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VOLTERRA INTEGRAL OPERATORS ON A FAMILY OF DIRICHLET–MORREY SPACES

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Abstract. A family of Dirichlet–Morrey spaces $\mathcal{D}_{\lambda,K}$ of functions analytic in the open unit disk $\mathbb{D}$ are defined in this paper. We completely characterize the boundedness of the Volterra integral operators T_g, I_g and the multiplication operator M_g on the space $\mathcal{D}_{\lambda,K}$. In addition, the compactness and essential norm of the operators T_g and I_g on $\mathcal{D}_{\lambda,K}$ are also investigated.

Keywords: Dirichlet–Morrey type space, Carleson measure, Volterra integral operators, bounded operator, essential norm.

Mathematics Subject Classification: 30H99, 47B38.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc in the complex plane and $H(\mathbb{D})$ be the set of all analytic functions in $\mathbb{D}$. Let H^∞ denote the space of all bounded analytic functions. For $\lambda > -1$, $0 < p < \infty$, a function $f \in H(\mathbb{D})$ belongs to the weighted Dirichlet space $\mathcal{D}^p_\lambda$ if

$$\|f\|_{\mathcal{D}^p_\lambda} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\lambda dA(z) \right)^{1/p} < \infty,$$

where dA denotes the normalized area measure on $\mathbb{D}$. When $\lambda = 1$, $p = 2$, the space $\mathcal{D}^p_\lambda$ coincides with the classical Hardy space H^2. When $\lambda = p$, the space $\mathcal{D}^p_\lambda$ becomes the Bergman space, denoted by A^p.

Let $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$. A function $f \in H(\mathbb{D})$ belongs to the space $F(p,q,s)$ if

$$\|f\|_{F(p,q,s)} = |f(0)| + \sup_{\alpha \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_\alpha(z)|^2)^s dA(z) \right)^{1/p} < \infty,$$

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where $\varphi_\alpha = \frac{\alpha - z}{1-\bar{\alpha}z}$ is a Möbius map that interchanges 0 and α. The space $F(p,q,s)$ was introduced by Zhao in [37]. From [37], when $q = p-2$, the space $F(p,p-2,s)$ coincides with the Bloch space $\mathcal{B}$ if $s > 1$. Furthermore, $F(p,p-2,0)$ is just the Besov space B_p. When $p = 2$, the space $F(p,p-2,s)$ becomes the Q_s space (see [32]). In particular, $F(2,0,1)$ is the BMOA space, the set of all analytic functions of bounded mean oscillation.

For $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$, a function $f \in F(p,q,s)$ belongs to the little space $F_0(p,q,s)$ if

$$\lim_{|\alpha|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_\alpha(z)|^2)^s dA(z) = 0.$$

Let $g, f \in H(\mathbb{D})$. The Volterra integral operator T_g and its associated operator I_g are defined by

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \quad I_g f(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

Obviously, $T_g f(z) = M_g f(z) - I_g f(z) - f(0)g(0)$, where $M_g f(z) = f(z)g(z)$ is the multiplication operator. These integral operators, as well as their various generalizations have attracted attention of many authors (see, e.g., [1–11, 15, 17–23, 26–28, 36] and the related references therein).

For any arc $I \subset \partial\mathbb{D}$, let $|I| = \frac{1}{\pi}\int_I |d\xi|$ be the normalized arc length of I and

$$S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |I| \le r < 1, e^{i\theta} \in I\}$$

be the Carleson box based on I. For $0 < s < \infty$, we say that a positive Borel measure μ on $\mathbb{D}$ is an s-Carleson measure if (see [17])

$$\|\mu\|_s = \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

For $0 \le \lambda \le 1$, a function $f \in H^2(\mathbb{D})$ belongs to the analytic Morrey space $\mathcal{L}^{2,\lambda}(\mathbb{D})$, which was introduced by Wu and Xie in [29], if

$$\sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_I |f(\eta) - f_I|^2 \frac{|d\eta|}{2\pi} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\eta) \frac{|d\eta|}{2\pi}.$$

Li, Liu and Lou showed that T_g is bounded on Morrey space $\mathcal{L}^{2,\lambda}(\mathbb{D})$ if and only if $g \in BMOA$ for $0 < \lambda < 1$ in [10]. Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing and right-continuous function, not identically equal to zero. In [28], Sun and Wulan defined a Morrey type space $\mathcal{D}_K^s$, which consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{D}_K^s} = |f(0)|^2 + \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^s}{K(1-|\alpha|^2)} \|f\circ\varphi_\alpha - f(\alpha)\|^2_{\mathcal{D}_s^2} < \infty.$$

They found some sufficient and necessary conditions for the identity operator I_d from $\mathcal{D}_K^s$ to $\mathcal{T}_K^s(\mu)$ to be bounded. Here $\mathcal{T}_K^s(\mu)$ is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{T}_K^s(\mu)} = \sup_{\alpha\in\mathbb{D}} \frac{1}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |f(z)-f(\alpha)|^2 \left(\frac{1-|\alpha|^2}{|1-\bar{\alpha}z|}\right)^{2s} d\mu(z) < \infty,$$

where $0 < s < \infty$ and μ is a positive Borel measure on $\mathbb{D}$. Morrey type spaces have received lots of attention and studied by many authors. See [3,12,13,18,28,29,31,33,34] and the references therein for more results on Morrey type spaces.

