ARCHIVE OF MECHANICAL ENGINEERING

VOL. LX 2013 Number 4

10.2478/meceng-2013-0030

Key words: Fourier's theory, heat conduction, ring-like body

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A ONE-DIMENSIONAL MODEL FOR THE HEAT CONDUCTION IN RING-LIKE BODY

A one-dimensional model based on the Fourier's theory of heat conduction is developed for ring-like bodies. The ring-like body is an incomplete or complete torus with arbitrary cross section. The thermal properties of considered rings are independent of the polar angle. Examples illustrate the application of model presented.

List of symbols

a	area of the cross section
B_i	Fourier's coefficients ($i = 0, 1, 2, \ldots$)
c	specific heat
C	thermal cross-sectional property defined by Eq. (14)
D_k, E_k	Fourier's coefficients ($i = 0, 1, 2, \ldots$)
$\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$, $\boldsymbol{e}_z$	unit vectors of the cylindrical coordinate system (O; r, φ, z)
$f_i = f_i(\varphi)$	eigenfunction ($i = 1, 2, \ldots$) defined by Eq. (42)
$F = F(r, \varphi, z)$	initial value of the temperature
h	internal heat source
H	heat source "resultant" defined by Eq. (16)
k_1, k_2	constants defined by Eq. (32)
$\boldsymbol{n}_1 = -\boldsymbol{e}_\varphi$	outer normal vector of ∂B_1
$\boldsymbol{n}_2 = \boldsymbol{e}_\varphi$	outer normal vector of ∂B_2
$\boldsymbol{n}_3 = n_r\boldsymbol{e}_r + n_z\boldsymbol{e}_z$	outer normal vector to the boundary surface segment ∂B_3
p_1, p_2, p_3	are prescribed surface heat sources defined on ∂B_1, ∂B_2 and ∂B_3 respectively
$q_i = \boldsymbol{q} \cdot \boldsymbol{n}_i$	heat flux ($i = 1, 2, 3$)

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p_1, p_2, p_3	are prescribed surface heat sources defined on ∂B_1, ∂B_2 and ∂B_3 respectively
$q_i = \boldsymbol{q} \cdot \boldsymbol{n}_i$	heat flux $(i = 1, 2, 3)$
$\boldsymbol{q}$	heat flux vector $(\boldsymbol{q} = q_r(r, \varphi, z)\boldsymbol{e}_r + q_\varphi(r, \varphi, z)\boldsymbol{e}_\varphi + q_z(r, \varphi, z)\boldsymbol{e}_z)$
Q	heat flux ”resultant” defined by Eq. (25)
r	radial coordinate
r_c	the radial coordinate of the centre of cross section
R	cross-sectional property of circular cross section defined by Eq. (28)
s	arc coordinate defined by boundary curve ∂A
$T = T(r, \varphi, z, \tau)$	temperature field
T_1, T_2	given temperature of the end cross sections used in Eq. (26)
$\boldsymbol{v} = v\boldsymbol{e}_\varphi$	given velocity field
z	polar coordinate

Greek symbols

α_1, α_2, α_3	convective heat transfer coefficient to surfaces ∂B_1, ∂B_2 and ∂B_3
β_1, β_2	cross-sectional properties defined by Eq. (19)
$\beta_0 = \beta_1 = \beta_2$	constants, used in Eqs. (38) and (40)
β_3	modified coefficient of heat transfer defined by Eq. (13)
γ_i	positive roots of Eq. (43), $(i = 1, 2, \dots)$
θ_1, θ_2, θ_3	given temperatures
λ	thermal conductivity
Λ	thermal cross-sectional property defined by Eq. (15)
ρ	density
τ	time
φ	polar coordinate
ϕ	polar coordinate of end cross section (for open ring $\phi < 2\pi$; for closed ring $\phi = 2\pi$)
ω	angular velocity

1. Introduction

The aim of this paper is to reduce the solution of the three-dimensional problem of heat conduction in an inhomogeneous ring-like body to the solution of a one-dimensional problem. The formulation of the one-dimensional theory of the heat conduction is based on the field equations and boundary conditions of the three-dimensional theory of heat conduction in solid body. The presented approach is similar to the solution of the three-dimensional elasticity problem of rod-shaped bodies by the application of the one-dimensional theory of elastic beams [5,6]. The approximate solution based on the one-dimensional solution can be applied for curved beam of uniform cross-section, whose cross section is much smaller than as the length of the center line of the beam like body. This statement is supported by Example 4.5, in which the presented method is compared with a FEM solution.

