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Roberta BONACINA and Daniel MISSELBECK-WESSEL

A FORMAL APPROACH TO MENGER'S THEOREM

A b s t r a c t. Menger's graph theorem equates the minimum size of a separating set for non-adjacent vertices a and b with the maximum number of disjoint paths between a and b. By capturing separating sets as models of an entailment relation, we take a formal approach to Menger's result. Upon showing that inconsistency is characterised by the existence of sufficiently many disjoint paths, we recover Menger's theorem by way of completeness.

1. Introduction

Consider a finite directed graph G, and let $a, b \in V(G)$ be distinct, non-adjacent vertices, fixed throughout the present note. Menger's theorem [21, 22], a classic result and cornerstone of graph theory, asserts that *the minimum number of vertices separating a from b in G is equal to the maximum number of pairwise internally vertex-disjoint paths from a to b in G*. A fair amount of proofs has been offered for several variants [1, 6, 7, 10–14, 16, 17, 20, 24, 27] (which list is by no means meant exhaustive), while computer-assisted formalisations have recently been carried out of McCuaig's [20] in Isabelle/HOL [8], and in Coq [9] of Göring's [10].

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Among the consequences of Menger's theorem [26] there is, e.g., the well-known Marriage Lemma (Hall's theorem) [15]. The latter has seen an elegant syntactical treatment by Coquand [4], using hyperresolution in the guise of Scott-style multi-conclusion entailment relations [3, 5, 30].

In a similar vein, the purpose of this note is to offer a change of perspective on Menger's theorem, thus providing further evidence for the applicability of formal methods in graph theory, as pioneered by Matiyasevich [18, 19]. Indeed we show that, once an appropriate entailment relation has been set up, Menger's theorem appears via completeness as the semantical counterpart of a syntactical criterion on inconsistency. The key lies in McCuaig's argument [20], which carries over almost verbatim to prove a crucial point (Proposition 3.3) towards our version (Proposition 3.1).

2. Entailment

Let S be a set. A relation $\vdash$ between finite subsets of S is an *entailment relation* [3] if it is

reflexive: $A \vdash B$ if $A \cap B$ is inhabited,

monotone: $A' \vdash B'$ if $A \vdash B$ and $A \subseteq A'$ and $B \subseteq B'$,

transitive: $A \vdash B$ if $A \vdash B, c$ and $A, c \vdash B$,

where the usual shorthand notation is at work, e.g., we write A, c where it should read $A \cup \{ c \}$. The *models* of $\vdash$ are the subsets T of S such that $T \cap B$ is inhabited whenever $T \supseteq A$ and $A \vdash B$, which requirement reduces to axioms where inductively generated entailment relations are concerned, as will be the case below. By way of the completeness theorem [3, 5, 30], entailment relations are determined by their models. This is to say that $A \vdash B$ already if $T \cap B$ is inhabited for every model $T \supseteq A$. In particular, if $\emptyset \nvdash \emptyset$, then $\vdash$ has a model.

3. A syntactical form of Menger's theorem

To fit the setting of Menger's theorem, we now take $S = V(G)$ to be our domain of discourse, i.e., we think of vertices as abstract tokens, and consider, for $n \geqslant 0$, the entailment relation $\vdash_n$ that is inductively generated by the following axioms:[1]

$$\vdash_n V(p) \qquad \text{where } p \in \mathrm{Path}(a, b) \tag{1}$$

$$U \vdash_n \qquad \text{whenever } |U| = n \tag{2}$$

[1]We take over from [29] the inductive generation of entailment relations by a rule-only approach.

with side conditions as indicated, where $\mathrm{Path}(a, b)$ is the set of ab-paths, and where $V(p)$ denotes the set of internal vertices of an ab-path p. The models T of $\vdash_n$ are precisely those sets of vertices that *separate* a *and* b (which is to say that every ab-path has an internal vertex in T) while having *fewer than* n elements. Note that $\vdash_0$ is *inconsistent* by its very definition, i.e., $\emptyset \vdash_0 \emptyset$.

Before we proceed, a terminological caveat is in order: "internally disjoint" means "pairwise internally vertex-disjoint" throughout.