Motivated by [28], in this paper we define a new Morrey type space $\mathcal{D}_{\lambda,K}$ as follows: for $-1 < \lambda < 0$, the Dirichlet–Morrey type space $\mathcal{D}_{\lambda,K}$ is defined as the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}_{\lambda,K}} = |f(0)| + \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \|f\circ\varphi_\alpha - f(\alpha)\|_{\mathcal{D}^1_\lambda} < \infty.$$

For $0 < s < 1$, if $K(x) = x^{(\lambda+1)s}$, the space $\mathcal{D}_{\lambda,K}$ coincides with the Dirichlet–Morrey space $\mathcal{D}_{\lambda,s}$ (see [5]).

In this paper, we always suppose that the following condition on K holds (see [30]):

$$\int_1^\infty \frac{\varphi_K(x)}{x^{1+\delta}} dx < \infty, \quad \delta > 0, \tag{1.1}$$

where

$$\varphi_K(x) = \sup_{0<s\le 1} \frac{K(sx)}{K(s)}, \quad 0 < x < \infty.$$

Obviously, $K(x) = x^p$ satisfies inequality (1.1) for $0 < p < \delta$.

This paper is organized as follows: Section 2 characterizes some properties for the Dirichlet–Morrey space $\mathcal{D}_{\lambda,K}$. The boundedness of the Volterra integral operators T_g, I_g and the multiplication operator M_g on the space $\mathcal{D}_{\lambda,K}$ is given in Section 3. In the last section, we study the essential norm of the operators T_g and I_g.

For two quantities A and B, we use the abbreviation $A \lesssim B$ whenever there is a positive constant C (independent of the associated variables) such that $A \le CB$. We write $A \approx B$, if $A \lesssim B \lesssim A$.

2. SOME BASIC PROPERTIES

In this section, some basic properties of the space $\mathcal{D}_{\lambda,K}$ are given. First, we state two lemmas as follows.

Lemma 2.1 ([16, Lemma 2.5]). *Let $r, t > 0$, $s > -1$ and $t + r - s > 2$. If $t < 2 + s < r$, then*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{\alpha}z|^r|1-\bar{\beta}z|^t} dA(z) \lesssim \frac{1}{(1-|\alpha|^2)^{r-s-2}|1-\bar{\alpha}\beta|^t}$$

for any $\alpha, \beta \in \mathbb{D}$.

Lemma 2.2 ([28, Remark 2.1]). *Let $0 < \alpha \le \beta < \infty$ and K satisfy* (1.1) *for some $\delta > 0$. Then for all sufficiently small positive constants $\varepsilon < \delta$,*

$$\frac{K(\beta)}{K(\alpha)} \le \left(\frac{\beta}{\alpha}\right)^{\delta-\varepsilon} \le \left(\frac{\beta}{\alpha}\right)^{\delta}.$$

Proposition 2.3. *Let $-1 < \lambda < 0$. Then $\mathcal{D}_{\lambda,K} \subseteq \mathcal{D}^1_\lambda$. Moreover, $\mathcal{D}_{\lambda,K} = \mathcal{D}^1_\lambda$ if and only if $K(0) > 0$.*

Proof. Let $f \in \mathcal{D}_{\lambda,K}$. Using the change of variables $w = \varphi_\alpha(z)$,

$$\begin{aligned}
\infty &> \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \|f\circ\varphi_\alpha - f(\alpha)\|_{\mathcal{D}^1_\lambda} \\
&= \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |(f\circ\varphi_\alpha)'(z)|(1-|z|^2)^\lambda dA(z) \\
&= \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |f'(w)|(1-|w|^2)^{-1}(1-|\varphi_\alpha(w)|^2)^{\lambda+1} dA(w) \\
&\ge \frac{1}{K(1)} \int_{\mathbb{D}} |f'(w)|(1-|w|^2)^{-1}(1-|w|^2)^{\lambda+1} dA(w) \\
&\gtrsim \int_{\mathbb{D}} |f'(w)|(1-|w|^2)^\lambda dA(w).
\end{aligned}$$

So $f \in \mathcal{D}^1_\lambda$, that is, $\mathcal{D}_{\lambda,K} \subseteq \mathcal{D}^1_\lambda$.

Next, we prove that $\mathcal{D}_{\lambda,K} = \mathcal{D}^1_\lambda$ if and only if $K(0) > 0$. First, we suppose that $f \in \mathcal{D}^1_\lambda$ and $K(0) > 0$. Using the monotonicity of K, we obtain that

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \|f\circ\varphi_\alpha - f(\alpha)\|_{\mathcal{D}^1_\lambda} \\
&\lesssim \frac{1}{K(0)} \int_{\mathbb{D}} |f'(z)|(1-|z|^2)^\lambda \frac{(1-|\alpha|^2)^{2\lambda+2}}{|1-\bar{\alpha}z|^{2\lambda+2}} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|(1-|z|^2)^\lambda dA(z) < \infty.
\end{aligned}$$

Therefore, $f \in \mathcal{D}_{\lambda,K}$. Furthermore, $\mathcal{D}_{\lambda,K} = \mathcal{D}^1_\lambda$.

