Dynamic Stability of Mechanically and Thermally Loaded Three-Layered Annular Plate with Viscoelastic Core

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Abstract

The problem of dynamic stability of composite three-layered annular plate with viscoelastic core is the subject of the consideration. Plate composed of thin outer layers and soft core is loaded quickly in time with forces compressing facings and with temperature gradient in radial direction. Two kinds of plate support system are analysed: plate slideably clamped in thermo-mechanical problem and plate clamped on both edges in thermal analysis. The analytical and numerical solution, which is based on the orthogonalization and finite difference methods includes axisymmetric and asymmetric forms of buckling and rheological properties of plate core.

Keywords: dynamic stability, composite plate, thermo-mechanical loading, rheological material

1. Introduction

The composite, annular plate in complex field of loading can be found in different applications, like in: aerospace industry, mechanical and nuclear engineering. Selected plate parameters and directional gradient of temperature field create examined case of problem as dedicated to specific applications. It is a current and still developed issue, for example presented in works [1,2] where critical buckling and dynamic postbuckling responses of composite plate structures and FGM annular plates with imperfections are considered. In this paper reactions of composite, three-layered annular plate subjected to dynamically increasing mechanical loads and located in variable temperature field will be shown.

2. Problem formulation

The evaluation of the dynamic reaction of three-layered, composite annular plate is the objective of the undertaken consideration. Plate structure is composed of thin steel facings and thicker foam core with rheological properties. Plate is linearly in time, thermally or/and mechanically loaded with forces or/and temperature difference between edges, respectively. The mechanical loading and temperature differences are expressed by:

$$p = st, \qquad \Delta T = at \tag{1}$$

where: p – compressive stress, s – rate of mechanical loading growth, ΔT – temperature difference, a – rate of temperature loading growth, t – time.

Plate with clamped-clamped (C-C) edges is loaded only thermally but complex thermo-mechanical loading exists for plate with both edges slideably clamped (SC-SC). Figure 1 shows the scheme of examined plate in thermal environment with temperatures T_i and T_o in the area of plate hole and outer perimeter, respectively. Undertaken dynamic stability problem requires to adopt the criterion of the loss of stability. The criterion presented by Volmir in work [3] was adopted. According to this criterion the loss of plate stability occurs at the moment when the speed of the point of maximum deflection reaches the first maximum value. Black dots shown in Figures mean the moment of the loss of plate dynamic stability.

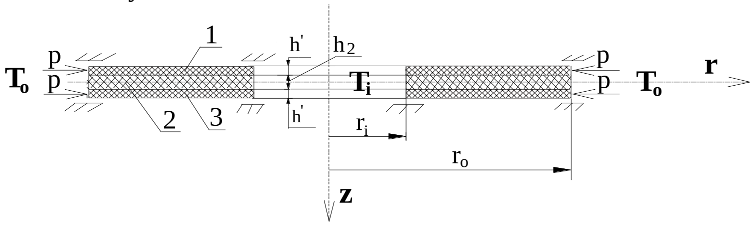

Figure 1. Scheme of three-layered annular plate composed of facings (layers 1,3) and core (layer 2) loaded with compressive stress *p* and subjected to axisymmetrical temperature field T_i, T_o

3. Methods of problem solution

The main solution method is based on the orthogonalization and finite difference ones (FDM). Results have been compared with ones obtained using numerical, finite element method (FEM). The main assumptions accepted in both FDM and FEM methods of solution are following: classical theory of sandwich is used, forces compress plate facings and are uniformly distributed on inner or outer edge of facings, thickness of the individual layers is fixed, plate structure has symmetrical cross-section, deformation of plate elastic facings is expressed by nonlinear geometry, plate layers are tied.
Assumptions connected with plate thermal environment are as follows: material constants do not depend on temperature, plate is subjected to the flat, time-dependent, axisymmetric field of temperature, heat flow exists in radial direction of plate facings, exchange of the heat on the plate surfaces is neglected, thermal isotropy exists.