Menger's theorem hinges on showing that if n is the minimum number of vertices separating a and b, then n internally disjoint ab-paths indeed exist. This being kept in mind, we swiftly recover Menger's from the completeness theorem on account of the following:

Proposition 3.1. *The following are equivalent.*

1. $\vdash_n$ *is inconsistent.*

2. *There are at least* n *internally disjoint* ab*-paths.*

In fact, if n is the minimum number of vertices separating a and b, then $\vdash_n$ does not have any model, whence $\emptyset \vdash_n \emptyset$ by completeness. This yields n internally disjoint ab-paths according to Proposition 3.1.

We concentrate now on a slight generalisation of Proposition 3.1, which describes the empty-conclusion instances of $\vdash_n$ in a direct, non-inductive manner through internally disjoint ab-paths:

Proposition 3.2. *The following are equivalent.*

1. $U \vdash_n$.

2. *There is a set* P *of internally disjoint* ab*-paths such that*

$$|P| + |U| \geqslant n \quad \text{and} \quad \bigcup_{p \in P} V(p) \cap U = \emptyset.$$

A moment's thought shows that Proposition 3.1 is the case $U = \emptyset$ of Proposition 3.2. To handle the crucial step in the proof of the latter proposition, it seems best to put an auxiliary result first, but which appears to be of some interest in itself:

Proposition 3.3. *Let* p *be an* ab*-path. Let* $m \geqslant 0$ *and suppose that, for every internal vertex* v *of* p*, there are* m *internally disjoint* ab*-paths, each of which avoids* v*. Then there are* $m + 1$ *internally disjoint* ab*-paths.*

Proposition 3.3 is even necessary for the former one. In fact, if, say, $V(p) = \{ v_0, \ldots, v_r \}$ and path-sets P_i were as assumed for $0 \leqslant i \leqslant r$, then Proposition 3.2 implied $v_i \vdash_{m+1}$ for $0 \leqslant i \leqslant r$. Since $\vdash_{m+1} V(p)$, transitivity yielded inconsistency of $\vdash_{m+1}$, which in turn implied that there were $m+1$ internally disjoint ab-paths, as claimed by Proposition 3.3.

For the sake of clarity in the proof of Proposition 3.3, we introduce some terminology. Suppose that p is an ab-path. A *p-bow* for a set of ab-paths $p_1, \ldots, p_m$ is given by a vertex x of p after a, along with an ax-path q whose inital arc is not on any p_i, and which does not meet any p_i sooner than in x.

Last but not least, here are the proofs.

Proof of Proposition 3.3. We follow very closely the argument of [20], which requires only little adaptation. To begin with, note that there are disjoint ab-paths $p_1, \ldots, p_m$ and a p-bow (p_{m+1}, x). (For instance, take $p_1, \ldots, p_m$ as given by the assumption on the first internal node of p, and take the initial arc of p as bow.) We assume that $p_1, \ldots, p_m, p_{m+1}$ have been chosen so that the distance from x to b on p is minimal. Again by assumption, there are disjoint ab-paths $q_1, \ldots, q_m$ each of which avoids x. We further suppose that $q_1, \ldots, q_m$ have been chosen so that a minimum number of arcs in $B = A(G) - \bigcup_{i=1}^{m+1} A(p_i)$ are used, where $A(G)$ and $A(p_i)$ denote the set of arcs of G and p_i, respectively.

Since $p_1, \ldots, p_{m+1}$ have pairwise distinct initial arcs, we can find a certain p_k among them whose initial arc does not coincide with any of the inital arcs of $q_1, \ldots, q_m$. Now let H be the directed graph consisting of the vertices and arcs of $q_1, \ldots, q_m$ together with the vertex x. Let y be the first vertex on p_k after a which is in $V(H)$. If $y = b$ we are done. Let's rule out the remaining cases: If $y = x$, then consider the xb-section r of p. Let z be the first vertex of r which is met by some q_j. The distance on p from z to b is less than the distance from x to b. But then the extension of p_k to z yields a p-bow for $q_1, \ldots, q_m$ contradicting the choice of $p_1, \ldots, p_{m+1}$. On the other hand, if y is an internal vertex of a certain q_i, then the ay-section of q_i has an arc in B. Replacing the ay-section of q_i by the ay-section of p_k, we get m internally disjoint ab-paths, each of which avoids x, but using less arcs in B than $q_1, \ldots, q_m$ do, which again is a contradiction. □

Proof of Proposition 3.2. Here we make use of a general principle to describe inconsistency, based on cut elimination [29], and linked to hyperresolution [5]. To do so, we introduce a shorthand notation: for finite subsets U of S, let $I(U)$ abbreviate the second item of the proposition.