Conversely, assume that $\mathcal{D}_{\lambda,K} = \mathcal{D}^1_\lambda$. For any $\gamma \in \mathbb{D}$, consider the function

$$f_\gamma(z) = (1-|\gamma|^2)\int_0^z \frac{dw}{(1-\bar{\gamma}w)^{3+\lambda}}, \quad z \in \mathbb{D}.$$

Applying Lemma 3.10 in [39], we get

$$\|f_\gamma\|_{\mathcal{D}^1_\lambda} \approx \int_{\mathbb{D}} |f'_\gamma(z)|(1-|z|^2)^\lambda dA(z) = \int_{\mathbb{D}} \frac{(1-|\gamma|^2)}{|1-\bar{\gamma}z|^{3+\lambda}}(1-|z|^2)^\lambda dA(z) \approx 1.$$

Thus, $f_\gamma \in \mathcal{D}^1_\lambda$. Then

$$\begin{aligned}
\infty > \|f_\gamma\|_{\mathcal{D}^1_\lambda} &\gtrsim \|f_\gamma\|_{\mathcal{D}_{\lambda,K}} \\
&\approx \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |f'_\gamma(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \\
&\gtrsim \frac{(1-|\gamma|^2)^{\lambda+1}}{K(1-|\gamma|^2)} \int_{\mathbb{D}} |f'_\gamma(z)|(1-|z|^2)^{-1}(1-|\varphi_\gamma(z)|^2)^{\lambda+1} dA(z) \\
&\approx \frac{1}{K(1-|\gamma|^2)},
\end{aligned}$$

which implies that $K(0) > 0$. □

Proposition 2.4. *Let $-1 < \lambda < 0$ and K satisfy* (1.1). *Then $\mathcal{D}_{\lambda,K} = F(1,-1,\lambda+1)$ if and only if $K(x) \approx x^{\lambda+1}$.*

Proof. Since

$$\|f\|_{F(1,-1,\lambda+1)} \approx \sup_{\alpha\in\mathbb{D}} \|f\circ\varphi_\alpha - f(\alpha)\|_{\mathcal{D}^1_\lambda} \lesssim \frac{K(1-|\alpha|^2)}{(1-|\alpha|^2)^{\lambda+1}}\|f\|_{\mathcal{D}_{\lambda,K}}, \quad \alpha \in \mathbb{D},$$

and

$$\|f\|_{\mathcal{D}_{\lambda,K}} \lesssim \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)}\|f\|_{F(1,-1,\lambda+1)},$$

the desired result follows immediately. □

Proposition 2.5. *Let $-1 < \lambda < 0$, $\gamma \in \mathbb{D}$ and K satisfy* (1.1) *for some $\delta > 0$ such that $\delta \le 2\lambda + 2$. Then the function*

$$f_\gamma(z) = \frac{K(1-|\gamma|^2)(1-|\gamma|^2)^{\lambda+1}}{(1-\bar{\gamma}z)^{2\lambda+2}}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{D}_{\lambda,K}$.

Proof. Using Lemmas 2.1 and 2.2, we have that

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |f'_\gamma(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \\
&\approx \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{2\lambda+2} K(1-|\gamma|^2)(1-|\gamma|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} \frac{(1-|z|^2)^{\lambda}}{|1-\bar{\gamma}z|^{2\lambda+3}|1-\bar{\alpha}z|^{2\lambda+2}} dA(z) \\
&\lesssim \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{2\lambda+2} K(1-|\gamma|^2)(1-|\gamma|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \frac{1}{(1-|\gamma|^2)^{\lambda+1}|1-\bar{\alpha}\gamma|^{2\lambda+2}} \\
&\lesssim \sup_{\alpha\in\mathbb{D}} \frac{K(1-|\gamma|^2)}{K(1-|\alpha|^2)} \left(\frac{1-|\alpha|^2}{|1-\bar{\alpha}\gamma|}\right)^{2\lambda+2} \\
&\lesssim \sup_{\alpha\in\mathbb{D}} \left(\frac{1-|\alpha|^2}{|1-\bar{\alpha}\gamma|}\right)^{2\lambda+2-\delta} \lesssim 1,
\end{aligned}$$

which means that $f_\gamma \in \mathcal{D}_{\lambda,K}$. □

Proposition 2.6. *Let* $-1 < \lambda < 0$ *and* K *satisfy* (1.1) *for some* $\delta > 0$ *such that* $\delta \leq \lambda + 1$. *Then for any* $f \in \mathcal{D}_{\lambda,K}$,

$$|f(\alpha)| \lesssim \frac{K(1-|\alpha|^2)}{(1-|\alpha|^2)^{\lambda+1}} \|f\|_{\mathcal{D}_{\lambda,K}}, \quad \alpha\in\mathbb{D}.$$

Proof. It is obvious that

$$\begin{aligned}
|f'(\alpha)| &\lesssim \frac{1}{(1-|\alpha|^2)} \int_{\mathbb{D}(\alpha,r)} |f'(z)|(1-|z|^2)^{-1} dA(z) \\
&\lesssim \frac{1}{(1-|\alpha|^2)} \int_{\mathbb{D}} |f'(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \\
&\lesssim \frac{K(1-|\alpha|^2)}{(1-|\alpha|^2)^{\lambda+2}} \|f\|_{\mathcal{D}_{\lambda,K}}.
\end{aligned}$$