In this paper, the interest is focused on heat conduction problems in ring-like bodies. The presented model is based on the Fourier's theory of heat conduction. The ring-like body is generated by the rotation of a plane figure whose inner and outer boundary points are taken from the set A and ∂A, respectively. The domain $\overline{A} = A \cup \partial A$ is the "cross section" of the ring-like body, which is a bounded plane domain. The ring-like body (torus) occupies the region $\overline{B} = B \cup \partial B$, $B = A \times (0, \phi)$, $\partial B = \partial B_1 \cup \partial B_2 \cup \partial B_3$, $\partial B_1 = \{(r, \varphi, z) \big| (r, z) \in A,\ \varphi = 0\}$, $\partial B_2 = \{(r, \varphi, z) \big| (r, z) \in A,\ \varphi = \phi\}$, $\partial B_3 = \partial A \times [0, \phi]$ which are referred to a given cylindrical coordinate system $(0;\, r, \varphi, z)$ whose z axis is the axis of rotation. The angle of rotation is ϕ $(0 < \phi < 2\pi)$, for complete ring $\phi = 2\pi$ and $\partial B = \partial B_3$.

It is assumed that the thermal properties do not depend on the angle coordinate φ, but they may depend on the cross-sectional coordinates r and z.

Fig. 1 illustrates the cross section of the ring-like body in a meridian plane φ = constant.

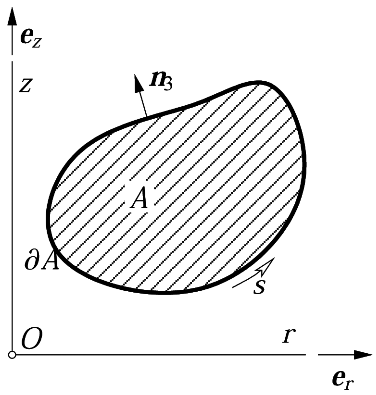

Fig. 1. Cross section of a ring-like body

The problem of heat conduction in the body is ruled by the balance equation [1, 2, 3]

$$-\nabla \cdot \boldsymbol{q} + h = c\rho \left(\frac{\partial T}{\partial \tau} + (\boldsymbol{v} \cdot \nabla) T \right) \tag{1}$$

and by the Fourier's law

$$\boldsymbol{q} = -\lambda \nabla T \tag{2}$$

in B with the boundary and initial conditions

$$q_1 = \boldsymbol{q} \cdot \boldsymbol{n}_1 = \alpha_1 (T - \theta_1) - p_1 \quad \text{on } \partial B_1, \tag{3}$$

$$q_2 = \boldsymbol{q} \cdot \boldsymbol{n}_2 = \alpha_2 (T - \theta_2) - p_2 \quad \text{on } \partial B_2, \tag{4}$$

$$q_3 = \boldsymbol{q} \cdot \boldsymbol{n}_3 = \alpha_3 (T - \theta_3) - p_3 \quad \text{on } \partial B_3, \tag{5}$$

$$T(r, \varphi, z, 0) = F(r, \varphi, z) \quad \text{in } B. \tag{6}$$

In Eqs. (3), (4) and (5) putting $\alpha_i = 0$ $(i = 1, 2, 3)$ we obtain the case of prescribed heat flux condition which is a Neumann's type boundary condition. The prescribed surface temperature, the case of Dirichlet's boundary condition can be obtained from Eqs. (3), (4), (5) by $\alpha_i \to \infty$ $(i = 1, 2, 3)$.

2. One-dimensional model

Eq. (1) in cylindrical coordinates r, φ, z can be written in the next form [4,5]

$$-\left[\frac{\partial}{\partial r}(r q_r) + \frac{\partial}{\partial z}(r q_z)\right] - \frac{\partial q_\varphi}{\partial \varphi} + r h = c \rho r \left(\frac{\partial T}{\partial \tau} + \omega \frac{\partial T}{\partial \varphi}\right). \tag{7}$$

Here, we have assumed that the velocity field is given by the equation

$$\boldsymbol{v} = v \boldsymbol{e}_\varphi = r \omega \boldsymbol{e}_\varphi, \quad (\omega = \text{constant}). \tag{8}$$

Integrating Eq. (7) over A and using the Stokes' theorem [4, 5] we get

$$-\int_{\partial A} q_3 r \, \mathrm{d}s - \int_A \frac{\partial q_\varphi}{\partial \varphi} \mathrm{d}A + \int_A r h \, \mathrm{d}A = \int_A c \rho r \left(\frac{\partial T}{\partial \tau} + \omega \frac{\partial T}{\partial \varphi}\right) \mathrm{d}A. \tag{9}$$

