POZNAN UNIVERSITY OF TECHNOLOGY ACADEMIC JOURNALS
No 97 Electrical Engineering 2019

DOI 10.21008/j.1897-0737.2019.97.0014

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ANALYSIS OF FRACTIONAL ELECTRICAL CIRCUIT WITH SINUSOIDAL INPUT SIGNAL USING CAPUTO AND CONFORMABLE DERIVATIVE DEFINITIONS

Abstract. This paper presents the fractional electrical circuit in the transient state described by the fractional-order state-space equations. General solutions to the fractional state-space equations containing two types of definitions of fractional derivative: Caputo definition and the conformable fractional derivative definition are given the solutions in the case of: 1) control in the form of sine function at zero initial states 2) control in the form of cosine function at zero initial states 3) control in the form of the sine function with phase shift at zero initial states. The solutions are shown for capacitor voltages for fractional derivative orders of 0.7; 0.8; 1.0. The results were compared using graphs..

KEYWORDS: fractional order system, Sinusoidal Signals, Caputo definition, conformable fractional derivative definition, fractional electrical circuit.

1. INTRODUCTION

Models of electrical circuits consist of resistors, coils, capacitors and voltage (current) sources increasingly using in calculations using fractional derivatives[4–5]. The state-space equations with fractional described by fractional order derivatives of Caputo, Riemann-Liouville and Grünwald-Letnikov types are widely analyzed in [9]. Solutions of the descriptor standard and fractional linear systems in are given [10–15].
Calculations were also done using the use method a new introduced definition by the authors Khalil, R., Al Horani, M., Yousef. A. and Sababheh, M called conformable fractional derivative (CDF) [11].

In this paper we will consider the solutions of the fractional circuit equations using the Caputo and CFD definitions with sine and cosine function. In all analog-based electronics, sinusoidal signals have a dominant role. The sinusoidal variable signal is one of the most commonly encountered electrical signals. It is easy to produce sinusoidal voltages and currents with devices such as generators or generators [6].The sinusoidal signals is determined by the fact that the utility

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grid supplies sinusoidal voltages. Sinusoidal signals are widely used in radiotechnics and telecommunications as carrier waves and synchronization signals. They are used in measurement, the basic type of measurement signal [7].

2. FRACTIONAL STATE-SPACE EQUATIONS

Equation of state has the following form [9]:

$$D^{\alpha} x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \le 1 \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, are the state, input and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$.

In the next sections we will use the fractional order state-space equations (1) with fractional order derivatives given by the Caputo and CFD definitions.

The Caputo fractional order derivative is given by [9, 7, 14]:

$${}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau. \tag{2}$$

where $n-1 < \alpha < n$, $n \in N, \Gamma(x)$ is the Euler gamma function and $f^{(n)}(t) = \frac{d^n f(t)}{dt^n}$.

The solution to the state-space equation (1) with derivative (2) is given by [2, 9, 14]:

$$x(t) = \Phi_0(t) x(0) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \tag{3a}$$

where

$$\Phi_0(t) = E_{\alpha}(At^{\alpha}) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \qquad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \tag{3b}$$

$E_{\alpha}(x)$ is the one parameter Mittag-Leffler function and $x(0)$ is initial condition.

If $n < \alpha \le n+1, n \in N_0$, then the conformable fractional derivative (CFD) of *n*- differentiable at *t* function *f* (where t > 0) is defined as [11]:

$${}_0^{CFD} D_t^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{(\lceil \alpha \rceil - 1)}\left(t + \varepsilon t^{\lceil \alpha \rceil - \alpha}\right) - f^{(\lceil \alpha \rceil - 1)}(t)}{\varepsilon}, \tag{4}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

The solution to the equation (1) with CFD definition of fractional order derivative (4) for $0 < \alpha \le 1$ is given by [1]:

$$x(t) = e^{A\frac{t^{\alpha}}{\alpha}} x(0) + e^{A\frac{t^{\alpha}}{\alpha}} \int_{0}^{t} e^{-A\frac{\tau^{\alpha}}{\alpha}} Bu(\tau)\tau^{\alpha-1} d\tau, \tag{5a}$$

where

$$e^{A\frac{t^{\alpha}}{\alpha}} = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\alpha^k k!}. \tag{5b}$$

3. FRACTIONAL ELECTRICAL CIRCUIT AND GENERAL DESCRIPTION OF THE PROBLEM

In this paper we will consider the fractional electrical circuit shown in Fig. 1 with resistor R, supercapacitor C and source voltage e.

