3.3.5), we get the following equation:

$$v_1(t) = Acoswt - \frac{1}{\omega}\left(-A\omega^2 + Aa + \frac{3bA^3}{4} + \frac{5cA^5}{8}\right)\left(\frac{1}{2}tsinwt\right) - \frac{1}{\omega}\left(\frac{5cA^5}{16} + \frac{1bA^3}{4}\right)\left(\frac{1}{8\omega}\right)(coswt - cos3wt) - \frac{1}{\omega}\left(\frac{A^5c}{16}\right)\left(\frac{1}{24\omega}\right)(coswt - cos5wt) \qquad (3.3.5)$$

$$-\frac{1}{\omega}\left(-A\omega^2 + Aa + \frac{3bA^3}{4} + \frac{5cA^5}{8}\right) = 0$$

which leads to the expression for the system's angular frequency.

$$\omega^2 = 1 + \frac{3bA^2}{4} + \frac{5cA^4}{8} \qquad (3.3.6)$$

$$\omega = \sqrt{1 + \frac{3bA^2}{4} + \frac{5cA^4}{8}} \qquad (3.3.7)$$

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The angular frequency of the problem third in Eq. (3.3.7) is also similar as calculated by the HPM [28] in Eq. (13) and gamma function method [29] in Eq. (3.7).

The Simulink comparison with the exact solution is as follows:

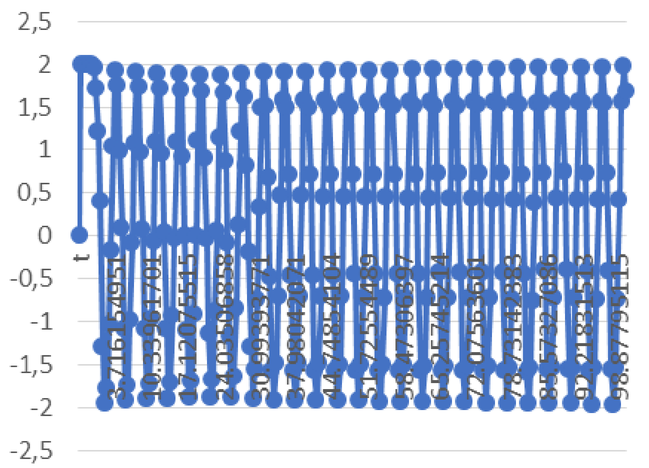


**Fig. 7.** Numerical results for Fifth order duffing equation obtained by Simulink A2

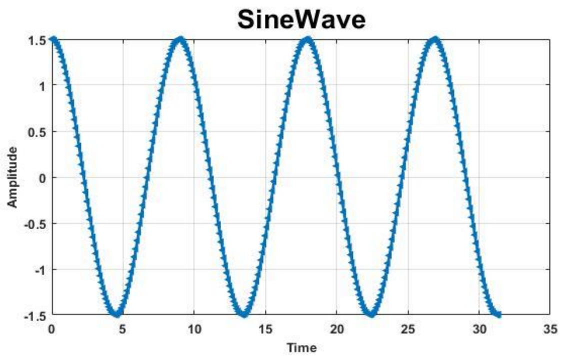


**Fig.8.** Sinewave plot for the solution of the fifth order Duffing equation against $\varepsilon = 0.3, A = 1.5$

**Tab.2.**Comparison of the approximate frequency of Eq. (3.3.6) with different A

| $A$ | GFM [29] | Ref [33] | Laplace-based VIM | Relative error |
|---|---|---|---|---|
| 0.1 | 1.003774 | 1.003759 | 1.003774 | 0.0015 |
| 0.2 | 1.015382 | 1.015135 | 1.015382 | 0.0243 |
| 0.4 | 1.065833 | 1.062073 | 1.065833 | 0.3540 |
| 0.6 | 1.162325 | 1.144771 | 1.162325 | 1.5334 |
| 0.8 | 1.317574 | 1.268069 | 1.317574 | 3.9040 |
| 1 | 1.541104 | 1.436141 | 1.541104 | 7.3087 |