3.1. Finite difference method

The FDM solution process is based on the solution proposed in work [4] for plate with slideably clamped edges (SC-SC) loaded mechanically. The main elements using in solution are following: formulation of dynamic equilibrium equations, formulation of the sectional forces and moments in facings, acceptance of the stress function to determine the resultant membrane forces, determination of the shape functions and form of plate predeflection, acceptance of dimensionless quantities and expressions, like for example: $\zeta_1=w_d/h$, where: w_d – additional plate deflection, $h=2h'+h_2$ – total plate thickness (h' - facing thickness, h_2 - core thickness;) and connected with mechanical loading (see, Eq. 1): $t^*=t\cdot K7$, $K7=s/p_{cr}$, where: p_{cr} – critical static load, and connected with thermal loading (see, Eq. 1): $t^*=t\cdot TK7$, $TK7=a/\Delta T_f$, where: ΔT_f – fixed temperature difference. Presented solution has been generalized on plate models subjected to temperature field and clamped-clamped (C-C) supported cases. The temperature distribution is a function of plate radius and is expressed by logarithmic equation according to theory presented in work [5]:

$$T_N = T_o + \frac{T_i - T_o}{\ln \rho_i} \ln \rho \tag{2}$$

where: T_i, T_o – temperatures of the inner and outer plate perimeters, $\rho=r/r_o$, $\rho_i=r_i/r_o$ - dimensionless plate radius and dimensionless inner plate radius, r_i, r_o – inner and outer plate radius, respectively.

Applied physical relations for viscoelastic material model of plate core correspond with the expressions of accepted three-elements, standard model. Kirchhoff's modulus $\tilde{G}_2$ is presented using the operator forms:

$$\tilde{G}_2\left(\tfrac{\partial}{\partial t}\right) = \frac{C_L + D_L \frac{\partial}{\partial t}}{E_L + F_L \frac{\partial}{\partial t}} \tag{3}$$

where: C_L, D_L, E_L, F_L - quantities expressed by the elastic constants G_2, G_2' and viscosity constant η of viscoelastic material of plate core.

Plate (SC-SC) with slideably clamped both edges can be loaded mechanically or thermally, or thermo-mechanically.

The main conditions of mechanically loaded plate edges (SC-SC) are expressed by the stress function Φ and formulae for radial stress σ_r in edge points of discretization and its derivative with respect to time t:

$$\text{for } r = r_{i(o)} \quad \sigma_r = \frac{1}{r}\Phi,_r = -s \cdot t \cdot d_{1(2)} \text{ and } \sigma_{r't} = -s \cdot d_{1(2)} \tag{4}$$

but for plate edges (SC-SC) subjected to only thermal loads conditions are expressed by:

$$\sigma_r\big|_{r=r_{i(o)}} = 0 \tag{5}$$

where: d_1, d_2 – quantities, equal to 0 or 1, determining the loading of the inner or/and outer plate perimeter.

The conditions for thermally loaded plate with edges (C-C) are expressed by the equations (7), (8) established in discrete points 0 and N+1, which are the points of the plate support. Equations (7), (8) have been obtained from equation (6), which is expressed using the relations of Hooke's law in plane stress state performed for normal forces with thermal elements in plate facings and after the elimination of the radial *(r)* and circumferential *(θ)* strains and acceptance of stress function Φ:

$$F,_{\rho\rho} - \frac{\nu}{\rho}\left(F,_{\rho} + \frac{1}{\rho}F,_{\theta\theta}\right) = -S \cdot T_N \tag{6}$$

where: $S = \frac{r_o^2}{h^2}\alpha$, F - dimensionless stress function $F = \frac{\Phi}{Eh^2}$, α, E, ν - linear expansion coefficient, Young's modulus and Poisson's ratio of facing material, respectively.

Approximating the derivatives in points 0 and N+1 with the use of FDM differences in front and back the following conditions and derivatives with respect to time t were established:

for $\rho=\rho_i$ $y_o=\dfrac{b\rho_i}{\rho_i+b\nu}\left(\dfrac{y_1}{b}+S(T_o+at)\right)$, $\dot{y}_o=\dfrac{b\rho_i}{\rho_i+b\nu}\left(\dfrac{\dot{y}_1}{b}+S\dfrac{a}{TK7}\right)$, (7)

for $\rho=\rho_o$ $y_{N+1}=\dfrac{b}{1-b\nu}\left(\dfrac{y_N}{b}-ST_o\right)$, $\dot{y}_{N+1}=\dfrac{b}{1-b\nu}\left(\dfrac{\dot{y}_N}{b}\right)$, (8)

where: $y_o, y_1, y_N, y_{N+1}, \dot{y}_o, \dot{y}_1, \dot{y}_N, \dot{y}_{N+1}$- elements of stress function vector $Y=F_{,\rho}$ and derivatives with respect to time t in discrete points 0,1,N,N+1.