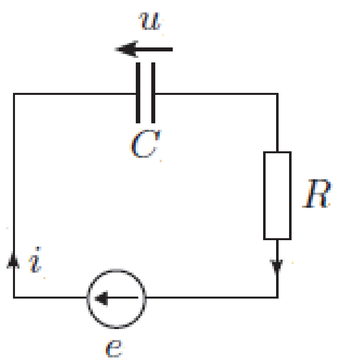

Fig. 1. The fractional electrical circuit

Using Kirchhoff's laws we obtain the equations describing the dynamics of voltages $u(t)$ on the respective capacitors in response to a control voltage $u(t)$. The appropriate circuit analysis has been concluded in [14].

The electrical circuit can be described by the state-space equation (1) with state vector $x(t) = [u(t)]$, input vector $u(t) = [e]$ and matrices [14]:

$$A = \left[-\frac{1}{RC_{\alpha}} \right], \qquad B = \left[\frac{1}{RC_{\alpha}} \right]. \tag{6}$$

The initial condition (the initial voltage across the capacitor) is given in the form:

$$x(0) = [u_0] = [u(0)]. \tag{7}$$

4. ANALYSIS OF THE SOLUTIONS WITH CAPUTO DEFINITION

4.1. Caputo definition for zero initial conditions $u_0=0$ and sine function input

Because $x(0)=[0]$, first component of the solution (3a) disappears.

$$x(t)=\int_0^t \Phi(t-\tau)Bu(\tau)d\tau. \tag{8}$$

Because control voltages can be both sine and cosine data consider the control voltage $u(\tau)$, giving a Marclaurin series for $\tau \geq 0$ [3]:

$$u(\tau)=1(\tau)\sum_{k=0}^{\infty} w_k \tau^k, \tag{9a}$$

where $w_k=\dfrac{u^{(k)}(0)}{k!}$ and

$$1(\tau)=\begin{cases} 1 & when \quad \tau \geq 0, \\ 0 & when \quad \tau < 0. \end{cases} \tag{9b}$$

Substituting formula (9a) to the solution (8):

$$x(t)=\int_0^t \Phi(t-\tau)B\sum_{k=0}^{\infty} w_k \tau^k d\tau. \tag{10}$$

In the formula (10) we change the order of summation with the integration and draw the coefficients w_k before the integral [3]:

$$x(t)=\sum_{k=0}^{\infty}\left[\int_0^t \Phi(t-\tau)B\tau^k d\tau\right]w_k. \tag{11}$$

We will replace the vector multiplication order B and τ^k in the equation (11). We have:

$$x(t)=\sum_{k=0}^{\infty}\left[\int_0^t \Phi(t-\tau)\tau^k d\tau\right]Bw_k. \tag{12}$$

In the next step we calculate the integral occurring in (12) using (3b):

$$\int_0^t \Phi(t-\tau)\tau^k d\tau=\int_0^t \sum_{l=0}^{\infty}\frac{A^l(t-\tau)^{(l+1)\alpha-1}}{\Gamma\left[(l+1)\alpha\right]}\tau^k d\tau. \tag{13}$$

Next change the order of summation with the integration and turn off the integral factors independent of τ:

$$\int_0^t (t-\tau)^{(l+1)\alpha-1}\tau^k d\tau = \left[\int_0^1 (1-\xi)^{(l+1)\alpha-1}\xi^k d\xi\right] t^{(l+1)\alpha+k} = B\big(k+1,\ (l+1)\alpha\big)t^{(l+1)\alpha+k}. \quad (14)$$

The integral in formula (14) describes the special function Beta [3]:

$$B\big(k+1,(l+1)\alpha\big) = \frac{\Gamma(k+1)\,\Gamma\big[(l+1)\alpha\big]}{\Gamma\big[k+1+(l+1)\alpha\big]} = \frac{k!\Gamma\big[(l+1)\alpha\big]}{\Gamma\big[k+1+(l+1)\alpha\big]}. \quad (15)$$