According to the Fourier's law we can write

$$q_\varphi = -\frac{\lambda}{r} \frac{\partial T}{\partial \varphi}. \tag{10}$$

The governing supposition is that the temperature does not depend on the cross-sectional coordinates r and z, i.e. $T = T(\varphi, \tau)$. Inserting this assumption into Eq. (10) and Eq. (9) we obtain the field equation of the one-dimensional model for the heat conduction in ring-like body as

$$C\left(\frac{\partial T}{\partial \tau} + \omega \frac{\partial T}{\partial \varphi}\right) = \Lambda \frac{\partial^2 T}{\partial \varphi^2} - \beta_3 (T - \theta_3) + H, \quad 0 < \varphi < 2\pi. \tag{11}$$

Here, we have used

$$\int_{\partial A} q_3 r \, \mathrm{d}s = \int_{\partial A} \boldsymbol{q} \cdot \boldsymbol{n}_3 r \, \mathrm{d}s = \int_{\partial A} \alpha_3 \left(T - \theta_3\right) \, r \, \mathrm{d}s - \int_{\partial A} r p_3 \, \mathrm{d}s, \tag{12}$$

where

$$\beta_3 = \int_{\partial A} \alpha_3 r \, \mathrm{d}s \tag{13}$$

and we have introduced the next thermal cross-sectional properties

$$C = \int_A \rho(r, z) c(r, z) \, r \, \mathrm{d}A, \tag{14}$$

$$\Lambda = \int_A \frac{\lambda(r, z)}{r} \mathrm{d}A \tag{15}$$

and H is defined as

$$H = \int_A r h \, \mathrm{d}A + \int_{\partial A} p_3 r \, \mathrm{d}s. \tag{16}$$

We note, the temperature of the surrounding medium depends only on φ and τ, i.e., $\theta_3 = \theta_3(\varphi, \tau)$.

3. Boundary and initial conditions

The boundary condition formulated by Eq. (5) is built in the field equation (11), but we note that in this model the case $\alpha_3 \to \infty$ is not possible since in this case, $T \to \theta_3$, i.e. the Eq. (11) cannot be satisfied. In order to obtain the one-dimensional version of boundary conditions (3) and (4) we integrate Eqs. (3) and (4) over A. The integration yields the result

$$\Lambda \frac{\partial T}{\partial \varphi} = \beta_1 \left(T - \hat{\theta}_1\right) - \hat{p}_1, \quad \varphi = 0, \tag{17}$$

$$-\Lambda \frac{\partial T}{\partial \varphi} = \beta_2 \left(T - \hat{\theta}_2\right) - \hat{p}_2, \quad \varphi = \phi, \quad \phi \neq 2\pi, \tag{18}$$

where

$$\hat{\theta}_i = \frac{1}{\beta_i} \int_A \alpha_i(r, z) \theta_i(r, z) \, \mathrm{d}A, \qquad \beta_i = \int_A \alpha_i \, \mathrm{d}A, \tag{19}$$

$$\hat{p}_i = \int_A p_i(r, z) \, \mathrm{d}A \qquad (i = 1, 2). \tag{20}$$

Boundary conditions above formulated are referred to open ring-like body ($\phi \neq 2\pi$).

In the case of closed ring ($\phi = 2\pi$) we do not have boundary conditions at $\varphi = 0$ and $\varphi = 2\pi$ as given in Eqs. (17) and (18). For closed ring we have

$$T(0) = T(2\pi), \qquad \left(\frac{\partial T}{\partial \varphi}\right)_{\varphi=0} = \left(\frac{\partial T}{\partial \varphi}\right)_{\varphi=2\pi}. \tag{21}$$

Eq. (21) expresses that T is periodic in φ with period 2π.

Integration of Eq. (6) gives the initial condition of the one-dimensional heat conduction model

$$T(\varphi, 0) = \frac{1}{a} \int_A F(r, \varphi, z)\, \mathrm{d}A, \tag{22}$$

since it was assumed that T does not depend on the cross-sectional coordinates r and z.