Solution process required a lot of algebraic operations and using the orthogonalization method. Then, approximating the derivatives with respect to ρ by the central differences in discrete points the following system of equations for three-layered annular plate with viscoelastic core in thermal environment has been obtained:

$$\boldsymbol{PU}+\boldsymbol{Q}+\boldsymbol{P}_L\dot{\boldsymbol{U}}+\boldsymbol{Q}_L-K\cdot\ddot{\boldsymbol{U}}=K\cdot\dddot{\boldsymbol{U}}\,, \tag{9}$$

$$\boldsymbol{M}_Y\boldsymbol{Y}=\boldsymbol{Q}_Y-\rho\cdot S\cdot T_{N',\rho}\,,\ \boldsymbol{M}_Y\dot{\boldsymbol{Y}}=\dot{\boldsymbol{Q}}_Y-S\rho T_{N',\rho t} \tag{10}$$

$$\boldsymbol{M}_{V(Z)}\boldsymbol{V}(Z)=\boldsymbol{Q}_{V(Z)}\,,\ \boldsymbol{M}_{V(Z)}\dot{\boldsymbol{V}}(\dot{Z})=\dot{\boldsymbol{Q}}_{V(Z)} \tag{11}$$

$$\boldsymbol{M}_{DL}\dot{\boldsymbol{D}}=\boldsymbol{M}_D\boldsymbol{D}+\boldsymbol{M}_U\boldsymbol{U}+\boldsymbol{M}_{UL}\dot{\boldsymbol{U}}+\boldsymbol{M}_G\boldsymbol{G}+\boldsymbol{M}_{GL}\dot{\boldsymbol{G}}\,, \tag{12}$$

$$\boldsymbol{M}_{GGL}\dot{\boldsymbol{G}}=\boldsymbol{M}_{GG}\boldsymbol{G}+\boldsymbol{M}_{GU}\boldsymbol{U}+\boldsymbol{M}_{GUL}\dot{\boldsymbol{U}}+\boldsymbol{M}_{GD}\boldsymbol{D}+\boldsymbol{M}_{GDL}\dot{\boldsymbol{D}}\,, \tag{13}$$

where: $K=TK7^2\cdot\dfrac{h'}{h}\cdot r_o h_2 M$; $M=2h'\mu+h_2\mu_2$; μ, μ_2 - facing and core mass density, respectively;

$\boldsymbol{U},\boldsymbol{Y},\boldsymbol{V},\boldsymbol{Z},\dot{\boldsymbol{U}},\ddot{\boldsymbol{U}},\dddot{\boldsymbol{U}},\dot{\boldsymbol{Y}},\dot{\boldsymbol{V}},\dot{\boldsymbol{Z}},\boldsymbol{Q},\boldsymbol{Q}_{\mathrm{L}},\boldsymbol{Q}_{\mathrm{Y}},\boldsymbol{Q}_{\mathrm{V}},\boldsymbol{Q}_{\mathrm{Z}},\dot{\boldsymbol{Q}}_{\mathrm{Y}},\dot{\boldsymbol{Q}}_{\mathrm{V}},\dot{\boldsymbol{Q}}_{Z},\boldsymbol{D},\boldsymbol{G},\dot{\boldsymbol{D}},\dot{\boldsymbol{G}}$ - vectors of initial and additional deflections, components of the stress function, geometric and material parameters, radius ρ, quantity b (b – length of the interval in FDM), coefficients δ, γ (differences of radial and circumferential displacements of points in the middle surfaces of facings) and number m of buckling waves and derivatives with respect to time t;

$\boldsymbol{P},\boldsymbol{P}_{\mathrm{L}},\boldsymbol{M}_{\mathrm{Y}}\boldsymbol{M}_{\mathrm{V}},\boldsymbol{M}_{\mathrm{Z}},\boldsymbol{M}_{\mathrm{D}},\boldsymbol{M}_{\mathrm{G}},\boldsymbol{M}_{\mathrm{GG}},\boldsymbol{M}_{\mathrm{GD}},\dot{\boldsymbol{M}}_{\mathrm{DL}},\dot{\boldsymbol{M}}_{\mathrm{GL}},\dot{\boldsymbol{M}}_{\mathrm{GGL}},\dot{\boldsymbol{M}}_{\mathrm{GDL}},\boldsymbol{M}_{\mathrm{U}},\boldsymbol{M}_{\mathrm{UL}},\boldsymbol{M}_{\mathrm{GU}},\boldsymbol{M}_{\mathrm{GUL}}$ - matrices with elements composed of geometric and material plate parameters, the quantity b, radius ρ, the number m and derivatives with respect to time t, respectively.