From the formulas (14) and (15) we have [3]:

$$\int_0^t (t-\tau)^{(l+1)\alpha-1}\tau^k d\tau = \frac{k!\Gamma\big[(l+1)\alpha\big]}{\Gamma\big[k+1+(l+1)\alpha\big]} t^{(l+1)\alpha+k}. \quad (16)$$

Substituting of (16) into (13) yields:

$$\int_0^t \Phi(t-\tau)\tau^k d\tau = \sum_{l=0}^{\infty} \frac{A^l t^{(l+1)\alpha+k}}{\Gamma\big[(l+1)\alpha\big]} \frac{k!\,\Gamma\big[(l+1)\alpha\big]}{\Gamma\big[k+1+(l+1)\alpha\big]} = \sum_{l=0}^{\infty} \frac{A^l k!\,t^{(l+1)\alpha+k}}{\Gamma\big[k+1+(l+1)\alpha\big]}. \quad (17)$$

Using (16) and (12) we have:

$$x(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{A^l B w_k k!\,t^{(l+1)\alpha+k}}{\Gamma\big[k+1+(l+1)\alpha\big]}. \quad (18)$$

When we replace the indexes *l*+1 to *l* :

$$x(t) = \sum_{l=1}^{\infty}\sum_{k=0}^{\infty} \frac{A^{l-1} B w_k k!}{\Gamma(k+1+l\alpha)} t^{l\alpha+k}. \quad (19)$$

Formula (19) is a solution of equation (1) for a control voltage that can be represented as a function series (9a).

4.2. Consider a case when control is given by the sine function $u(\tau)=E1(\tau)\sin(\omega\tau)$ with zero initial conditions

Then the coefficients of the Maclaurin series (9a) of function $u(\tau) = E\sin(\omega\tau)$ for $\tau \ge 0$, are [3]:

$$w_k = \frac{E\sin^{(k)}(\omega\tau)\big|_{\tau=0}}{k!} = \begin{cases} (-1)^p \dfrac{\omega^{2p+1}}{(2p+1)\,!} E & when \quad k = 2p+1 \ \text{ i } \ p \in \mathrm{N}_0, \\ 0 & when \quad k = 2p \ \text{ i } \ p \in \mathrm{N}_0. \end{cases} \quad (20)$$

We skip components with even indexes *k* in equation (19):

$$x(t)=\sum_{l=1}^{\infty}\sum_{k=1,3,5\ldots}\frac{A^{l-1}Bw_k k!}{\Gamma(l\alpha+k+1)}t^{l\alpha+k}. \tag{21}$$

index k was substituted $2p+1$

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{A^{l-1}Bw_{2p+1}(2p+1)!}{\Gamma(l\alpha+2p+2)}t^{l\alpha+2p+1}. \tag{22}$$

Next the coefficients (21) are substituted into the formula (22):

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{A^{l-1}B(2p+1)!}{\Gamma(l\alpha+2p+2)}(-1)^p\frac{\omega^{2p+1}}{(2p+1)!}Et^{l\alpha+2p+1}. \tag{23}$$

Then, we get

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{A^{l-1}BE(-1)^p\omega^{2p+1}}{\Gamma(l\alpha+2p+2)}t^{l\alpha+2p+1}. \tag{24}$$

4.3. Consider a case when control is given by the cosine function $u(\tau)=E1(\tau)\cos(\omega\tau)$ with zero initial conditions

Then the coefficients of the Maclaurin series (9a) of function $u(\tau)=E\cos(\omega\tau)$ for $\tau\geq 0$, are [3]:

$$w_k=\frac{E\cos^{(k)}(\omega\tau)\big|_{\tau=0}}{k!}=\begin{cases}(-1)^p\dfrac{\omega^{2p}}{(2p)\;!}E & \textit{when} \quad k=2p \text{ and } p\in \mathrm{N}_0,\\ 0 & \text{when} \quad k=2p+1 \text{ and } p\in \mathrm{N}_0.\end{cases} \tag{25}$$