Then Lemma 2.2 yields that there exists a constant $c \in (0, \delta)$ such that

$$\begin{aligned}
|f(\alpha) - f(0)| = \left|\alpha \int_0^1 f'(\alpha z) dz\right| &\lesssim \|f\|_{\mathcal{D}_{\lambda,K}} \int_0^1 \frac{|\alpha| K(1-|\alpha z|^2)}{(1-|\alpha z|^2)^{\lambda+2}} dz \\
&\lesssim \|f\|_{\mathcal{D}_{\lambda,K}} \frac{K(1-|\alpha|)}{(1-|\alpha|)^{\delta-c}} \int_0^1 (1-|\alpha z|)^{\delta-c-\lambda-2}|\alpha| dz \\
&\lesssim \frac{K(1-|\alpha|)}{(1-|\alpha|)^{\lambda+1}} \|f\|_{\mathcal{D}_{\lambda,K}},
\end{aligned}$$

which implies the desired result. □

3. BOUNDEDNESS

In this section, we characterize the boundedness of Volterra integral operators T_g and I_g on the space $\mathcal{D}_{\lambda,K}$. We begin this section with the definition of p-Carleson measure for $\mathcal{D}^1_\lambda$. For $-1<\lambda<0<p<\infty$, a positive Borel measure μ on $\mathbb{D}$ is called a p-Carleson measure for $\mathcal{D}^1_\lambda$ if for any $f\in\mathcal{D}^1_\lambda$, the identity operator $I_d:\mathcal{D}^1_\lambda\to L^p(d\mu)$ is bounded, that is, there exists a positive constant C such that

$$\int_{\mathbb{D}}|f(z)|^p d\mu(z)\le C\|f\|^p_{\mathcal{D}^1_\lambda}$$

for all functions $f\in\mathcal{D}^1_\lambda$. Using Theorem 9 in [14], we immediately obtain the following result.

Lemma 3.1. *Let $-1<\lambda<0$ and μ be a positive Borel measure on $\mathbb{D}$. Then μ is a $(\lambda+1)$-Carleson measure if and only if μ is a 1-Carleson measure for $\mathcal{D}^1_\lambda$, that is, for all functions $f\in\mathcal{D}^1_\lambda$,*

$$\int_{\mathbb{D}}|f(z)|d\mu(z)\lesssim|f(0)|+\int_{\mathbb{D}}|f'(z)|(1-|z|^2)^\lambda dA(z)\approx\|f\|_{\mathcal{D}^1_\lambda}.$$

The following theorem is the main result in this section.

Theorem 3.2. *Let $-1<\lambda<0$ and K satisfy (1.1) for some $\delta>0$ such that $\delta\le\lambda+1$. Then $T_g:\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}$ is bounded if and only if*

$$g\in F(1,-1,\lambda+1).$$

Proof. First, assume that $T_g:\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}$ is bounded. For each fixed arc $I\subset\partial\mathbb{D}$, let $\gamma=(1-|I|)\xi$, ξ be the midpoint of I. Then for $z\in S(I)$,

$$|1-\bar{\gamma}z|\approx1-|\gamma|^2\approx|I|=1-|\gamma|.$$

Consider the test function f_γ, defined in Proposition 2.5. Then

$$\begin{aligned}\infty&>\sup_{\alpha\in\mathbb{D}}\frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)}\int_{\mathbb{D}}|(T_gf_\gamma)'(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1}dA(z)\\&\approx\sup_{\alpha\in\mathbb{D}}\frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)}\int_{\mathbb{D}}|f_\gamma(z)||g'(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1}dA(z)\\&\gtrsim\frac{1}{|I|^{\lambda+1}}\int_{S(I)}|g'(z)|(1-|z|^2)^\lambda dA(z),\end{aligned}$$

which implies that $g\in F(1,-1,\lambda+1)$ (see [37]).

Conversely, suppose that $g\in F(1,-1,\lambda+1)$. Then

$$d\mu_g=|g'(z)|(1-|z|^2)^\lambda dA(z)$$

is a $(\lambda+1)$-Carleson measure (see [37]). Let $f \in \mathcal{D}_{\lambda,K}$. For each fixed arc $I \subset \partial\mathbb{D}$, let $\alpha = (1-|I|)\xi$, ξ be the midpoint of I. Then

$$\begin{aligned}
\|T_g f\|_{\mathcal{D}_{\lambda,K}} &\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{\lambda+1}}{K(1-|a|^2)} \\
&\quad \times \int_{\mathbb{D}} |(T_g f)'(z)|(1-|z|^2)^{-1}(1-|\varphi_a(z)|^2)^{\lambda+1} dA(z) \\
&\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{\lambda+1}}{K(1-|a|^2)} \\
&\quad \times \int_{\mathbb{D}} |f(z)||g'(z)|(1-|z|^2)^{-1}(1-|\varphi_a(z)|^2)^{\lambda+1} dA(z) \\
&\lesssim \sup_{a\in\mathbb{D}} \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} |f(z)-f(a)| \left(\frac{1-|a|^2}{|1-\bar{a}z|}\right)^{2\lambda+2} d\mu_g(z) \\
&\quad + \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{\lambda+1}}{K(1-|a|^2)} \\
&\quad \times \int_{\mathbb{D}} |f(a)||g'(z)|(1-|z|^2)^{-1}(1-|\varphi_a(z)|^2)^{\lambda+1} dA(z) \\
&\lesssim E + F.
\end{aligned}$$