4. Examples

4.1. Heat flow in incomplete torus

A steady state heat conduction problem is considered, the ring-like body is an incomplete torus, i.e. $0 \leq \varphi \leq \phi < 2\pi$. Let $p_3 = 0$ $\alpha_3 = 0$, $\omega = 0$ and $h = 0$. In this case we have

$$\frac{\partial^2 T}{\partial \varphi^2} = 0, \qquad 0 < \varphi < \phi, \tag{23}$$

that is

$$T = T_1 + \frac{T_2 - T_1}{\phi} \varphi, \qquad 0 \leq \varphi \leq \phi, \tag{24}$$

where $T_1 = T(0)$, $T_2 = T(\phi)$. The heat flux resultant Q is defined as

$$Q = \int_A q_\varphi\, \mathrm{d}A = -\int_A \frac{\lambda}{r} \frac{\partial T}{\partial \varphi}\, \mathrm{d}A. \tag{25}$$

A simple calculation gives

$$Q = -\frac{\Lambda}{\phi} (T_2 - T_1). \tag{26}$$

If the material is homogeneous then we have

$$\Lambda = \frac{\lambda a}{R}, \qquad \frac{a}{R} = \int_A \frac{\mathrm{d}A}{r}. \tag{27}$$

For solid circular cross section R can be obtained by the next formula [6] (Fig. 2):

$$R = \frac{b^2}{2\left(r_1 - \sqrt{r_1^2 - b^2}\right)}. \tag{28}$$

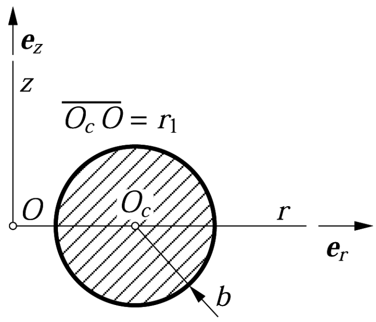

Fig. 2. Circular cross section

4.2. Heat flow in closed ring caused by a rotating line heat source

In this example, we determine the temperature field for a closed ring which is caused by a rotating line heat source which acts on a meridian section of ∂B_3 at time τ and its definition is given as

$$\int_{\partial A} p_3 r \,\mathrm{d}s = P_3 \delta(\varphi - \omega\tau), \qquad P_3 = \text{contant} \tag{29}$$

Here, we note the unit of line heat source P_3 is [Watt].

In Eq. (29) $\delta(\varphi)$ is the Dirac function. According to Eq. (16) we can write

$$H = P_3 \delta(\varphi - \omega\tau), \tag{30}$$

since $h = 0$. Let the function $f = f(\varphi)$ be defined as

$$f(\varphi) = \delta(\varphi - \vartheta) \quad 0 \le \varphi \le 2\pi, \text{ and } f(\varphi) = f(\varphi + 2n\pi) \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{31}$$

Representation of $f(\varphi)$ by its Fourier' series is as follows [7]

$$f(\varphi) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{i=1}^{\infty} \cos(i\varphi - i\vartheta)\right]. \tag{32}$$

Combination of Eq. (30) with Eq. (32) ($\vartheta = \omega\tau$) gives

$$H = \frac{P_3}{\pi}\left[\frac{1}{2} + \sum_{i=1}^{\infty} \cos(i\varphi - i\omega\tau)\right]. \tag{33}$$

The ring looses heat by convention into the surrounding medium at temperature $\theta_3 = 0$ according to the Newton's law of cooling. For the present problem Eq. (11) has the form

$$C\left(\frac{\partial T}{\partial \tau} + \omega\frac{\partial T}{\partial \varphi}\right) = \Lambda\frac{\partial^2 T}{\partial \varphi^2} - \beta_3 T + \frac{P_3}{\pi}\left[\frac{1}{2} + \sum_{i=1}^{\infty} \cos\left(i\varphi - i\omega\tau\right)\right]. \tag{34}$$

We consider only the "large time solution" of Eq. (34). The large time solution is valid for $\tau >> 0$ and its determination does not require initial condition [1]. The temperature field belonging to $\tau >> 0$ is a periodic function of the time coordinate τ.

We assume that

$$T(\varphi, \tau) = T_0 + \sum_{i=1}^{\infty} T_i \cos\left(i\varphi - i\omega\tau\right). \tag{35}$$

Substitution of Eq. (35) into Eq. (34) gives

$$T_0 = \frac{P_0}{2\pi\beta_3}, \tag{36}$$

$$T_i = \frac{P_3}{i^2\Lambda + \beta_3\pi} \qquad (i = 1, 2, \dots) \tag{37}$$

The final formula of the large time solution is represented as

$$T(\varphi, \tau) = \frac{P_3}{2\pi\beta_3}\left[1 + \sum_{i=1}^{\infty} \frac{2\beta_3}{\Lambda i^2 + \beta_3} \cos\left(i\varphi - i\omega\tau\right)\right]. \tag{38}$$