We skip components with odd indexes k in equation (19):

$$x(t)=\sum_{l=1}^{\infty}\sum_{k=0,2,4\ldots}\frac{A^{l-1}Bw_k k!}{\Gamma(l\alpha+k+1)}t^{l\alpha+k}. \tag{26}$$

index k was substituted $2p$

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{A^{l-1}Bw_{2p}(2p)!}{\Gamma(l\alpha+2p+1)}t^{l\alpha+2p}. \tag{27}$$

We substitute the coefficients with the formula (25) for solving (27):

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p A^{l-1}BE\omega^{2p}}{\Gamma(l\alpha+2p+1)}t^{l\alpha+2p}. \tag{28}$$

4.4. Consider the control in the form of a sine function with phase shift *φ* at zero initial conditions

$$u(\tau)=E1(\tau)\sin(\omega\tau+\phi). \tag{29}$$

For non-negative times, we can use reductive formulas and write control functions in the form of a combination of linear sine and cosine functions.

$$u(\tau)=E\cos\phi\sin(\omega\tau)+E\sin\phi\cos(\omega\tau). \tag{30}$$

Consequently, the voltage across the capacitor will be the linear combination of the solutions given by the formulas (24) and (28), with the coefficients $E\cos\phi$ and $E\sin\phi$:

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{A^{l-1}BE\cos\phi(-1)^{p}\,\omega^{2p+1}}{\Gamma(l\alpha+2p+2)}t^{l\alpha+2p+1}+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{A^{l-1}BE\sin\phi(-1)^{p}\,\omega^{2p}}{\Gamma(l\alpha+2p+1)}t^{l\alpha+2p}=$$

$$=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{A^{l-1}(-1)^{p}\,\omega^{2p}}{\Gamma(l\alpha+2p+1)}\left(\frac{\omega\cos\phi}{l\alpha+2p+1}t^{l\alpha+2p+1}+t^{l\alpha+2p}\sin\phi\right)BE. \tag{31}$$

5. ANALYSIS OF THE SOLUTIONS OF CFD DEFINITION

5.1. CFD definition for zero initial conditions $u_0=0$

Because $x(0)=[0]$, first component of the solution (5a) disappears.

$$x(t)=e^{\frac{At^{\alpha}}{\alpha}}\int_{0}^{t}e^{-\frac{A\tau^{\alpha}}{\alpha}}Bu(\tau)\tau^{\alpha-1}d\tau=\int_{0}^{t}e^{\frac{A(t^{\alpha}-\tau^{\alpha})}{\alpha}}Bu(\tau)\tau^{\alpha-1}d\tau. \tag{32}$$

Put the formula (9a) to the solution (32)

$$x(t)=\int_{0}^{t}e^{\frac{A(t^{\alpha}-\tau^{\alpha})}{\alpha}}B\sum_{k=0}^{\infty}w_{k}\tau^{k}\;\tau^{\alpha-1}d\tau. \tag{33}$$

In the formula (33) we change the order of summation with the integration and we draw the coefficients w_k before the integral:

$$x(t)=\sum_{k=0}^{\infty}\int_{0}^{t}e^{\frac{A(t^{\alpha}-\tau^{\alpha})}{\alpha}}Bw_{k}\tau^{k-1+\alpha}d\tau. \tag{34}$$

We change the order of vector multiplication Bw_k and $\tau^{k-1+\alpha}$ in (34). We have:

$$x(t)=\sum_{k=0}^{\infty}\left(\int_0^t e^{\frac{A\left(t^{\alpha}-\tau^{\alpha}\right)}{\alpha}}\tau^{k-1+\alpha}d\tau\right)Bw_k\ . \tag{35}$$

In the next step we calculate the integral occurring in (35) using (5b):

$$\int_0^t e^{\frac{A\left(t^{\alpha}-\tau^{\alpha}\right)}{\alpha}}\tau^{k-1+\alpha}d\tau=\int_0^t\sum_{l=0}^{\infty}\frac{A^l\left(t^{\alpha}-\tau^{\alpha}\right)^l}{\alpha^l l!}\tau^{k-1+\alpha}d\tau. \tag{36}$$