Proposition 2.6 yields that

$$\begin{aligned}
F &\lesssim \|f\|_{\mathcal{D}_{\lambda,K}} \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{\lambda+1}}{K(1-|a|^2)} \\
&\quad \times \int_{\mathbb{D}} \frac{K(1-|a|^2)}{(1-|a|^2)^{\lambda+1}} |g'(z)|(1-|z|^2)^{-1}(1-|\varphi_a(z)|^2)^{\lambda+1} dA(z) \\
&\lesssim \|f\|_{\mathcal{D}_{\lambda,K}} \|g\|_{F(1,-1,\lambda+1)}.
\end{aligned}$$

Next, we need to prove that

$$E \lesssim \|f\|_{\mathcal{D}_{\lambda,K}}.$$

For this purpose, we consider the function

$$F_{\alpha,K}(z) = \frac{(1-|\alpha|^2)^{2\lambda+2}(f(z)-f(\alpha))}{K(1-|\alpha|^2)(1-\bar{\alpha}z)^{2\lambda+2}}, \quad \alpha, z \in \mathbb{D}.$$

We will prove that $F_{\alpha,K} \in \mathcal{D}^1_\lambda$ and $\sup_{\alpha\in\mathbb{D}} \|F_{\alpha,K}\|_{\mathcal{D}^1_\lambda} \lesssim \|f\|_{\mathcal{D}_{\lambda,K}}$. It is obvious that

$$\begin{aligned}\sup_{\alpha\in\mathbb{D}} \|F_{\alpha,K}\|_{\mathcal{D}^1_\lambda} &= \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{2\lambda+2}}{K(1-|\alpha|^2)} \\ &\quad\times \left(|f(\alpha)-f(0)| + \int_{\mathbb{D}} \left| \left(\frac{f(z)-f(\alpha)}{(1-\bar{\alpha}z)^{2\lambda+2}} \right)' \right| (1-|z|^2)^\lambda dA(z) \right) \\ &= \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{2\lambda+2}}{K(1-|\alpha|^2)} |f(\alpha)-f(0)| + G,\end{aligned}$$

where

$$G = \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{2\lambda+2}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} \left| \left(\frac{f(z)-f(\alpha)}{(1-\bar{\alpha}z)^{2\lambda+2}} \right)' \right| (1-|z|^2)^\lambda dA(z).$$

Applying Proposition 2.6, we obtain that

$$\sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{2\lambda+2}}{K(1-|\alpha|^2)} |f(\alpha)-f(0)| \lesssim \sup_{\alpha\in\mathbb{D}} (1-|\alpha|^2)^{\lambda+1} \|f\|_{\mathcal{D}_{\lambda,K}} \lesssim \|f\|_{\mathcal{D}_{\lambda,K}}.$$

For the second term, we have that

$$\begin{aligned}G &\lesssim \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{2\lambda+2}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} \left| \frac{f'(z)}{(1-\bar{\alpha}z)^{2\lambda+2}} \right| (1-|z|^2)^\lambda dA(z) \\ &\quad + \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{2\lambda+2}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} \left| \frac{f(z)-f(\alpha)}{(1-\bar{\alpha}z)^{2\lambda+3}} \right| (1-|z|^2)^\lambda dA(z) = G_1 + G_2.\end{aligned}$$

It is obvious that

$$G_1 = \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |f'(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \lesssim \|f\|_{\mathcal{D}_{\lambda,K}}.$$

By the change of variables $z = \varphi_\alpha(w)$, we get that

$$\begin{aligned}G_2 &= \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |f(z)-f(\alpha)| \frac{(1-|z|^2)^{-1}}{|1-\bar{\alpha}z|} (1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \\ &= \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |f\circ\varphi_\alpha(w) - f(\alpha)| \frac{(1-|w|^2)^\lambda}{|1-\bar{\alpha}w|} dA(w).\end{aligned}$$

It is well known that

$$|f\circ\varphi_\alpha(z) - f(\alpha)| \lesssim \int_{\mathbb{D}} |(f\circ\varphi_\alpha)'(u)| \frac{(1-|u|^2)^2}{|1-\bar{u}z|^3} dA(u).$$

Therefore, employing Fubini's theorem and Lemma 2.1, we have

$$\begin{aligned}
G_2 &\lesssim \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}}\int_{\mathbb{D}} |(f\circ\varphi_\alpha)'(u)| \frac{(1-|u|^2)^2}{|1-\bar{u}z|^3} dA(u) \frac{(1-|z|^2)^\lambda}{|1-\bar{\alpha}z|} dA(z) \\
&\lesssim \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |(f\circ\varphi_\alpha)'(u)|(1-|u|^2)^2 dA(u) \\
&\quad \times \int_{\mathbb{D}} \frac{(1-|z|^2)^\lambda}{|1-\bar{u}z|^3|1-\bar{\alpha}z|} dA(z) \\
&\lesssim \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |(f\circ\varphi_\alpha)'(u)|(1-|u|^2)^2 \frac{1}{(1-|u|^2)^{1-\lambda}|1-\bar{\alpha}u|} dA(u) \\
&\lesssim \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |(f\circ\varphi_\alpha)'(u)|(1-|u|^2)^\lambda dA(u) \\
&\lesssim \|f\|_{\mathcal{D}_{\lambda,K}}.
\end{aligned}$$