For homogeneous curved beam of solid circular cross section (Fig. 2) by the data $\lambda = 400\,\mathrm{W/mK}$, $\alpha_3 = 5\,\mathrm{W/m^2K}$, $b = 15\,\mathrm{mm}$, $r_1 = 35\,\mathrm{mm}$, $\omega = 0.5$ 1/s the graph of the function $\hat{T}(\varphi, \tau) = \dfrac{2\pi\beta_3}{P_3}\widetilde{T}(\varphi, \tau)$ is shown in Fig. 3, where

$$\widetilde{T} = \frac{P_3}{2\pi\beta_3}\left[1 + \sum_{i=1}^{10} \frac{2\beta_3}{\Lambda i^2 + \beta_3} \cos\left(i\varphi - i\omega\tau\right)\right] \tag{39}$$

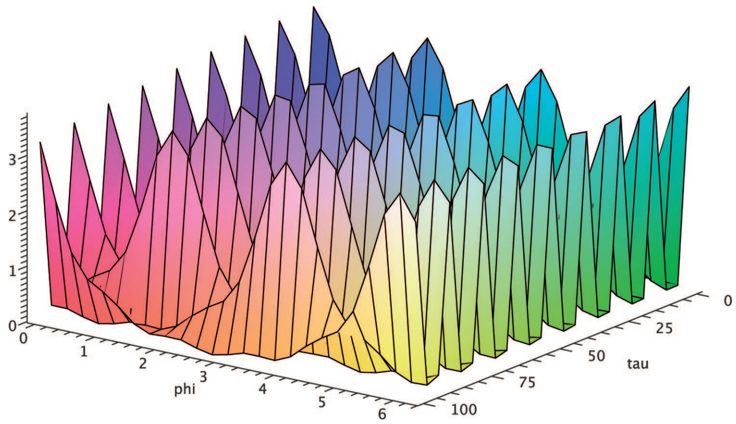

Fig. 3. The graph of $\hat{T}(\varphi, \tau)$

4.3. Closed ring is allowed to cool by convention

The surrounding medium has zero temperature $\theta_3 = 0$ and the initial temperature of the ring is given by its Fourier series as

$$T(\varphi, 0) = \sum_{k=0}^{\infty} (D_k \cos k\varphi + E_k \sin k\varphi) . \tag{40}$$

In the present case, the ring is fixed $\omega = 0$ and it is assumed that there are no internal heat source in B and there is no surface heat source on ∂B_3, that is, we have $h = p_3 = 0$. It is very easy to check all the conditions of cooling problem considered are satisfied if

$$T(\varphi, \tau) = \sum_{k=0}^{\infty} \exp\left[-\left(k^2\Lambda + \beta_3\right)\frac{\tau}{C}\right](D_k \cos k\varphi + E_k \sin k\varphi) . \tag{41}$$

We note, if

$$T(\varphi, 0) = T_0 = \text{constant}, \quad (0 \le \varphi \le 2\pi) \tag{42}$$

then (41) becomes

$$T(\varphi, \tau) = T_0 \exp\left(-\frac{\beta_3 \tau}{C}\right). \tag{43}$$

4.4. Split ring is allowed to cool by convection

The closed ring considered in Section 4.3 at $\varphi = 0$ is opened by a splitting. Same data are used as in the case of closed ring. The boundary conditions according to Eqs. (17), (18) at $\varphi = 0$ and $\varphi = 2\pi$ are as follows for the case of $\hat{p}_1 = \hat{p}_2 = 0$ and $\hat{\theta}_1 = \hat{\theta}_2 = 0$:

$$-\Lambda \frac{\partial T}{\partial \varphi} + \beta_0 T = 0 \qquad \varphi = 0, \tag{44}$$

$$\Lambda \frac{\partial T}{\partial \varphi} + \beta_0 T = 0 \qquad \varphi = 2\pi. \tag{45}$$

Here, we assume that $\beta_1 = \beta_2 = \beta_0$.