We change the order of summation with the integration and turn off the integral factors independent of τ:

$$\int_0^t e^{\frac{A\left(t^{\alpha}-\tau^{\alpha}\right)}{\alpha}}\tau^{k-1+\alpha}d\tau=\sum_{l=0}^{\infty}\frac{A^l}{\alpha^l l!}\int_0^t\left(t^{\alpha}-\tau^{\alpha}\right)^l\tau^{k-1+\alpha}d\tau. \tag{37}$$

Calculate the integral under the sum of the formula (37) substituting $\tau=\xi^{\frac{1}{\alpha}}t$, where $0\le\xi\le1$:

$$\begin{aligned}\int_0^t\left(t^{\alpha}-\tau^{\alpha}\right)^l\tau^{k-1+\alpha}d\tau&=\int_0^1\left[t^{\alpha}-\left(\xi^{\frac{1}{\alpha}}t\right)^{\alpha}\right]^l\left(\xi^{\frac{1}{\alpha}}t\right)^{k-1+\alpha}t\frac{1}{\alpha}\xi^{\frac{1}{\alpha}-1}d\xi=\\&=t\frac{1}{\alpha}\int_0^1\left(t^{\alpha}-\xi t^{\alpha}\right)^l\xi^{\frac{k-1+\alpha}{\alpha}}t^{k-1+\alpha}\xi^{\frac{1}{\alpha}-1}d\xi=\\&=\frac{1}{\alpha}t^{(l+1)\alpha+k}\int_0^1(1-\xi)^l\xi^{\frac{k-1+\alpha+1}{\alpha}-1}d\xi=\frac{1}{\alpha}t^{(l+1)\alpha+k}\int_0^1(1-\xi)^l\xi^{\frac{k}{\alpha}}d\xi.\end{aligned} \tag{38}$$

The last integral in formula (38) describes the beta function, which is expressed by the gamma function [11]:

$$\int_0^1(1-\xi)^l\xi^{\frac{k}{\alpha}}d\xi=B\left(\frac{k}{\alpha}+1,l+1\right)=\frac{\Gamma\left(\frac{k}{\alpha}+1\right)\Gamma(l+1)}{\Gamma\left(\frac{k}{\alpha}+l+2\right)}=\frac{\Gamma\left(\frac{k}{\alpha}+1\right)l!}{\Gamma\left(\frac{k}{\alpha}+l+2\right)}. \tag{39}$$

By substituting the final results (44) into formula (43) we obtain:

$$\int_0^t\left(t^{\alpha}-\tau^{\alpha}\right)^l\tau^{k-1+\alpha}d\tau=\frac{\Gamma\left(\frac{k}{\alpha}+1\right)l!}{\alpha\Gamma\left(\frac{k}{\alpha}+l+2\right)}t^{(l+1)\alpha+k}. \tag{40}$$

To the formula (37) we insert the calculated integral (40):

$$\int_0^t e^{\frac{A\left(t^\alpha-\tau^\alpha\right)}{\alpha}} \tau^{k-1+\alpha} d\tau = \sum_{l=0}^{\infty} \frac{A^l}{\alpha^l l!} \frac{\Gamma\left(\frac{k}{\alpha}+1\right) l!}{\alpha \Gamma\left(\frac{k}{\alpha}+l+2\right)} t^{(l+1)\alpha+k} = \sum_{l=0}^{\infty} \frac{A^l \Gamma\left(\frac{k}{\alpha}+1\right)}{\alpha^{l+1} \Gamma\left(\frac{k}{\alpha}+l+2\right)} t^{(l+1)\alpha+k}. \quad (41)$$

Result (41) will be inserted (35):

$$x(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{A^l B w_k \Gamma\left(\frac{k}{\alpha}+1\right)}{\alpha^{l+1} \Gamma\left(\frac{k}{\alpha}+l+2\right)} t^{(l+1)\alpha+k}. \quad (42)$$

When we replace the indexes l to l-1:

$$x(t) = \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{A^{l-1} B w_k \Gamma\left(\frac{k}{\alpha}+1\right)}{\alpha^{l} \Gamma\left(\frac{k}{\alpha}+l+1\right)} t^{l\alpha+k}. \quad (43)$$