Thus, we see that $F_{\alpha,K} \in \mathcal{D}_\lambda^1$ and $\sup_{\alpha\in\mathbb{D}} \|F_{\alpha,K}\|_{\mathcal{D}_\lambda^1} \lesssim \|f\|_{\mathcal{D}_{\lambda,K}}$. Since μ_g is a $(\lambda+1)$-Carleson measure, using Lemma 3.1, we obtain that

$$E = \sup_{\alpha\in\mathbb{D}} \int_{\mathbb{D}} |F_{\alpha,K}| d\mu_g(z) \le C \sup_{\alpha\in\mathbb{D}} \|F_{\alpha,K}\|_{\mathcal{D}_\lambda^1} \lesssim \|f\|_{\mathcal{D}_{\lambda,K}}.$$

This means that $T_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is bounded. □

Theorem 3.3. *Let $-1<\lambda<0$ and K satisfy* (1.1) *for some $\delta>0$ such that $\delta \le \lambda+1$. Then $I_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is bounded if and only if $g \in H^\infty$.*

Proof. First, suppose that $I_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is bounded. For $r>0$ and each $\gamma \in \mathbb{D}$, let $\mathbb{D}(\gamma,r)$ be the Bergman metric disc centered at γ with radius r, that is, $\mathbb{D}(\gamma,r) = \{z\in\mathbb{D} : \beta(\gamma,z) < r\}$. From [39] we have

$$\frac{(1-|\gamma|^2)^2}{|1-\bar{\gamma}z|^4} \approx \frac{1}{(1-|\gamma|^2)^2} \approx \frac{1}{(1-|z|^2)^2}, \quad z \in \mathbb{D}(\gamma,r).$$

Consider the function

$$f_\gamma(z) = \frac{K(1-|\gamma|^2)(1-|\gamma|^2)^{\lambda+1}}{\bar{\gamma}(1-\bar{\gamma}z)^{2\lambda+2}}, \quad \gamma, z \in \mathbb{D}.$$

Clearly, $f_\gamma \in \mathcal{D}_{\lambda,K}$ by Proposition 2.5. By the assumption we obtain that

$$\begin{aligned}
\infty &> \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \int_{\mathbb{D}} |(I_g f_\gamma)'(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \\
&\gtrsim \frac{(1-|\gamma|^2)^{\lambda+1}}{K(1-|\gamma|^2)} \int_{\mathbb{D}} |f_\gamma'(z)||g(z)|(1-|z|^2)^{-1}(1-|\varphi_\gamma(z)|^2)^{\lambda+1} dA(z) \\
&\approx \int_{\mathbb{D}} \frac{(1-|\gamma|^2)^{2\lambda+2}}{|1-\bar{\gamma}z|^{2\lambda+3}} |g(z)|(1-|z|^2)^{-1}(1-|\varphi_\gamma(z)|^2)^{\lambda+1} dA(z) \\
&\gtrsim \frac{1}{(1-|\gamma|^2)} \int_{\mathbb{D}(\gamma,r)} |g(z)|(1-|z|^2)^{-1} dA(z) \gtrsim |g(\gamma)|.
\end{aligned}$$

The arbitrariness of γ implies $g \in H^\infty$.

Conversely, we suppose that $g \in H^\infty$. Let $f \in \mathcal{D}_{\lambda,K}$. Then

$$\begin{aligned}
\|I_g f\|_{\mathcal{D}_{\lambda,K}} &\approx \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \\
&\quad \times \int_{\mathbb{D}} |(I_g f)'(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \\
&\approx \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \\
&\quad \times \int_{\mathbb{D}} |f'(z)||g(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \\
&\lesssim \|g\|_{H^\infty} \sup_{\alpha\in\mathbb{D}} \frac{(1-|\alpha|^2)^{\lambda+1}}{K(1-|\alpha|^2)} \\
&\quad \times \int_{\mathbb{D}} |f'(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z) \\
&\lesssim \|g\|_{H^\infty} \|f\|_{\mathcal{D}_{\lambda,K}},
\end{aligned}$$

which means that $I_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is bounded. □

Theorem 3.4. *Let* $-1 < \lambda < 0$ *and* K *satisfy* (1.1) *for some* $\delta > 0$ *such that* $\delta \le \lambda + 1$. *Then* $M_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ *is bounded if and only if* $g \in F(1,-1,\lambda+1) \cap H^\infty$.

Proof. Suppose first that $g \in F(1,-1,\lambda+1) \cap H^\infty$. Employing Theorems 3.2 and 3.3, we obtain that both T_g and I_g are bounded on $\mathcal{D}_{\lambda,K}$. Therefore, $M_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is bounded.

Conversely, suppose that $M_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is bounded. For $\gamma \in \mathbb{D}$, set

$$f_\gamma(z) = \frac{K(1-|\gamma|^2)(1-|\gamma|^2)^{\lambda+1}}{(1-\bar{\gamma}z)^{2\lambda+2}}, \quad z \in \mathbb{D}.$$

By Proposition 2.5, f_γ is bounded in $\mathcal{D}_{\lambda,K}$. Applying the assumption we obtain that $M_g f_a \in \mathcal{D}_{\lambda,K}$. By Proposition 2.6, we have

$$|g(z)f_\gamma(z)| = |M_g f_\gamma(z)| \lesssim \frac{K(1-|z|^2)\|M_g f_\gamma\|_{\mathcal{D}_{\lambda,K}}}{(1-|z|^2)^{\lambda+1}} \lesssim \frac{K(1-|z|^2)\|M_g\|_{\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}}}{(1-|z|^2)^{\lambda+1}}.$$