Way of solution to the problem of static stability of plate loaded thermally or mechanically is performed in works [4,6] in detail, respectively.

3.2. Finite element method

Annular plate model composed of shell and solid elements has been built using the finite element method. The outer surfaces of facing mesh elements are tied with the outer surfaces of core elements using the surface contact interaction. The calculations were carried out at the ACC CYFRONET-CRACOW using Dynamic option of the ABAQUS system (KBN/SGI_ORIGIN_2000/PŁódzka/030/1999).

4. Numerical examples

Dynamic reaction of three-layered annular plate will be shown for plate with slideably clamped SC-SC both edges loaded thermo-mechanically and for plate with both clamped edges C-C loaded thermally.

4.1. Calculation data

The accepted in numerical calculations material, geometrical and loading parameters of examined plate models are as follows:
inner radius: r_i=0.2 m, outer radius: r_o=0.5 m, facing thickness: h'=0.001 m, core thickness: h_2=0.0025 m, 0.005 m, 0.01 m, steel for facing material: Young's modulus $E=2.1\cdot10^5$ MPa, Poisson's ratio ν=0.3, mass density $\mu=7.85\cdot10^3$ kg/m^3, linear expansion coefficient α=0.000012 1/K, two kinds of polyurethane foam of core material treated as isotropic one: Kirchhoff's modulus G_2=5 MPa, Young's modulus E_2=13 MPa, viscosity constants: G_2'=3.13 MPa, $\eta'=212.92\cdot10^4$ MPa·s and G_2=15.82 MPa, G_2'=69.59 MPa, $\eta=7.93\cdot10^4$ MPa·s, Poisson's ratio ν=0.3, mass density μ_2=64 kg/m^3, linear expansion coefficient α=0.00007 1/K, the rates of mechanical loading growth is equal to: $s\approx931$ MPa/s (K7=20 1/s) – for plate loaded on outer edge, the rate of thermal loading growth is equal to: a=200 K/s; rates *K7*, *TK7* are equal *K7=TK7*. Thermal environment is characterized by axisymmetric, flat temperature field with positive ($T_i>T_o$) or negative ($T_i<T_o$) gradient (see, Fig 1).

4.2. Convergence analysis

The calculations carried out using the finite difference method have been proceeded by the selection of number *N* of discrete points. Table 1 presents the critical, static ΔT_{cr} and dynamic temperature difference ΔT_{crdyn} versus different plate modes *m* for FDM plate model with different number *N* of the discrete points, equal to N=11,14,17,21,26. Plate is subjected to the temperature field with positive temperature gradient.

Table 1. Critical static ΔT_{cr} and dynamic temperature differences ΔT_{crdyn} of FDM plate model C-C with core parameters: G_2=15.82 MPa, G_2'=69.59 MPa, $\eta=7.93\cdot10^4$ MPa·s loaded thermally with positive gradient versus different number *N* of the discrete points

m	ΔT_{cr} / ΔT_{crdyn}, K				
	N=11	N=14	N=17	N=21	N=26
0	42.17 / 42.3	42.21 / 41	42.23 / 40	42.24 / 39.6	42.51 / 38.9
5	44.31 / 35.2	44.50 / 34,4	44.60 / 34.2	44.68 / 34.1	44.74 / 33.5
6	45.45 / 34.4	45.72 / 33,6	45.85 / 33.5	45.95 / 33.8	46.02 / 33.3
7	46.93 / 34.3	47.18 / 33,5	47.33 / 33.4	47.44 / 33.3	47.53 / 33.2
8	48.67 / 34.2	48.93 / 34	49.09 / 33.9	49.22 / 33.3	49.31 / 33.3

The analysis of differences of ΔT_{crdyn} values, which depend on the number *N* shows small fluctuations (5 % of technical error). The number N=14 has been chosen in numerical calculations.