Note that $I(U)$ implies $U \vdash_n$. In fact, if there are paths $p_1, \ldots, p_m$ as indicated, where $m + |U| \geqslant n$, then by (1) we have that $\vdash_n V(p_i)$ for $1 \leqslant i \leqslant m$, while by (2) we know that $U, v_1, \ldots, v_m \vdash_n$ for *every* choice of elements $v_i \in V(p_i)$. Repeated application of transitivity (induction on m) yields $U \vdash_n$. Moreover, it is easy to see that I is monotone, i.e., if $I(U)$ and $U \subseteq U'$, then $I(U')$.

Conversely, to show that $U \vdash_n$ implies $I(U)$—and thus to prove Proposition 3.2—

it suffices [31, Lemma 1] to check the following criteria, corresponding to the generating axioms: (i) if $|U| = n$, then $I(U)$; as well as that (ii) if p is a path from a to b and $I(U, v)$ for every $v \in V(p)$, then $I(U)$. The former is trivial: $P = \emptyset$ will do. As regards the latter, we may assume that $U \cap V(p) = \emptyset$, for otherwise $I(U)$ will be immediate. Accordingly, suppose that, for every internal node v of p, there is a set P_v of internally disjoint ab-paths with $|P_v| + |U| + 1 \geqslant n$, and such that every $p \in P_v$ avoids both v and U. Let $m = \min\{\, |P_v| \mid v \in V(p)\,\}$. By deleting the vertices of U we pass to a subgraph G' in which Proposition 3.3 yields $m + 1$ internally disjoint ab-paths witnessing $I(U)$. □

Intuitively, extending a set of vertices so that it separates a and b requires that we pick for each ab-path p an internal vertex, and, if need be, adjoin the latter to the vertices chosen thus far. However, if this cannot be carried out consistently, then we need to be able to spot a problem already at an earlier stage of the construction. The final step in the proof of Proposition 3.2 makes this precise and shows a form of heredity. It is quite common [2, 23, 25, 28, 29] that semantical extension principles can be recast in this way, once focus has been shifted to a syntactical representation.

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References

[1] T. Böhme, F. Göring, and J. Harant, Menger's Theorem, *J. Graph Theory* **37:1** (2001), 35–36.

[2] R. Bonacina and D. Wessel, Ribenboim's order extension theorem from a constructive point of view, *Algebra Universalis* **81:5** (2020), https://doi.org/10.1007/s00012-019-0634-0.

[3] J. Cederquist and T. Coquand, Entailment relations and distributive lattices, in: Logic Colloquium '98. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic, Prague, Czech Republic, August 9–15, 1998, S. R. Buss, P. Hájek and P. Pudlák (Eds.), Lect. Notes Logic, A. K. Peters, Natick, MA, 2000, pp.127–139.

[2] The opinions expressed in this paper are those of the authors and do not necessarily reflect the views of these foundations.

[4] T. Coquand, A syntanctical proof of the Marriage Lemma, *Theoret. Comput. Sci.* **290:1** (2003), 1107–1113.

[5] T. Coquand and Guo-Qiang Zhang, Sequents, frames, and completeness, in: Computer Science Logic (Fischbachau, 2000), P. G. Clote and H. Schwichtenberg (Eds.), volume 1862 of Lecture Notes in Comput. Sci., Springer, Berlin 2000, pp. 277–291.

[6] R. Diestel. Graph Theory, volume 173 of Graduate Texts in Mathematics, fifth edition, Springer, Berlin 2017.

[7] G.A. Dirac, Short proof of Menger's graph theorem, *Mathematika* **13:1** (1966), 42–44.