Both the "field" equation (11) and boundary conditions (44), (45) and initial condition (22) are satisfied if

$$T(\varphi, \tau) = \sum_{i=1}^{\infty} B_i \exp\left(-\frac{\gamma_i^2 \Lambda + \beta_3}{C}\tau\right) f_i(\varphi), \tag{46}$$

where

$$f_i(\varphi) = \beta_0 \sin \gamma_i \varphi + \Lambda \gamma_i \cos \gamma_i \varphi, \quad 0 \le \varphi \le 2\pi, \quad (i = 1, 2, \dots), \tag{47}$$

and γ_i is the positive roots of the transcendental equation

$$\tan 2\pi\gamma = \frac{2\beta_0 \gamma \Lambda}{(\gamma\Lambda)^2 - \beta_0^2} \tag{48}$$

and

$$B_i = \frac{1}{D_i} \int_0^{2\pi} T(\varphi, 0) f_i(\varphi)\, \mathrm{d}\varphi \tag{49}$$

$$D_i = \int_0^{2\pi} f_i^2(\varphi)\, \mathrm{d}\varphi = \pi \left[\gamma_i^2 \Lambda^2 + \beta_0^2\right] + \Lambda \beta_0 \sin^2 2\pi\gamma_i + \frac{\sin 4\pi\gamma_i}{4\gamma_i}\left[-\beta_0^2 + \gamma_i^2 \Lambda^2\right], \quad (i = 1, 2, \dots). \tag{50}$$

We note, if

$$T(\varphi, 0) = T_0 = \text{constant} \tag{51}$$

then

$$B_i = \frac{\beta_0 (1 - \cos 2\pi\gamma_i) + \Lambda \gamma_i \sin 2\pi\gamma_i}{\gamma_i D_i} T_0 \quad (i = 1, 2, \dots). \tag{52}$$

The derivation of solution (46) is based on the integral transform technique described in [1].

4.5. Numerical solution of heat conduction problem for split ring

In this section, a numerical solution is presented for the transient heat conduction problem for a composite split ring of rectangular cross section. The numerical solution is based on the formulae derived in Section 4.4 of this paper. The considered cross section is shown in Fig. 3. The following data are used:

$$r_1 = 0.8\,\mathrm{m}, \quad r_2 = 0.85\,\mathrm{m}, \quad r_3 = 0.9\,\mathrm{m}, \quad t = 0.01\,\mathrm{m}$$

$$\rho_1 = 7850\,\frac{\mathrm{kg}}{\mathrm{m}^3}, \quad \rho_2 = 8960\,\frac{\mathrm{kg}}{\mathrm{m}^3}, \quad \lambda_1 = 400\,\frac{\mathrm{W}}{\mathrm{mK}}, \quad \lambda_2 = 60\,\frac{\mathrm{W}}{\mathrm{mK}},$$

$$c_1 = 385\,\frac{\mathrm{J}}{\mathrm{kg\,K}}, \quad c_2 = 460\,\frac{\mathrm{J}}{\mathrm{kg\,K}}$$

$$\alpha_1 = 20\,\frac{\mathrm{W}}{\mathrm{m^2K}} \quad \text{for} \quad r_1 < r < r_2,\ |z| < \frac{t}{2},\ \varphi = 0;$$

$$\alpha_1 = 10\,\frac{\mathrm{W}}{\mathrm{m^2K}} \quad \text{for} \quad r_2 < r < r_3,\ |z| < \frac{t}{2},\ \varphi = 0;$$

$$\alpha_2 = 20\,\frac{\mathrm{W}}{\mathrm{m^2K}} \quad \text{for} \quad r_1 < r < r_2,\ |z| < \frac{t}{2},\ \varphi = 2\pi;$$

$$\alpha_2 = 10\,\frac{\mathrm{W}}{\mathrm{m^2K}} \quad \text{for} \quad r_2 < r < r_3,\ |z| < \frac{t}{2},\ \varphi = 2\pi;$$

$$\alpha_3 = 20\,\frac{\mathrm{W}}{\mathrm{m^2K}} \quad \text{for} \quad r = r_1,\ 0 < \varphi < 2\pi,\ |z| < \frac{t}{2}, \quad \text{and}$$
$$\text{for}\ \ r_1 < r < r_2,\ \ 0 < \varphi < 2\pi,\ |z| = \frac{t}{2};$$

$$\alpha_3 = 10\,\frac{\mathrm{W}}{\mathrm{m^2K}} \quad \text{for} \quad r = r_3,\ 0 < \varphi < 2\pi,\ |z| < \frac{t}{2}, \quad \text{and}$$
$$\text{for}\ \ r_2 < r < r_3,\ \ 0 < \varphi < 2\pi,\ |z| = \frac{t}{2}.$$

For the FEM solution the ring-like-body is discretized by three-dimensional forced convection/diffusion linear elements (`DCC3D8`). The total number of variables in the whole model is 3682. The distribution of elements in radial, circumferential and axial directions is quasi uniform.