5.2. Consider a case when control is given by the sine function $u(\tau)=E1(\tau)\sin(\omega\tau)$ with zero initial conditions

The coefficients in the function series where $u(\tau)$ are given by the formula (20). In formula (43) we omit components with indices $k = 2p$ (even):

$$x(t) = \sum_{l=1}^{\infty} \sum_{k=1,3,5,\ldots} \frac{A^{l-1} B w_k \Gamma\left(\frac{k}{\alpha}+1\right)}{\alpha^{l} \Gamma\left(\frac{k}{\alpha}+l+1\right)} t^{l\alpha+k}. \quad (44)$$

index k (44) was substituted $2p+1$

$$x(t) = \sum_{l=1}^{\infty} \sum_{p=0}^{\infty} \frac{A^{l-1} B w_{2p+1} \Gamma\left(\frac{2p+1}{\alpha}+1\right)}{\alpha^{l} \Gamma\left(\frac{2p+1}{\alpha}+l+1\right)} t^{l\alpha+2p+1}. \quad (45)$$

Substitute coefficients given by the formulas (20) in (45):

$$x(t) = \sum_{l=1}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma\left(\frac{2p+1}{\alpha}+1\right) A^{l-1} B E \omega^{2p+1}}{(2p+1)!\, \Gamma\left(\frac{2p+1}{\alpha}+l+1\right) \alpha^l} t^{l\alpha+2p+1}. \quad (46)$$

5.3. Consider a case when control is given by the cosine function $u(\tau)=E1(\tau)cos(\omega\tau)$ with zero initial conditions

The coefficients w_k in the function series where $u(\tau)$ are given by the formula (25). In the formula (46) we omit components with indices $k=2p+1$ (odd):

$$x(t)=\sum_{l=1}^{\infty}\sum_{k=0,2,4,\ldots} w_k \frac{A^{l-1}Bw_k\Gamma\left(\frac{k}{\alpha}+1\right)}{\alpha^l\Gamma\left(\frac{k}{\alpha}+l+1\right)}t^{l\alpha+k} \tag{47}$$

index k was substituted $2p$

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{A^{l-1}Bw_{2p}\Gamma\left(\frac{2p}{\alpha}+1\right)}{\alpha^l\Gamma\left(\frac{2p}{\alpha}+l+1\right)}t^{l\alpha+2p}. \tag{48}$$

We substitute the data coefficients with patterns (25) to solve (48):

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p\,\Gamma\left(\frac{2p}{\alpha}+1\right)A^{l-1}BE\omega^{2p}}{\Gamma\left(\frac{2p}{\alpha}+l+1\right)(2p)\;!\alpha^l}t^{l\alpha+2p}. \tag{49}$$

5.4. Consider the control in the form of a sine function with phase shift φ (30) at zero initial conditions

For non-negative times, we can use reductive formulas and write control functions in the form of a combination of linear sine and cosine functions.

$$u(\tau)=E\cos\phi\sin(\omega\tau)+E\sin\phi\cos(\omega\tau) \tag{50}$$

Consequently, the voltage across the capacitor will be the linear combination of the solutions given by the formulas (51) and (54), with the coefficients $E\cos\phi$ and $E\sin\phi$:

$$x(t)=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p\,\Gamma\left(\frac{2p}{\alpha}+1\right)A^{l-1}BE\omega^{2p}\sin\phi}{\Gamma\left(\frac{2p}{\alpha}+l+1\right)(2p)\;!\alpha^l}t^{l\alpha+2p}+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p\,\Gamma\left(\frac{2p+1}{\alpha}+1\right)A^{l-1}BE\omega^{2p+1}\cos\phi}{(2p+1)\;!\Gamma\left(\frac{2p+1}{\alpha}+l+1\right)\alpha^l}t^{l\alpha+2p+1}=$$

$$=\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p\,A^{l-1}BE\omega^{2p}}{(2p)\;!\alpha^l}\left[\frac{\Gamma\left(\frac{2p}{\alpha}+1\right)\sin\phi}{\Gamma\left(\frac{2p}{\alpha}+l+1\right)!}t^{l\alpha+2p}+\frac{\Gamma\left(\frac{2p+1}{\alpha}+1\right)\omega\cos\phi}{(2p+1)\Gamma\left(\frac{2p+1}{\alpha}+l+1\right)}t^{l\alpha+2p+1}\right]. \tag{51}$$