Since γ is arbitrary, by setting $\gamma = z$, we get

$$|g(z)| \lesssim \|M_g\|_{\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}},$$

which means that $g \in H^\infty$. Theorem 3.3 yields that the operator I_g is bounded on $\mathcal{D}_{\lambda,K}$. Since $T_g f(z) = M_g(z) - I_g f(z) - f(0)g(0)$, then the operator T_g is also bounded on $\mathcal{D}_{\lambda,K}$. We immediately obtain that $g \in F(1,-1,\lambda+1)$. □

4. ESSENTIAL NORM OF INTEGRAL OPERATORS

In this section, we study the essential norm of the operators T_g and I_g on $\mathcal{D}_{\lambda,K}$. Recall that the essential norm of a bounded linear operator $L : W \to Q$ is defined by

$$\|L\|_{e,W\to Q} = \inf_S \{\|L-S\|_{W\to Q} : S \text{ is compact from } W \text{ to } Q\},$$

where $(W, \|\cdot\|_W)$, $(Q, \|\cdot\|_Q)$ are Banach spaces. Clearly, $L : W \to Q$ is compact if and only if $\|L\|_{e,W\to Q} = 0$. For some resent works on estimating essential norms of integral-type and some related operators, we refer [4, 25, 35, 38].

Let A and W be Banach spaces such that $A \subset W$. Given $f \in W$, the distance of f to A denoted by $\mathrm{dist}_W(f, A)$, is defined by $\mathrm{dist}(f, A) = \inf_{g\in A} \|f-g\|_W$.

The following lemma gives the distance from the space $F(1,-1,\lambda+1)$ to its little space $F_0(1,-1,\lambda+1)$ (see [5]).

Lemma 4.1. *If* $g \in F(1,-1,\lambda+1)$, *then*

$$\limsup_{|\alpha|\to 1} \int_{\mathbb{D}} |g'(z)|(1-|z|^2)^{-1}(1-|\varphi_\alpha(z)|^2)^{\lambda+1} dA(z)$$
$$\approx \mathrm{dist}_{F(1,-1,\lambda+1)}(g, F_0(1,-1,\lambda+1)) \approx \limsup_{r\to 1^-} \|g-g_r\|_{F(1,-1,\lambda+1)}.$$

Here $g_r(z) = g(rz)$, $0 < r < 1, z \in \mathbb{D}$.

Lemma 4.2. *Let* $-1 < \lambda < 0$ *and* K *satisfy* (1.1) *for some* $\delta > 0$ *such that* $\delta \le \lambda + 1$. *If* $g \in F_0(1,-1,\lambda+1)$, *then* $T_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ *is compact.*

Proof. Since $F_0(1,-1,\lambda+1)$ is the closure of polynomials in the norm of $F(1,-1,\lambda+1)$, there exist polynomials P_n such that $\|g-P_n\|_{F(1,-1,\lambda+1)} \to 0$. From the proof of Theorem 3.2, we see that

$$\|T_g - T_{P_n}\|_{\mathcal{D}_{\lambda,K}} = \|T_{g-P_n}\|_{\mathcal{D}_{\lambda,K}} \lesssim \|g-P_n\|_{F(1,-1,\lambda+1)} \to 0$$

as $n \to \infty$. For a polynomial P, noting that T_P is the product of the multiplication operator $f \to fP'$, which is bounded by the boundedness of P' on $\mathbb{D}$, with the integration operator $f \to \int_0^z f(\xi)d\xi$, which is compact on $\mathcal{D}_{\lambda,K}$ (see [1]), we obtain that $T_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is compact. □

Lemma 4.3. *Let $-1 < \lambda < 0$ and K satisfy* (1.1) *for some $\delta > 0$ such that $\delta \le \lambda + 1$. If $g \in F(1, -1, \lambda + 1)$, then $T_{g_r} : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is compact.*

Proof. Since $g \in F(1, -1, \lambda + 1)$, then $g_r \in F_0(1, -1, \lambda + 1)$. Lemma 4.2 gives that $T_{g_r} : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is compact. □

Theorem 4.4. *Let $-1 < \lambda < 0$ and K satisfy* (1.1) *for some $\delta > 0$ such that $\delta \le \lambda+1$. If $g \in H(\mathbb{D})$ and $T_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is bounded, then*

$$\begin{aligned}\|T_g\|_{e,\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} &\approx \operatorname{dist}_{F(1,-1,\lambda+1)}(g, F_0(1, -1, \lambda + 1)) \\ &\approx \limsup_{r\to 1^-} \|g - g_r\|_{F(1,-1,\lambda+1)}.\end{aligned}$$

Proof. Let $\{\alpha_n\}$ be a bounded sequence in $\mathbb{D}$ such that $\lim_{n\to\infty} |\alpha_n| = 1$. Set

$$f_n(z) = \frac{K(1 - |\alpha_n|^2)(1 - |\alpha_n|^2)^{\lambda+1}}{(1 - \bar{\alpha}_n z)^{2\lambda+2}}, \quad z \in \mathbb{D}.$$