For $\tau_0 = 0\,\mathrm{s}$, $\tau_1 = 50\,\mathrm{s}$, $\tau_2 = 500\,\mathrm{s}$, $\tau_3 = 1000\,\mathrm{s}$ and $\tau_4 = 10\,000\,\mathrm{s}$ the temperature fields calculated from Eq. (46) for open ring are shown in Figs. 5-9 compared with FEM solutions which were obtained by the use of Abaqus finite element software. In Figs. 5-9 $\overline{T}_{\mathrm{FEM}}$ is defined as

$$\overline{T}_{\mathrm{FEM}} = \overline{T}_{\mathrm{FEM}}(\varphi) = \frac{1}{a}\int\limits_A T_{\mathrm{FEM}}\,\mathrm{d}A, \tag{53}$$

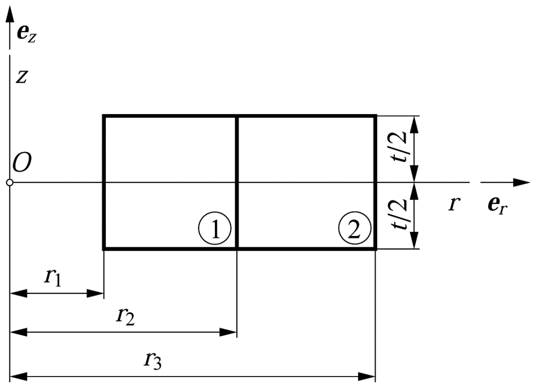

Fig. 4. Composite rectangular cross section

where T_{FEM} denotes the temperature field extracted from FEM computations. For the "numerical analytical" solution in Eq. (46) instead of infinite number of term only ten terms were considered.