Nomenclature Table: $\omega$ – angular frequency, $\lambda$ – Lagrange multiplier, $t$ – time, $v$ – angle of oscillation or deformation of elastic system, $\varepsilon v^3$ – perturbation non-linear term, $A\cos\omega t$ – initial trial function, $\mathcal{L}$ – Laplace transform, $\Omega T$– non-dimensional time, VIM – variational iteration method, LVIM – Laplace variational iteration method, ZK – Zakharov–Kuznetsov, GKdVE– geophysical Korteweg–de Vries equation, HBM – Hirota bilinear method, FG – functionally graded

## 4. CONCLUDING REMARKS

In this article, we use the Laplace transform to find the Lagrange multiplier quickly and easily. Laplace transform is a very powerful mathematical tool that is used in many areas of science and engineering. As engineering problems become significantly more difficult, Laplace transforms can help solve them using a simple method. Scientists who are trying to solve nonlinear problems can use the VIM by finding the Lagrange multiplier. This article will talk about how Laplace transforms are used in physics, and then, it will talk about how they are used in the simple pendulum, mass and spring oscillator. In the field of power systems engineering, a more complicated way to use load frequency control is also discussed in my recent research.The LVIM method is applied to solve the nonlinear problems, but there are a lot of nonlinear problems which are not solvable by this method because the limitations of our work are given as follows: if only the first derivative appears in the differential equation or only a constant term is added in the differential equation, noninformality occurs in the solution two times, and we obtained two different values of angular frequency and that is why we are unable to find the amplitude frequency relationship. Moreover, the Lagrange multiplier of higher order for this method is not established; so,the researcher tries to find the Lagrange multiplier of higher order and solves the highly nonlinear problems in future.

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## APPENDIX

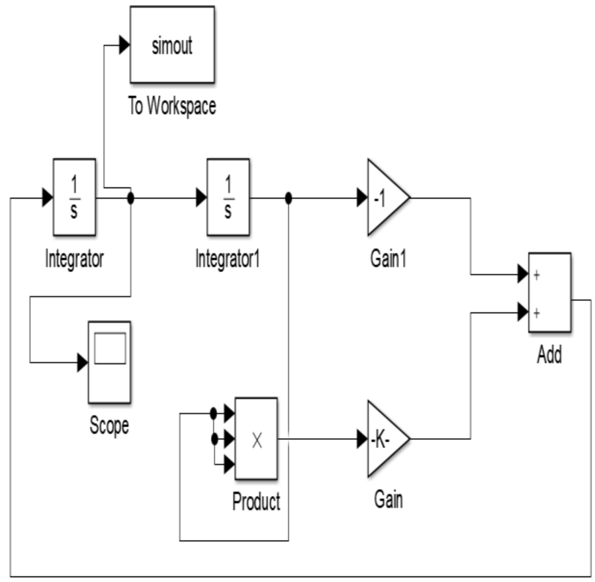


**Fig. A1.** Simulink model for duffing equation

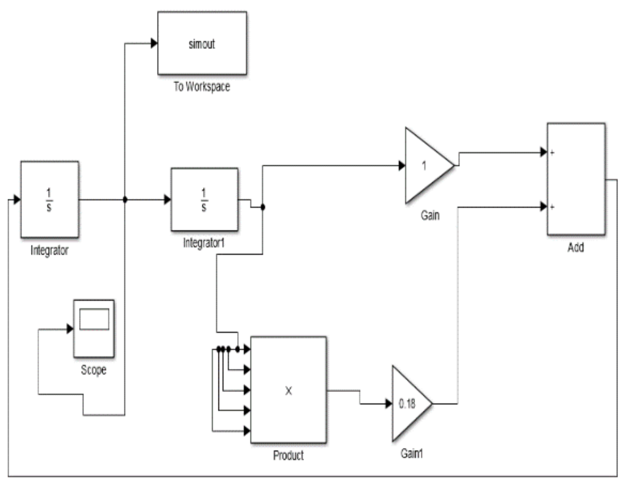


**Fig.A2.** Simulink model for duffing equation against $\varepsilon = 0.18$