4.3 Plate C-C loaded thermally

Results presented in Figures 2,3,4,5 were calculated for plates loaded with positive temperature gradient. Figures 2 shows the comparison between the dynamic responds of plate with two kinds of viscoelastic core parameters represented by Kirchhoff's modulus: G_2=5 MPa and G_2=15.82 MPa, respectively. Presented results show the dynamic reaction of axisymmetric m=0 and asymmetric m≠0 plate modes. Character of behaviour is similar. Minimal value of dynamic critical temperature difference ΔT_{crdyn} is for waved plate mode. Plate with stiffer core (G_2=15.82 MPa) loses dynamic stability with higher value of temperature difference ΔT_{crdyn} calculated for plate mode with greater number of circumferential buckling waves (see, Table 2). Figure 3 shows the influence of viscoelastic core thickness on run of curves $\zeta_{1max}=f(t^*)$ for axisymmetric m=0 plate mode and asymmetric m≠0 ones, which correspond with minimal value of ΔT_{crdyn}. Plate core parameters are G_2=15.82 MPa, G_2'=69.59 MPa, η'=7.93·10^4 MPa·s. Of course with thicker core the value of ΔT_{crdyn} is higher and number *m* of circumferential buckling waves increases. For plate with core thickness h_2=0.01 m number *m* is equal to m=6. Figure 4 presents the influence of viscosity parameter η (eta) of the same plate core G_2=5 MPa on dynamic respond of plate, whose mode number *m* is equal to m=4. The influence of values of viscosity constant η is weak. Just, strongly decrease of values η changes plate reaction.

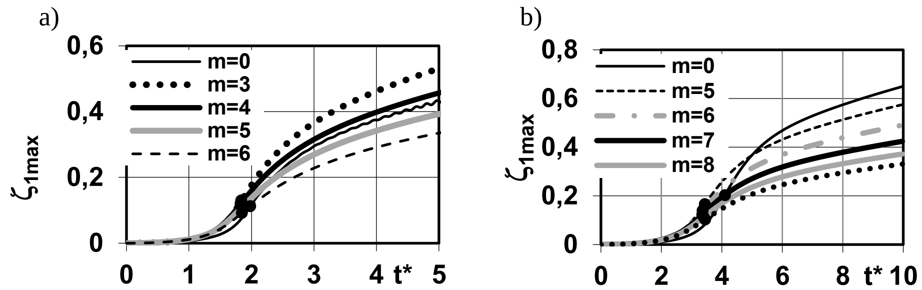

Figure 2. Time histories of deflections of FDM plate model with viscoelastic core: a) G_2=5 MPa, b) G_2=15.82 MPa depending on mode *m*

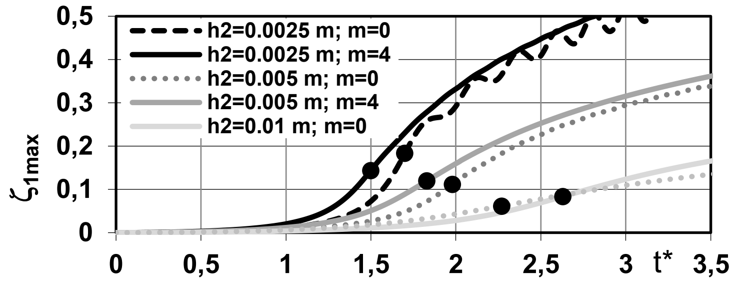

Figure 3. Time histories of deflections of FDM plate model with viscoelastic core G_2=5 MPa depending on core thickness h_2

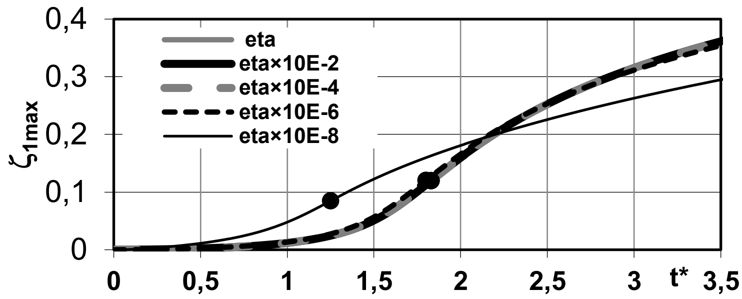

Figure 4. Time histories of deflections of FDM plate model with viscoelastic core G_2=5 MPa depending on viscosity parameter η' (eta)

Table 2 shows the example results for FDM plate model with different viscoelastic material and thickness of the core. FDM plate model with core parameters: G_2=5 MPa, h_2=0.005 m is compared with FEM model. Dynamic critical temperature differences ΔT_{crdyn} and buckling mode *m* depend on core parameters. The differences of ΔT_{crdyn} values for FEM plate with different mode *m* are less than observed for FDM model. Some range of ΔT_{crdyn} values exists for FEM plate model. Figure 5 shows the example run of curves of displacement and velocity of displacement, and buckling mode of FEM plate. Presented buckling case is for m=2.