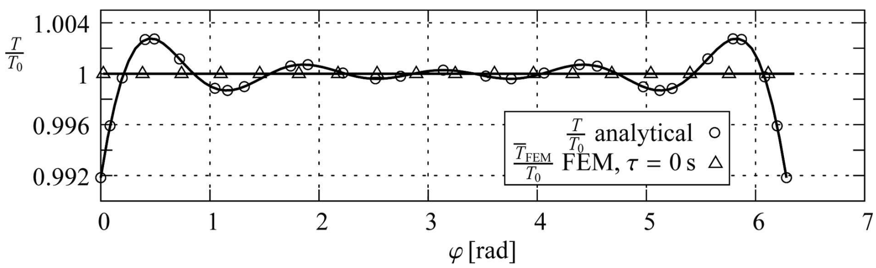

Fig. 5. Temperature field of open ring at $\tau = \tau_0$

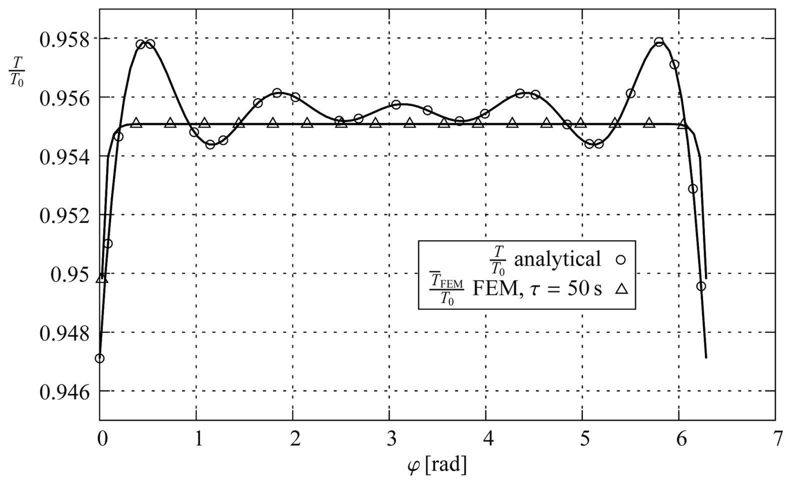

Fig. 6. Temperature field of open ring at $\tau = \tau_1$

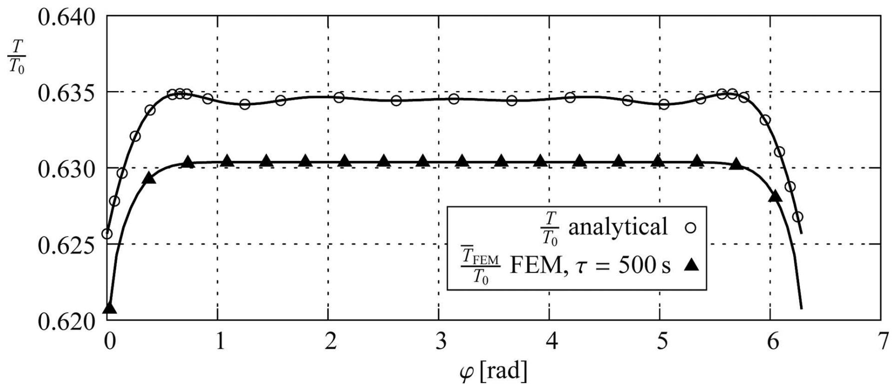

Fig. 7. Temperature field of open ring at $\tau = \tau_2$

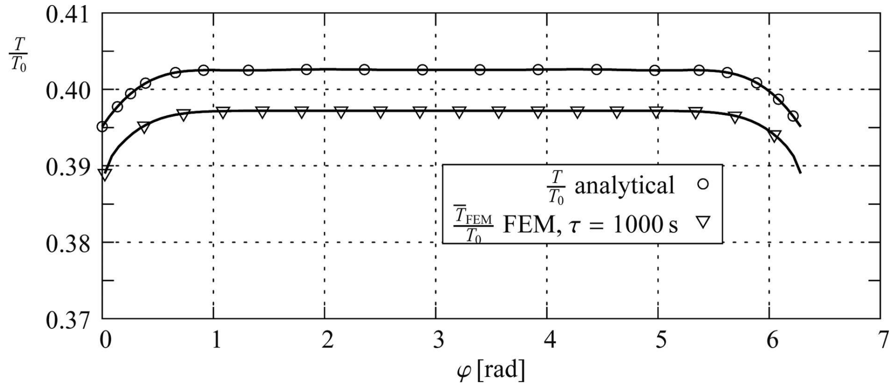

Fig. 8. Temperature field of open ring at $\tau = \tau_3$

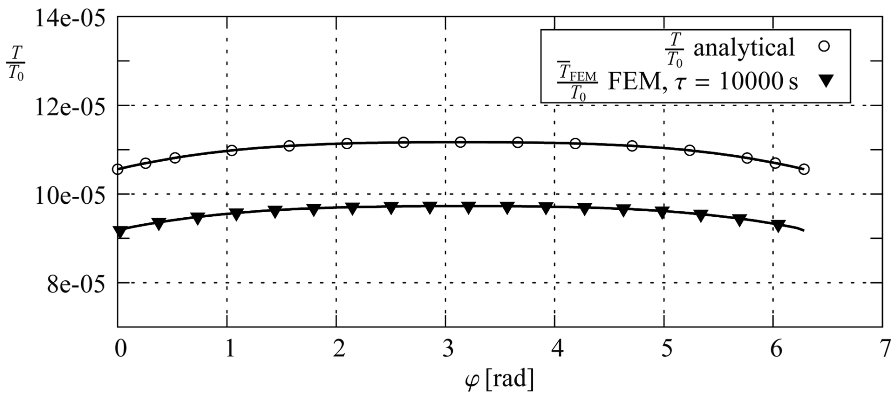

Fig. 9. Temperature field of open ring at $\tau = \tau_4$

5. Conclusions

A one-dimensional model based on Fourier's theory of heat conduction is presented. The main assumption is that the temperature field depends only on the polar angle, i.e., in a cross section the temperature does not change.

The ring-like body may be a complete or incomplete torus with arbitrary cross section. The thermal properties may depend on the cross sectional coordinates, the ring-like body in circumferential direction is homogeneous. It is possible that the ring-like body is a composite of different homogeneous materials, so that the thermal properties are piecewise constants in the cross section. The discontinuity of thermal properties should not effect to the validity of the presented approximation.

Examples show how we can use the one-dimensional model developed to solve steady and unsteady heat conduction problems in closed and open composite ring-like bodies.

Acknowledgment

This research was carried out as part of the TAMOP 4.2.1.B-10/2/KONV-2010-0001 project with support by European Union, co-financed by European Social Fund.

Manuscript received by Editorial Board, May 29, 2012;
final version, April 18, 2013.

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Jednowymiarowy model przewodnictwa cieplnego w ciele o kształcie pierścieniowym

S t r e s z c z e n i e

Opracowano jednowymiarowy model przewodnictwa cieplnego dla ciał o kształcie pierścieniowym oparty o fourierowską teorię przewodnictwa. Rozważane ciała mają kształt pełnego lub niepełnego torusa o dowolnym przekroju. Właściwości cieplne rozważanych ciał są niezależne od kąta biegunowego. Przedstawiono przykłady ilustrujące zastosowanie modelu.