6. NUMERICAL ANALYSIS

The parameters of simulations are conductances R=5 Ω; capacitance C=0.5 F; initial voltages u_0=0.0 V and source constant voltage E=1.0 V, $\phi = 0.5$ rad, $\omega = 1.0$ rad/s. The solutions using the Caputo definition for voltage across capacitor C for different fractional orders is shown in Fig.2. Solutions for CFD definitions for α=0.7; 0.8; 1.0 are shown in Fig. 3 for the first capacitor. The comparison of the solutions for Caputo and CFD definitions are presented in Fig. 4.

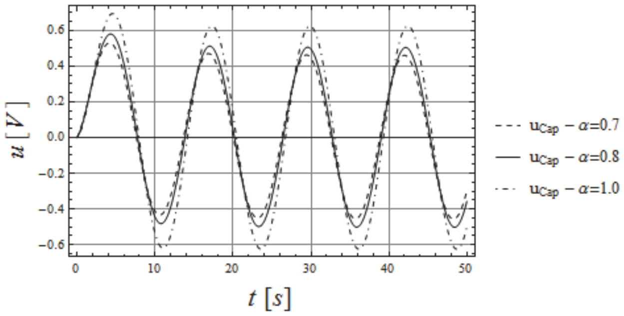

Fig. 2. Solution using the Caputo definition for the first capacitor for α=0.7; 0.8; 1.0

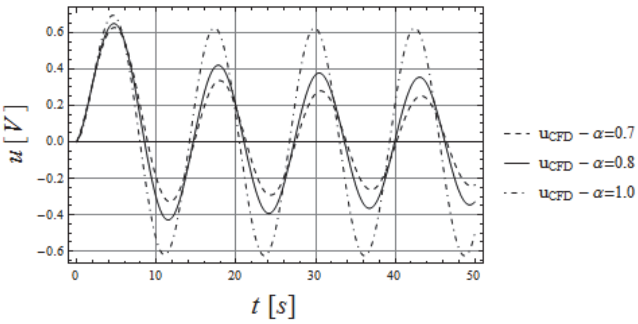

Fig. 3. Solution using the CDF definition for the first capacitor for α=0.7; 0.8; 1.0

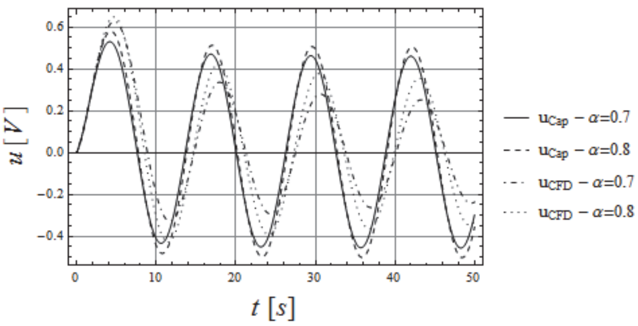

Fig. 4. Comparison of solutions using Caputo and CFD definitions for the first capacitor for α=0.7; 0.8

7. SUMMARY

The paper presents the method of calculating the voltage on the elements of the fractional electrical circuit. It was found the following conclusions: a) for control voltages of the form (42) voltage capacitors for times *t*, become similar to the corresponding sine function with a certain phase shift and amplitude. The period of the signal is identical to the control function. This correctness is observed both for the derivative of a non-integer derivative according to Caputo and for the definition CDF, b) in the case of the Caputo and CDF derivatives, the higher the order derivative, then higher the amplitude of sine function to which the graphs for large time are similar. For derivatives of the order more than zero and less than one, the amplitude of sinusoidal functions to which the solutions for large times converge are bigger for Caputo than CFD solution, c) for times the capacitance on the first and second capacitors calculated using the Caputo derivative have the phase shift the larger, the smaller is the order derivative. A similar rule was observed for voltage first capacitor, according to the CDF definition.

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(Received: 11.01.2019, revised: 04.03.2019)