Table 2. Critical dynamic temperature differences ΔT_{crdyn} of FDM and FEM plate model C-C with viscoelastic core loaded thermally with positive gradient versus different mode *m* and structure parameters G_2, h_2

m	ΔT_{crdyn}, K				
	FDM plate model				FEM plate model
	G_2, MPa / h_2, m				
	5 / 0.005	15.82 / 0.005	5 / 0.025	5 / 0.01	5 / 0.005
0	19,8	41	17	26,3	17
1	19,9	40,9	17,1	26,9	17
2	19,1	39	16,3	25,7	17
3	18,4	36,9	15,4	24,8	17
4	18,3	35,5	15	24	18
5	18,5	34,4	15,1	22,9	19

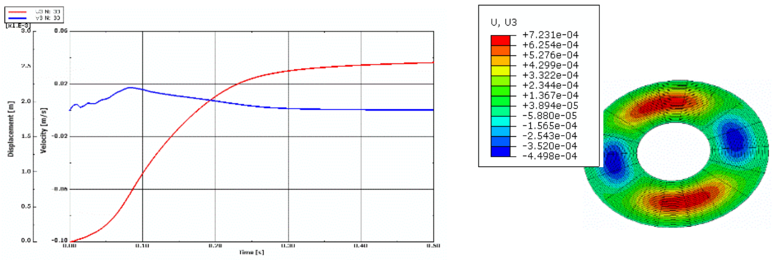

Figure 5. Time histories of deflection and velocity of deflection and buckling mode m=2 of FEM plate model with viscoelastic core G_2=5 MPa

4.4 Plate SC-SC loaded thermally and mechanically

Figure 6 presents values of both critical static p_{cr} and dynamic p_{crdyn} loads calcualted for FDM plate SC-SC with viscoelastic core characterized by the value G_2 equal to G_2=5 MPa versus number m of buckling mode. Plate is subjected to positive and negative tempartature gradiend. Figure shows the influence of thermal fields on dynamic reaction of plates mechanically loaded. Additionally, the comparison with results obtained in basic static analysis for plates loaded only mechanically is presented.
The dynamic respond of plate subjected to temperature field differs for plates with various buckling mode m. For axisymmetric m=0 plates and asymmetric ones with number m=1÷3 of circumferential, buckling waves direction of temperature gradient influences the values of p_{crdyn}. With the increase of number m the differences disappear. There are not observed significant differences between the values of p_{crdyn} calculated for plates subjected to temperature field and not located in thermal environment (see, mode m=7 in Figure 6).

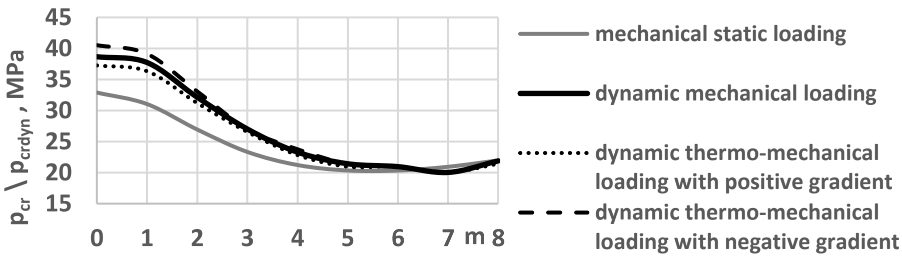

Figure 6. Distribution of values of static p_{cr} and dynamic p_{crdyn} loads for FDM plates SC-SC with viscoelastic core G_2=5 MPa for two temperature fields versus buckling mode m

5. Conclusions

The way of analytical and numerical solution to the problem of dynamic thermo-mechanical loading and dynamic sensitivity of composite plate with viscoelastic core have been presented. Results show the significant meaning of geometrical and material structure parameters, less influence of rheological core properties, values fluctuations and dynamic respond character of structure with different buckling waves, influence of temperature gradient on thermal and mechanical dynamic critical state of plate.

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