[8] C. Dittmann, Menger's Theorem, *Archive of Formal Proofs*, 2017. http://isa-afp.org/entries/Menger.html, Formal proof development.

[9] Ch. Doczkal, Short proof of Menger's Theorem in Coq (Proof Pearl), Technical report, 2019. URL: http://www-sop.inria.fr/members/Christian.Doczkal/pdf/menger.pdf.

[10] F. Göring, Short proof of Menger's theorem, *Discrete Math.* **219** (2000), 295–296.

[11] F. Göring, A proof of Menger's theorem by contraction. *Discuss. Math. Graph Theory* **22** (2002), 111–112.

[12] T. Grünwald (later Gallai), Ein neuer Beweis eines Mengerschen Satzes, *J. Lond. Math. Soc.* **13** (1938), 188–192.

[13] G. Hajós, Zum Mengerschen Graphensatz, *Acta Sci. Math. (Szeged)* **7** (1934-35), 44–47.

[14] R. Halin, Über trennende Eckenmengen in Graphen und den Mengerschen Satz, *Math. Ann.* **157** (1964), 34–41.

[15] P. Halmos and H. E. Vaughan, The marriage problem, *Amer. J. Math.* **72** (1950), 214–215.

[16] D. Kőnig, Über trennende Knotenpunkte in Graphen (nebst Anwendungen auf Determinanten und Matrizen), *Acta Sci. Math. (Szeged)* **6:2-3**, (1932-34), 155–179.

[17] L. Lovász, A remark on Menger's theorem, *Acta Math. Acad. Sci. Hungar.* **21:3–4** (1970), 365– 368.

[18] Y. V. Matiyasevich, The application of the methods of the theory of logical derivation to graph theory, *Math. Notes Acad. Sci. USSR* **12:6** (1972), 904–908.

[19] Y. V. Matiyasevich, Metamathematical approach to proving theorems of discrete mathematics, *J. Soviet Math.* **10** (1978), 517–533.

[20] W. McCuaig, A simple proof of Menger's theorem, *J. Graph Theory* **8** (1984), 427–429.

[21] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* **10:1** (1927), 96–115.

[22] K. Menger, Kurventheorie, Teubner, Hrsg. unter Mitarb. von Georg Nöbeling, Leipzig, 1932.

[23] C. J. Mulvey and J. Wick-Pelletier, A globalization of the Hahn–Banach theorem, *Adv. Math.* **89** (1991), 1–59.

[24] C. St. John Alvah Nash-Williams and W. T. Tutte, More proofs of Menger's theorem, *J. Graph Theory* **1** (1977), 13–17.

[25] S. Negri, J. von Plato and T. Coquand, Proof-theoretical analysis of order relations, *Arch. Math. Logic* **43** (2004), 297–309.

[26] H. Perfect, Applications of Menger's graph theorem, *J. Math. Anal. Appl.* **22** (1968), 96–111.

[27] J. S. Pym, A proof of Menger's theorem, *Monatsh. Math.* **73** (1969), 81–83.

[28] D. Rinaldi, P. Schuster and D. Wessel, Eliminating disjunctions by disjunction elimination, *Indag. Math. (N.S.)* **29:1** (2018), 226–259.

[29] D. Rinaldi and D. Wessel, Cut elimination for entailment relations, *Arch. Math. Logic* **58:5–6** (2019), 605–625.

[30] D. Scott, Completeness and axiomatizability in many-valued logic, in: Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971), L. Henkin, J. Addison, C.C. Chang, W. Craig, D. Scott, and R. Vaught (Eds.), Amer. Math. Soc., Providence, RI, 1974, pp. 411–435

[31] D. Wessel, Point-free spectra of linear spreads, in: Mathesis Universalis, Computability and Proof, Synthese Library, S. Centrone, S. Negri, D. Sarikaya, and P. Schuster (Eds.), Springer, Cham, 2019, pp. 353–374

Roberta Bonacina
Carl Friedrich von Weizsäcker-Zentrum, Universität Tübingen

r.bonacina92@gmail.com

Daniel Misselbeck-Wessel
Dipartimento di Informatica, Università di Verona

daniel.wessel@univr.it