Then $\{f_n\}$ is a bounded sequence in $\mathcal{D}_{\lambda,K}$ and $f_n \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $n \to \infty$. For each compact operator $S : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$, similar to [24,25] we have that $\lim_{n\to\infty} \|Sf_n\|_{\mathcal{D}_{\lambda,K}} = 0$. Employing Proposition 4.13 in [39], we get that

$$\begin{aligned}&\|T_g - S\|_{\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} \\ &\gtrsim \limsup_{n\to\infty} \|(T_g - S)(f_n)\|_{\mathcal{D}_{\lambda,K}} \\ &\gtrsim \limsup_{n\to\infty} (\|T_g f_n\|_{\mathcal{D}_{\lambda,K}} - \|Sf_n\|_{\mathcal{D}_{\lambda,K}}) \\ &= \limsup_{n\to\infty} \|T_g f_n\|_{\mathcal{D}_{\lambda,K}} \\ &\gtrsim \limsup_{n\to\infty} \frac{(1 - |\alpha_n|^2)^{\lambda+1}}{K(1 - |\alpha_n|^2)} \int_{\mathbb{D}} |f_n(z)||g'(z)|(1 - |z|^2)^{-1}(1 - |\varphi_{\alpha_n}(z)|^2)^{\lambda+1} dA(z) \\ &\gtrsim \limsup_{n\to\infty} \int_{\mathbb{D}(\alpha_n,r)} |g'(z)|(1 - |z|^2)^{-1}(1 - |\varphi_{\alpha_n}(z)|^2)^{\lambda+1} dA(z).\end{aligned}$$

Since α_n is arbitrary, we obtain that

$$\|T_g\|_{e,\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} \gtrsim \limsup_{n\to\infty} \int_{\mathbb{D}} |g'(z)|(1 - |z|^2)^{-1}(1 - |\varphi_{\alpha_n}(z)|^2)^{\lambda+1} dA(z).$$

Conversely, Lemma 4.3 yields that $T_{g_r} : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is compact when $0 < r < 1$. So

$$\begin{aligned}\|T_g\|_{e,\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} &\le \|T_g - T_{g_r}\|_{\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}}\\ &= \|T_{g-g_r}\|_{\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}}\\ &\lesssim \|g-g_r\|_{F(1,-1,\lambda+1)}.\end{aligned}$$

Employing Lemma 4.1, we get that

$$\begin{aligned}\|T_g\|_{e,\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} &\lesssim \limsup_{r\to 1} \|g-g_r\|_{F(1,-1,\lambda+1)}\\ &\approx \operatorname{dist}_{F(1,-1,\lambda+1)}(g, F_0(1,-1,\lambda+1)).\end{aligned}$$

□

We immediately get the following corollary by Theorem 4.4.

Corollary 4.5. *Let $-1 < \lambda < 0$ and K satisfy* (1.1) *for some $\delta > 0$ such that $\delta \le \lambda+1$. If $g \in H(\mathbb{D})$, then $T_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is compact if and only if $g \in F_0(1,-1,\lambda+1)$.*

Theorem 4.6. *Let $-1 < \lambda < 0$ and K satisfy* (1.1) *for some $\delta > 0$ such that $\delta \le \lambda+1$. If $g \in H(\mathbb{D})$ and $I_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is bounded, then*

$$\|I_g\|_{e,\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} \approx \|g\|_{H^\infty}.$$

Proof. We define S and $\{\alpha_n\}$ as in the proof of Theorem 4.4. Set

$$F_n(z) = \frac{K(1-|\alpha_n|^2)(1-|\alpha_n|^2)^{\lambda+1}}{\bar{\alpha}_n(1-\bar{\alpha}_n z)^{2\lambda+2}}, \quad z \in \mathbb{D}, \alpha_n \neq 0.$$

Then we have that $\|F_n\|_{\mathcal{D}_{\lambda,K}} \lesssim 1$ by Proposition 2.5. Since $S : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is compact, we have that $\lim_{n\to\infty} \|SF_n\|_{\mathcal{D}_{\lambda,K}} = 0$. Thus

$$\begin{aligned}\|I_g - S\|_{\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} &\gtrsim \limsup_{n\to\infty} \|(I_g - S)(F_n)\|_{\mathcal{D}_{\lambda,K}}\\ &\gtrsim \limsup_{n\to\infty} (\|I_g F_n\|_{\mathcal{D}_{\lambda,K}} - \|SF_n\|_{\mathcal{D}_{\lambda,K}})\\ &= \limsup_{n\to\infty} \|I_g F_n\|_{\mathcal{D}_{\lambda,K}}.\end{aligned}$$

From the proof of Theorem 3.3 we obtain that $\|I_g F_n\|_{\mathcal{D}_{\lambda,K}} \gtrsim |g(\alpha_n)|$. Then

$$\|I_g\|_{e,\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} \gtrsim \|g\|_{H^\infty}.$$

Conversely, by Theorem 3.3 again, we have that

$$\begin{aligned}\|I_g\|_{e,\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} &= \inf_S \|I_g - S\|_{\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}}\\ &\lesssim \|I_g\|_{\mathcal{D}_{\lambda,K}\to\mathcal{D}_{\lambda,K}} \lesssim \|g\|_{H^\infty}.\end{aligned}$$

This finishes the proof. □

By Theorem 4.6, we immediately get the following corollary.

Corollary 4.7. *Let $-1 < \lambda < 0$ and K satisfy* (1.1) *for some $\delta > 0$ such that $\delta \le \lambda+1$. If $g \in H(\mathbb{D})$, then $I_g : \mathcal{D}_{\lambda,K} \to \mathcal{D}_{\lambda,K}$ is compact if and only if $g = 0$.*

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